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PROOF OF A CONJECTIURE ON THE NORMAL EXTENSIONS OF Q^{ω}

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In this paper is proved that every normal extension X of Q^{ω} has the following property :

If H is an F_{σ}-subset of X and H cuts X between some two points of Q^{ω}, then H is strongly infinite dimensional in the sense of Smirnov.

This result proves one conjecture of Hadjiivanov and generalizes some facts.

A system $\{F_{-i}, F_{+i}\}_{i=1}^{n}$ of closed subsets of a given topological space Y will be called essential in Y if $F_{-i}\cap F_{+i}=\varnothing$, $i=1, 2, \dots, n$ and for any choice of partitions C_i in Y between F_{-i} and F_{+i} we have $\bigcap_{i=1}^{n} C_i \neq \varnothing$.

Let us remind that C is a partition in Y between the sets A and B if $Y=P\cup C\cup Q$, $A\subset P$, $B\subset Q$, the system $\{P, C, Q\}$ is disjoint and P and Q are open. When P and Q are closed then C is called an open partition in Y between A and B.

A system $\{F_{-i}, F_{+i}\}_{i=1}^{\infty}$ will be called ω-system in Y if $F_{-i}\cap F_{+i}=\varnothing$, F_{-i} and F_{+i} are closed subsets of Y, $i=1, 2, 3, \dots$, and for any choice of partitions C_i in Y between F_{-i} and F_{+i} the system $\{C_i\}_{i=1}^{\infty}$ has the finite intersectional property.

Following Yu. Smirnov [15, p. 528] a space X will be called S-weakly infinite dimensional if there is no ω-system in it, and is S-strongly infinite dimensional in the opposite case. It is clear that every closed subset of any S-weakly infinite dimensional space is also S-weakly dimensional.

Following N. Hadjiivanov a space X will be called S-connected if $X=\Phi_1\cup\Phi_2$, where Φ_n are closed in X and $\Phi_n\neq X$, $n=1, 2$, implies that the set $\Phi_1\cap\Phi_2$ is S-strongly infinite dimensional, and X will be called strongly S-connected if $X=\bigcup_{n=1}^{\infty}\Phi_n$, where Φ_n are closed in X and $\Phi_n\neq X$, $n=1, 2, \dots$, implies that the set $\bigcup_{i\neq j}(\Phi_i\cap\Phi_j)$ is S-strongly infinite dimensional [1].

Obviously the strong S-connectedness does imply S-connectedness, but the opposite does not always hold true even in the compact case, as is shown in [2]. It is clear that in every non-trivial Hausdorff space the S-connectedness implies S-strongly infinite dimensionality.

We will denote by Q^{ω} the set of those points of the Hilbert cube $[0,1]^{\aleph_0}$ at most finitely many of whose coordinates are different from zero. It is clear that $Q^{\omega}=\bigcup_{n=1}^{\infty}Q^n$, where $Q=[0, 1]$.

B. T. Levshenko [3] proved, that the space Q^{ω} is S-strongly infinite dimensional. E. G. Sklyarenko [4] proved that every compactification of Q^{ω} is also S-strongly infinite dimensional. Since the Hilbert cube is a compactification of Q^{ω} it follows that the final result is a generalization of a theorem of W. Hurewicz according to which the Hilbert cube is S-strongly infinite dimensional [5, p. 75]. N. Hadjiivanov proved that the Hilbert cube is even strongly S-connected [6]. Later he considerably improved his result by applying his method [6]. He proved that every compact (and even normal and countably compact) extension of Q^{ω} is also strongly S-connected. His proof remained unpublished. Later on he got to a second, shorter proof of the same fact, where the method applied combined some ideas [6] and [4] by E. G. Sklyarenko. Subsequently this method was used by N. Hadjiivanov and myself in two papers [7] and [8]. In the first one we found that every normal extension of the space Q^{ω} is S-connected, and in the second one we further improved this result by proving that if X is a normal extension of Q^{ω} and $X=\bigcup_{n=1}^{\infty}\Phi_n$, where Φ_n are closed in X and $\Phi_n\neq X$, $n=1, 2, \ldots$, then supposition, that the set $\bigcup_{i\neq j}(\Phi_i\cap\Phi_j)$ is S-weakly infinite dimensional, implies $\operatorname{Int}\Phi_n=\emptyset$ for each natural n.

In such a way we obtained the above mentioned theorem of N. Hadjiivanov, and also that every normal extension of Q^{ω}, which is of second category in itself is also strongly S-connected. N. Hadjiivanov made the following conjecture:

(x) *Every normal extension of Q^{ω} is strongly S-connected.*

The truth of this conjecture easily follows from Theorem 1 below and [7], according to the scheme elaborated by N. Hadjiivanov [9; 10; 11] and developed later in our jointwork [12].

Certainly, by a normal extension of Q^{ω} we mean a normal space X, with $Q^{\omega}\subset X$ and $\bar{Q}^{\omega}=X$.

Let X be a topological space, $M\subset X$ and $a_-, a_+\in X$. We shall say that M cuts X between the points a_- and a_+ if $a_-, a_+\in X\setminus M$ and there is no continuum K with $a_-, a_+\in K$ and $K\subset X\setminus M$.

Theorem 1. *Let X be a normal extension of Q^{ω}, H be an F_σ subset of X and $a_-, a_+\in Q^{\omega}$. Then if H cuts X between the points a_- and a_+, then H is S-strongly infinite dimensional,*

Proof of the supposition (x). Let us assume the contrary. Then $X=\bigcup_{n=1}^{\infty}\Phi_n$, where Φ_n are closed in X, $\Phi_n\neq X$, $n=1, 2, \ldots,$ and the set $H=\bigcup_{i\neq j}(\Phi_i\cap\Phi_j)$ is S-weakly infinite dimensional. We shall note first of all that a non-empty set O, open in Q^{ω}, for which $O\subset H$, does not exist. Really it there is such a set O it is easy to see that there exists a set of the type $P^{\omega}=Q^{\omega}\cap(\Delta_1\times\Delta_2\times\cdots\times\Delta_s\times Q\times Q\times\cdots)$, where Δ_i are closed and non-trivial subintervals of Q, $i=1, 2, \ldots, s$, for which $P^{\omega}\subset O$. Then $P^{\omega}\subset H$ and hence the space $Y=[P^{\omega}]_H$ is a normal extension of P^{ω}. Since Y is a closed subset of H and hence Y is S-weakly infinite dimensional. On the other hand, Y is a normal extension of P^{ω} and P^{ω} is homeomorphic to Q^{ω}, hence it follows by [7] lhat Y is S-strongly infinite dimensional. The obtained contradiction implies $\operatorname{Int}_{Q^{\omega}}(Q^{\omega}\cap H)=\emptyset$.

We shall note now that there does not exist an index n such that $Q^{\omega}\subset H\cup\Phi_n$. Reallyi f $Q^{\omega}\subset H\cup\Phi_n$, then $Q^{\omega}\subset\bar{\Phi}_n=\Phi_n$, because $\operatorname{Int}_{Q^{\omega}}(Q^{\omega}\cap H)=\emptyset$ Hence $X=\bar{Q}^{\omega}\subset\Phi_n$, which contradicts our condition $\Phi_n\neq X$.

So, from the statement just proved, it follows that card $(Q^{\omega}\setminus H)\geq 2$. Let us fix a point $a\in Q^{\omega}\setminus H$ and an index n_0, for which $a\in\Phi_{n_0}$. We shall prove that there exists a point $b\in Q^{\omega}\setminus H$, $b\neq a$ such that H cuts X between a and b. Thus we would conclude according to Theorem 1 that H is S-strongly infinite dimensional, which would contradict our assumption. The obtained contradiction proves the validity of supposition (x).

To prove the existence of a point b with the above properties we shall assume the contrary. Then for every $b \in Q^\omega \setminus (H \cup \{a\})$ there exists a continuum C_b, for which $a, b \in C_b$ and $C_b \subset X \setminus H$. Since $C_b = \bigcup_{n=1}^{\infty} (C_b \cap \Phi_n)$ and $(C_b \cap \Phi_i) \cap (C_b \cap \Phi_j) \subset C_b \cap H = \emptyset$ if $i \neq j$ the Sierpinski's theorem concerning continua implies the existence of an index $s = s(b)$, for which $C_b = C_b \cap \Phi_s$ and $C_b \cap \Phi_{s'} = \emptyset$ for all $s' \neq s$. But $a \in C_b \cap \Phi_{n_0}$, hence $s(b) = n_0$, $C_b = C_b \cap \Phi_{n_0}$, $C_b \subset \Phi_{n_0}$ and hence $b \in \Phi_{n_0}$. Thus we obtain $Q^\omega \setminus (H \cup \{a\}) \subset \Phi_{n_0}$ and hence $Q^\omega \subset H \cup \Phi_{n_0}$, which is impossible as it was shown above. In this way the validity of hypothesis (x) is proved.

To make the proof of Theorem 1 easier we will give some notions and lemmas.

Let us consider the n-dimensional Euclidean space R^n. If $p \in R^n$ and $M \subset R^n$ the symbol $\mathrm{con}(M, p)$ will denote a cone with base M and vertex p. Let us take the points $v_\varepsilon = (0, 0, \ldots, 0, \varepsilon 1)$, $\varepsilon = \mp$ from R^n and the subset $L = \{(x_1, x_2, \ldots, x_n) \in R^n : -1 \leq x_i \leq 1, i = 1, 2, \ldots, n-1$ and $x_n = 0\}$ of R^n. We put $P_n = L_- \cup L_+$, where $L_\varepsilon = \mathrm{con}(L, v_\varepsilon)$, $\varepsilon = \mp$. If $h : P^n \to Z$ is a homeomorphism "onto" then the space $Z = h(P^n)$ will be called n-dimensional double cone (n-dim d. c.) with vertices $a_\varepsilon = h(v_\varepsilon)$, $\varepsilon = \mp$. Let $L_{\varepsilon i} = \{(x_1, \ldots, x_n) \in L : x_i = \varepsilon 1\}$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$ and $P^n_{\varepsilon i} = \mathrm{con}(L_{\varepsilon i}, v_-) \cup \mathrm{con}(L_{\varepsilon i}, v_+)$. The sets $h(P^n_{\varepsilon i})$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$ are called sides of Z. It is clear that $h(P^n_{-i}) \cap h(P^n_{+i}) = \{a_-, a_+\}$, $i = 1, 2, \ldots, n-1$. Every system $\{F^n_{\varepsilon i} : \varepsilon = \mp, i = 1, 2, \ldots, n-1\}$ of closed subsets of Z, which satisfies conditions: a) $F^n_{-i} \cap F^n_{+i} = \{a_-, a_+\}$, $i = 1, 2, \ldots, n-1$ and b) $h(P^n_{\varepsilon i} \setminus \{v_-, v_+\}) \subset \mathrm{Int}\, F^n_{\varepsilon i}$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$ we shall call a system of thick sides of n-dim. d. c. Z with vertices a_- and a_+.

We shall prove the following lemma using some ideas from [13].

Lemma 1. *Let X be a compact space and the system $\{F_{-i}, F_{+i}\}_{i=1}^{s}$ be essential in X. We lay $\mathring{F}_{\varepsilon i} = F_{\varepsilon i} \setminus \Phi$, $\varepsilon = \mp$, $i = 1, 2, \ldots, s-1$, where $\Phi = F_{-s} \cup F_{+s}$. Then if C_i are partitions in X between the sets $\mathring{F}_{-i}$ and $\mathring{F}_{+i}$, $i = 1, 2, \ldots, s-1$, then there exists a continuum K in $\bigcap_{i=1}^{s-1} C_i$ connecting F_{-s} and F_{+s}, i. e. $K \subset \bigcap_{i=1}^{s-1} C_i$ and $F_{-s} \cap K \neq \emptyset \neq K \cap F_{+s}$.*

Proof. Suppose the contrary, then for every component C_α of $C = \bigcap_{i=1}^{s-1} C_i$ we can obtain either $C_\alpha \cap F_{-s} = \emptyset$ or $C_\alpha \cap F_{+s} = \emptyset$. In a compact space (such as C) the components coincide with the quasi-components. Therefore for every α there exists in C an open-closed set O_α such that $C_\alpha \subset O_\alpha$ and either $O_\alpha \cap F_{-s} = \emptyset$ or $O_\alpha \cap F_{+s} = \emptyset$.

Let us choose a finite subcover $\{O_{\alpha_v} : v = 1, \ldots, m\}$ of the cover $\{O_\alpha\}_\alpha$ of the compact space C. Let $C_- = \cup \{O_{\alpha_v} : O_{\alpha_v} \cap F_{+s} = \emptyset\}$ and $C_+ = C \setminus C_-$. Then $C = C_- \cup C_+$, $C_- \cap C_+ = \emptyset$ and C_-, C_+ are closed in C. Clearly the sets $F_{-s} \cup C_-$ and $F_{+s} \cup C_+$ are closed in X and disjoint. Therefore there exists a partition C_s in X between $F_{-s} \cup C_-$ and $F_{+s} \cup C_+$ and hence $\bigcap_{i=1}^{s} C_i = \emptyset$. Let $X = U_{-s} \cup C_s \cup U_{+s}$, where the system $\{U_{-s}, C_s, U_{+s}\}$ is disjoint, U_{-s}, U_{+s} are open in X and $F_{-s} \cup C_- \subset U_{-s}$, $F_{+s} \cup C_+ \subset U_{+s}$. Let V_{-s} and V_{+s} be open and $F_{-s} \cup C_- \subset V_{-s} \subset \bar{V}_{-s}$ and $F_{+s} \cup C_+ \subset V_{+s} \subset \bar{V}_{+s} \subset U_{+s}$. We put $X_0 = X \setminus (V_{-s} \cup V_{+s})$. Noting the fact that C_i are partitions in X between $\mathring{F}_{-i}$ and $\mathring{F}_{+i}$ we may state that

$\widetilde{C}_i = C_i \cap X_0$ are partitions in X_0 between $X_0 \cap \overset{\circ}{F}_{-i}$ and $X_0 \cap \overset{\circ}{F}_{+i}$, $i=1, 2, \ldots, s-1$ and moreover $\widetilde{C}_i$ are partitions in X_0 between $X_0 \cap F_{-i}$, since it is easily seen that $X_0 \cap F_{\varepsilon i} \subset X_0 \cap \overset{\circ}{F}_{\varepsilon i}$, $\varepsilon = \mp$, $i=1, 2, \ldots, s-1$. By the above-mentioned fact $\bigcap_{i=1}^{s} C_i = \emptyset$ we get $\bigcap_{i=1}^{s-1} \widetilde{C}_i \cap C_s = \emptyset$. It is well-known that there exist open sets O_i in X such that $\widetilde{C}_i \subset O_i$ and $\bigcap_{i=1}^{s-1} O_i \cap C_s = \emptyset$. Moreover those sets O_i can be taken so small that $O_i \cap ((X_0 \cap F_{-i}) \cup (X_0 \cap F_{+i})) = \emptyset$, $i=1, 2, \ldots, s-1$. From here and the fact that $\widetilde{C}_i$ are partitions in X_0 between $X_0 \cap F_{-i}$ and $X_0 \cap F_{+i}$ and $\widetilde{C}_i \subset O_i$ it easily follows that X_0 may be decomposed in to the form $X_0 = X_{-i} \cup (X_0 \cap O_i) \cup X_{+i}$, where the system $\{X_{-i}; X_0 \cap O_i; X_{+i}\}$ is disjoint, X_{-i} and X_{+i} are closed in X_0 (and hence in X), $X_0 \cap F_{-i} \subset X_{-i}$, $X_0 \cap F_{+i} \subset X_{+i}$, $i=1, 2, \ldots, s-1$.

Obviously the sets $F_{-i} \cup X_{-i}$ and $F_{+i} \cup X_{+i}$ are colsed in X. Let us prove that they are disjoint. We have $(F_{-i} \cup X_{-i}) \cap (F_{+i} \cup X_{+i}) = (F_{-i} \cap F_{+i}) \cup (F_{-i} \cap X_{+i}) \cup (X_{-i} \cap F_{+i}) \cup (X_{-i} \cap X_{+i}) = (F_{-i} \cap X_{+i}) \cup (X_{-i} \cap F_{+i})$, because the pairs $\{F_{-i}, F_{+i}\}$ $\{X_{-i}, X_{+i}\}$, are disjoint. But $F_{-i} \cap X_{+i} = F_{-i} \cap X_0 \cap X_{+i} \subset X_{-i} \cap X_{+i} = \emptyset$ and analogously $F_{+i} \cap X_{-i} = \emptyset$. Thus it is shown that the sets $F_{-i} \subset X_{-i}$ and $F_{+i} \cup X_{+i}$ are disjoint. Then let Π_i be a partition in X between them, $i=1, 2, \ldots, s-1$. This means in particular that $\Pi_i \cap (X_{-i} \cup X_{+i}) = \emptyset$, so $\Pi_i \cap X_0 \subset X_0 \cap O_i \subset O_i$. From here, letting $\Pi_s = C_s$ and according to $\bigcap_{i=1}^{s-1} O_i \cap C_s = \emptyset$ and $C_s \subset X_0$ we get $\bigcap_{i=1}^{s} \Pi_i = C_s \cap \bigcap_{i=1}^{s-1} \Pi_i = C_s \cap \bigcap_{i=1}^{s-1} (\Pi_i \cap X_0) \subset C_s \cap \bigcap_{i=1}^{s-1} O_i = \emptyset$.

The set Π_i is a partition in X between $F_{-i} \cup X_{-i}$ and $F_{+i} \cup X_{+i}$ and hence is a partition in X between F_{-i} and F_{+1}, $i=1, 2, \ldots, s-1$. Analogously Π_s is a partition in X between F_{-s} and F_{+s}. Hence by the conditions it must be $\bigcap_{i=1}^{s} \Pi_i \neq \emptyset$, which contradicts to the obtained above $\bigcap_{i=1}^{s} \Pi_i = \emptyset$. This contradiction proves the lemma.

Let X be a compact space and $\{F_{-i}; F_{+i}\}_{i=1}^{s}$, $2 \leq s$ be an essential system in it. We introduce a relation of equivalence R in X in the following manner: xRy if and only if, when either $x, y \in F_{-s}$, or $x, y \in F_{+s}$. Let $Y = X/R$ be the quotient space and $\varphi: X \to Y$ be the natural mapping onto Y. Let ξ_- be the equivalence class determined by some element of F_{-s} and ξ_+ — analogously. It is clear that $\varphi^{-1}(\xi_-) = F_{-s}$ and $\varphi^{-1}(\xi_+) = F_{+s}$. The sets $\varphi(F_{\varepsilon i}) = \Phi_{\varepsilon i}$ are closed in Y, because φ is a closed mapping. It is clear that $\Phi_{-i} \cap \Phi_{+i} \subset \{\xi_-, \xi_+\}$, $i=1, 2, \ldots, s-1$. Let $\overset{\circ}{\Phi}_{\varepsilon i} = \Phi_{\varepsilon i} \setminus \{\xi_-; \xi_+\}$.

Lemma 2. *If L_i are partitions in Y between $\overset{\circ}{\Phi}_{-i}$ and $\overset{\circ}{\Phi}_{+i}$, then the intersection $L = \bigcap_{i=1}^{s-1} L_i$ contains a continuum K, which connects ξ_- and ξ_+, i. e. $K \subset L$ and $\xi_-, \xi_+ \in K$.*

Proof. By our assumption $Y = U_{-i} \cup L_i \cup U_{+i}$, where the system $\{U_{-i}, L_i, U_{+i}\}$ is disjoint, U_{-i}, U_{+i} are open in Y and $\overset{\circ}{\Phi}_{-i} \subset U_{-i}$, $\overset{\circ}{\Phi}_{+i} \subset U_{+i}$, $i=1, \ldots, s-1$. In that case $X = \varphi^{-1}(Y) = \varphi^{-1}(U_{-i}) \cup \varphi^{-1}(L_i) \cup \varphi^{-1}(U_{+i}) = V_{-i} \cup \Pi_i \cup V_{+i}$, where we put $V_{\varepsilon i} = \varphi^{-1}(U_{\varepsilon i})$, $\Pi_i = \varphi^{-1}(L_i)$, $\varepsilon = \mp$, $i=1, 2, \ldots, s-1$. The inclusion $\overset{\circ}{\Phi}_{\varepsilon i} \subset U_{\varepsilon i}$ implies $\varphi^{-1}(\overset{\circ}{\Phi}_{\varepsilon i}) \subset \varphi^{-1}(U_{\varepsilon i})$ and by the obvious inclusion $F_{\varepsilon i} \setminus (F_{-s} \cup F_{+s}) \subset \varphi^{-1}(\overset{\circ}{\Phi}_{\varepsilon i})$ we may conclude that Π_i is a partition in X between $F_{-i} \setminus (F_{-s} \cup F_{+s})$ and $F_{+i} \setminus (F_{-s} \cup F_{+s})$, $i=1, 2, \ldots, s-1$. According to Lemma 1 there exists a continuum K_1 for which $K_1 \subset \bigcap_{i=1}^{s-1} \Pi_i$ and $K_1 \cap F_{-s} \neq \emptyset \neq K_1 \cap F_{+s}$. By setting $K = \varphi(K_1)$ we obtain the desired continuum and Lemma 2 is proved.

Let $T^n = h(P^n)$ be an n-dim. d. c. with vertices $a_\varepsilon = h(v_\varepsilon)$, $\varepsilon = \mp$, and sides, $T^n_{\varepsilon i} = h(P^n_{\varepsilon i})$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$. Put $\mathring{T}^n_{\varepsilon i} = T^n_{\varepsilon i} \setminus \{a_-, a_+\}$, $\varepsilon = \mp$, $i = 1, 2, \ldots, s-1$.

Lemma 3. *If C_i are partitions in T^n between $\mathring{T}^n_{-i}$ and $\mathring{T}^n_{+i}$, then the intersection $\bigcap_{i=1}^{n-1} C_i$ contains a continuum K, which connects a_- and a_+.*

Proof. Obviously it is sufficient to prove this lemma in the particular case when $T^n = P^n$, i. e. h is the identity. But then the assertion of lemma 3 is an immediate consequence of Lemma 2 and the fact that P^n is a quotient space which may be obtained from I^n by identifying some pair of its opposite sides into the points v_- and v_+.

Let $Y = X/R$ and $\Phi_{\varepsilon i}$ be the same as in Lemma 2 and $\Gamma_{\varepsilon i}$ are open in Y with $\mathring{\Phi}_{\varepsilon i} \subset \Gamma_{\varepsilon i}$ and $\overline{\Gamma}_{-i} \cap \overline{\Gamma}_{+i} \subset \{\xi_-, \xi_+\}$, $i = 1, 2, \ldots, s-1$.

We shall prove the following lemma using some ideas from [13].

Lemma 4. *If Y is hereditarily normal space and $M \subset Y$ cuts Y between the points ξ_- and ξ_+, then the system $\{M \cap \overline{\Gamma}_{-i}; M \cap \overline{\Gamma}_{+i}\}_{i=1}^{s-1}$ is essential in M.*

Proof. Assume the contrary and let M_i be partition in M between $M \cap \overline{\Gamma}_{-i}$ and $M \cap \overline{\Gamma}_{+i}$ with empty intersection, i. e. $M = M_{-i} \cup M_i \cup M_{+i}$, where the system $\{M_{-i}, M_i, M_{+i}\}$ is disjoint, M_{-i}, M_{+i} are open in M, $M \cap \overline{\Gamma}_{-i} \subset M_{-i}$ $M \cap \overline{\Gamma}_{+i} \subset M_{+i}$, $i = 1, 2, \ldots, s-1$ and $\cap_{i=1}^{s-1} M_i = \emptyset$. By virtue of normality of M there exist partitions $O_i \supset M_i$, open in M, with void intersection, i. e. $M = H_{-i} \cup O_i \cup H_{+i}$, where the system $\{H_{-i}, O_i, H_{+i}\}$ is disjoint, H_{-i}, H_{+i} are closed in $M \cap \overline{\Gamma}_{-i} \subset H_{-i}$, $M \cap \overline{\Gamma}_{+i} \subset H_{+i}$, $i = 1, 2, \ldots, s-1$ and $\bigcap_{i=1}^{s-1} O_i = \emptyset . M$,

Now, we assert

$$\overline{\mathring{\Phi}_{-i} \cup H_{-i}} \cap (\mathring{\Phi}_{+i} \cup H_{+i}) = \emptyset, \quad (\mathring{\Phi}_{-i} \cup H_{-i}) \cap \overline{\mathring{\Phi}_{+i} \cup H_{+i}} = \emptyset. \tag{1}$$

By reason of symmetry it is sufficient to show the first equality. We have $\overline{\mathring{\Phi}_{-i} \cup H_{-i}} \cap (\mathring{\Phi}_{+i} \cup H_{+i}) = (\overline{\mathring{\Phi}}_{-i} \cup \overline{H}_{-i}) \cap (\mathring{\Phi}_{+i} \cup H_{+i}) = (\overline{\mathring{\Phi}}_{-i} \cap \Phi_{+i}) \cup (\overline{\mathring{\Phi}}_{-i} \cap H_{+i}) \cup (\overline{H}_{-i} \cap \mathring{\Phi}_{+i}) \cup (\overline{H}_{-i} \cap H_{+i})$. Further $\overline{\mathring{\Phi}}_{-i} \cap \mathring{\Phi}_{+i} \subset \Phi_{-i} \cap \mathring{\Phi}_{+i} = \emptyset$; $\overline{H}_{-i} \cap H_{+i} = \overline{H}_{-i} \cap M \cap H_{+i} = [H_{-i}]_M \cap H_{+i} = H_{-i} \cap H_{+1} = \emptyset$. Also $H_{-i} \cap \Gamma_{+i} = H_{-i} \cap M \cap \Gamma_{+i} \subset H_{-i} \cap M \cap \overline{\Gamma}_{+i} \subset H_{-i} \cap H_{+1} = \emptyset$ and hence $\overline{H}_{-i} \cap \mathring{\Phi}_{+i} = \emptyset$, because Γ_{+i} is an open neighbourhood of $\mathring{\Phi}_{+i}$ and $H_{-i} \cap \Gamma_{+i} = \emptyset$. Finally, $\overline{\mathring{\Phi}}_{-i} \cap H_{+i} \subset \Phi_{-i} \cap H_{+i} = \Phi_{-i} \cap M \cap H_{+i} = \mathring{\Phi}_{-i} \cap M \cap H_{+i}$, because M cuts Y between the points ξ_- and ξ_+ and hence $\xi_-, \xi_+ \notin M$. Thus $\Phi_{-i} \cap H_{+i} \subset \mathring{\Phi}_{-i} \cap M \cap H_{+i} \subset \Gamma_{-i} \cap M \cap H_{+i} \subset \overline{\Gamma}_{-i} \cap M \cap H_{+i} \subset H_{-i} \cap H_{+i} = \emptyset$. In this way we proved (1). Then there exist open sets U_{-i}, U_{+i} with $\mathring{\Phi}_{-i} \cup H_{-i} \subset U_{-i}$, $\mathring{\Phi}_{+i} \cup H_{+i} \subset U_{+i}$ and $U_{-i} \cap U_{+i} = \emptyset$, $i = 1, \ldots, s-1$ in conformity with the hereditary normality of Y. Letting $L_i = Y \setminus (U_{-i} \cup U_{+i})$, $i = 1, 2, \ldots, s-1$ we get partitions in Y between $\mathring{\Phi}_{-i}$ and $\mathring{\Phi}_{+i}$ such that $M \cap \bigcap_{i=1}^{s-1} L_i = \bigcap_{i=1}^{s-1} (M \cap L_i) \subset \bigcap_{i=1}^{s-1} O_i = \emptyset$. In accordance with Lemma 2 there exists a continuum $K \subset \bigcap_{i=1}^{s-1} L_i$, which connects ξ_- and ξ_+. But this contradicts with our condition that M cuts Y between the points ξ_- and ξ_+, because $K \subset \bigcap_{i=1}^{s-1} L_i$ and $M \cap \bigcap_{i=1}^{s-1} L_i = \emptyset$ imply $K \cap M = \emptyset$. Thus Lemma 4 is proved.

The following Lemma is an immediate consequence of the above assertion.

Lemma 5. *Let T^n be an n-dim d. c. with vertices a_- and a_+ and thick sides $F^n_{\varepsilon i}$. Let $M\subset T^n$* cuts *T^n between the points a_- and a_+. Then the system $\{M\cap F^n_{-i};\ M\cap F^n_{+i}\}_{i=1}^{n-1}$ is essential in M.*

Lemma 6. *Let T^n be an n-dim d. c. in Q^n with vertices a_- and a_+ and let $\{F^n_{\varepsilon i}:\varepsilon=\mp,\ i=1,2,\ldots,n-1\}$ be a system of its thick sides and $\varphi:T^n\to[0,1]$ be a continuous function with $\varphi^{-1}(0)=\{a_-,a_+\}$. Then the set $T^{n+1}=\{(x,t)\in Q^n\times Q=Q^{n+1}:x\in T^n$ and $0\leq t\leq\varphi(x)\}$ is $(n+1)$ — dim d. c. in Q^{n+1} with the same vertices a_- and a_+. Moreover the system of sets*

$$F^{n+1}_{\varepsilon i}=\{(x,t)\in T^{n+1}:x\in F^n_{\varepsilon i}\ \text{and}\ 0\leq t\leq\varphi(x)\},\ \varepsilon=\mp,\ i=1,2,\ldots,n-1,$$

$$F^{n+1}_{-n}=\{(x,t)\in T^{n+1}:x\in T^n\ \text{and}\ 0\leq t\leq\frac{1}{3}\varphi(x)\};\tag{2}$$

$$F^{n+1}_{+n}=\{(x,t)\in T^{n+1}:x\in T^n\ \text{and}\ \frac{2}{3}\varphi(x)\leq t\leq\varphi(x)\},$$

is a system of thick sides of T^{n+1}.

The simple proof of this lemma will be omitted.

For the proof of Theorem 1 we have to construct ω-system in H. This will be done by induction and the following lemma is essentially the inductive step.

Lemma 7. *Let T^n be an n-dim d. c. in Q^n with vertices a_- and a_+ and some system $\{F^n_{\varepsilon i}:\varepsilon=\mp,i=1,2,\ldots,n-1\}$ of its thick sides. We lay $\mathring{F}_{\varepsilon i}=F_{\varepsilon i}\setminus\{a_-,a_+\}$. Let $O_{\varepsilon i}$ be open sets in $Q^{n+1}\setminus\{a_-,a_+\}$ and $\mathring{F}_{\varepsilon i}\subset O_{\varepsilon i}$, $\varepsilon=\mp$, $i=1,2,\ldots,n-1$. Then there exists a continuous function $\varphi:T^n\to[0,1]$ with $\varphi^{-1}(0)=\{a_-,a_+\}$ and if we get the $(n+1)$-dim d. c. T^{n+1} in Q^{n+1} with vertices a_- and a_+ (as in Lemma 6) and its system (2) of thick sides (as in Lemma 6) then $\mathring{F}^{n+1}_{\varepsilon i}\subset O_{\varepsilon i}$, $\varepsilon=\mp$, $i=1,2,\ldots,n-1$, where as usual $\mathring{F}^{n+1}_{\varepsilon i}=F^{n+1}_{\varepsilon i}\setminus\{a_-,a_+\}$.*

Proof of Lemma 7. Let us fix one $i=1,2,\ldots,n-1$. It is not difficult to construct a continuous function $\varphi_i:T^n\to[0,1]$, such that $\varphi_i^{-1}(0)=\{a_-,a_+\}$ and if $x\in\mathring{F}^n_{\varepsilon i}$ for some $\varepsilon=\pm$ and $0\leq t\leq\varphi(x)$, then $(x,t)\in O_{\varepsilon i}$. Let $\varphi=\min_i\varphi_i$. Then it is clear that φ is the desired mapping.

Proof of Theorem 1. Let us fix one n with $a_-,a_+\in Q^n$. Let $T^n=h_n(P^n)$ be an n-dim d. c. in Q^n with vertices a_- and a_+. Let $\{F^n_{\varepsilon i}:\varepsilon=\mp,i=1,2,\ldots,n-1\}$ be some system of thick sides of T^n. The space $\widetilde{H}=H\cup(Q^\omega\setminus\{a_-,a_+\})$ is normal, because it is F_σ-subset of a normal space X. The sets $\mathring{F}^n_{\varepsilon i}$ are obviously closed subsets of $\widetilde{H}$ and $\mathring{F}^n_{-i}\cap\mathring{F}^n_{+i}=\emptyset$, $i=1,2,\ldots,n-1$. Hence, there exist open in $\widetilde{H}$ sets $U^n_{\varepsilon i}$ such that $\mathring{F}^n_{\varepsilon i}\subset U^n_{\varepsilon i}$ and $[U^n_{-i}]_{\widetilde{H}}\cap[U^n_{+i}]_{\widetilde{H}}=\emptyset$. Put $A_{\varepsilon i}=[U^n_{\varepsilon i}]_{\widetilde{H}}$, $\varepsilon=\mp$, $i=1,\ldots,n-1$. By our condition H cuts X between the points a_- and a_+, hence, $H\cap T^n$ cust T^n between a_- and a_+. Since $H\cap F^n_{\varepsilon i}=H\cap\mathring{F}^n_{\varepsilon i}$, then in accordance with Lemma 5 the system $\{H\cap\mathring{F}^n_{-i};\ H\cap\mathring{F}^n_{+i}\}_{i=1}^{n-1}$ is essential in $H\cap T^n$. Then it is essential in H too, because $H\cap T^n$ is closed subset of H. As $H\cap\mathring{F}_{\varepsilon i}\subset H\cap A_{\varepsilon i}$, then the system $\{H\cap A_{-i};\ H\cap A_{+i}\}_{i=1}^{n-1}$ of closed sets of H is essential in H.

Let us make the next inductive supposition. There exists n-dim d. c. in Q^n with vertices a_- and a_+, and its thick sides $F^n_{\varepsilon i}$ and open in $\widetilde{H}$ sets $U^n_{\varepsilon i}$ such that $\mathring{F}^n_{\varepsilon i} \subset U^n_{\varepsilon i}$ and $[U^n_{-i}]_{\widetilde{H}} \cap [U^n_{+i}]_{\widetilde{H}} = \varnothing$, $i=1, 2, \ldots, n-1$, and such that the system $\{H \cap A_{-i};\ H \cap A_{+i}\}_{i=1}^{n-1}$ is essential in H, where we put $A^{*}_{\varepsilon i} = [U^n_{\varepsilon i}]_{\widetilde{H}}$.

In accordance with Lemma 7 there exists continuous function $\varphi: T^n \to [0, 1]$ with $\varphi^{-1}(0) = \{a_-, a_+\}$ and if we get the $(n+1)$-dim d. c. in Q^{n+1} with vertices a_- and a_+ (as in Lemma 6) $T^{n+1} = h_{n+1}(P^{n+1}) = \{(x, t) \in Q^n \times Q = Q^{n+1} : x \in T^n$ and $0 \leqq t \leqq \varphi(x)\}$ with thick sides $F^{n+1}_{\varepsilon i}$ (2), then $\mathring{F}^{n+1}_{\varepsilon i} \subset U^n_{\varepsilon i} \cap (Q^{n+1} \setminus \{a_-, a_+\}) \subset U^n_{\varepsilon i}$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$. Let $U^{n+1}_{\varepsilon i} = U^n_{\varepsilon i}$, $\varepsilon = \mp$, $i = 1, 2, \ldots, n-1$. Since $\mathring{F}^{n+1}_{-n}$ and $\mathring{F}^{n+1}_{+n}$ are obviuously closed and disjoint subsets of $\widetilde{H}$, then there exist open subsets U^{n+1}_{-n} and U^{n+1}_{+n} in $\widetilde{H}$, for which $\mathring{F}^{n+1}_{\varepsilon n} \subset U^{n+1}_{\varepsilon n}$, $\varepsilon = \mp$ and $[U^{n+1}_{-n}]_{\widetilde{H}} \cap [U^{n+1}_{+n}]_{\widetilde{H}} = \varnothing$. Let $A_{\varepsilon n} = [U^{n+1}_{\varepsilon n}]$, $\varepsilon = \mp$. That the system $\{H \cap A_{-i};\ H \cap A_{+i}\}_{i=1}^{n}$ is essential in H we can conclude in the same manner as we did before.

In this way by a method of mathematical induction we have constructed a system $\{A_{-i}, A_{+i}\}_{i=1}^{\infty}$ of closed disjoint pairs in $\widetilde{H}$ for which the system $\{H \cap A_{-i};\ H \cap A_{+i}\}_{i=1}^{m}$ is essential in H for every natural number m. Hence, the system $\{H \cap A_{-i};\ H \cap A_{+i}\}_{i=1}^{\infty}$ is ω-system in H and thus H is S-strongly infinite dimensional. This completes the proof of Theorem 1.

It is easy to see that the same arguments are appliable for proving the following more general:

Theorem 2. *Let X be a normal extension of Q^{ω}, $H \subset Q^{\omega}$, $a_-, a_+ \in Q^{\omega}$ and the space $\widehat{H} = H \cup (Q^{\omega} \setminus \{a_-, a_+\})$ be normal. Then if for every sufficiently large m the set $H \cap Q^m$ cuts Q^m between the points a_- and a_+, then H is S-strongly infinite dimensional.*

From here, we easily obtain:

Corollary 1. *If X is hereditarily normal extension of Q^{ω}, $H \subset X$, a_-, $a_+ \in Q^{\omega}$ and for every sufficiently large m the set $H \cap Q^m$ cuts Q^m between the points a_- and a_+, then H is S-strongly infinite dimensional.*

Corollary 2 (see [14]). *If $M \subset Q^{\omega}$ cuts Q^{ω} between the points a_- and a_+ from Q^{ω}, then M is S-strongly infinite dimensional.*

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