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ON THE BASIS OF THE IDENTITIES OF THE MATRIX ALGEBRA OF SECOND ORDER OVER A FIELD OF CHARACTERISTIC ZERO

BUI TUONG TKI

In this work is obtained a basis of the identities of the matrix algebra of second order over a field of characteristic zero, which contains four identities.

Throughout this work, we shall denote by K an arbitrary field of characteristic zero. It is well known [1, Th. 4] that the following nine polynomial identities:

$$(1)\quad f_1=\sum_{\sigma\in\Sigma_3}(-1)^{\varepsilon(\sigma)}[x_5, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_4]-\sum_{\sigma\in\Sigma_3}(-1)^{\varepsilon(\sigma)}[x_5, x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]=0;$$

$$(2)\qquad f_2=[[z, y], [x, y], y]=0;$$

Remark. After linearization of f_2, one obtains a multilinear polynomial identity and we can denote it by $f_2(x_1, x_2, x_3, x_4, x_5)=0$;

$$(3)\qquad f_3=\sum_{\sigma\in\Sigma_3}(-1)^{\varepsilon(\sigma)}[x_5, x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]-[x_5, [x_1, x_2], x_3, x_4]$$

$$-[x_5, x_3, [x_1, x_2], x_4]+[x_4, [x_1, x_2], x_5, x_3]+[x_4, x_3, x_5, [x_1, x_2]]=0,$$

where $\varepsilon(\sigma)=0,1$ depends on the signature of the permutation $\sigma\in\Sigma_3$.

$$(4)\qquad 4[z, x](v_1\circ v_2)=[z, v_1, v_2, x]+[z, v_2, v_1, x]-[x, v_1, z, v_2]-[x, v_2, z, v_1],$$

where $v_1=[t_1, t_2]$, $v_2=[t_3, t_4]$;

$$(5)\qquad H(x_1, x_2, x_3, x_4, x_5)=[[x_1, x_2]\circ[x_3, x_4], x_5]=0;$$

(6) the standard identity $S_4(x_1, x_2, x_3, x_4)=0$;

$$(7)\qquad f_1'(x_1, x_2, x_3, x_4, x_5, x_6)=0;$$

$$(8)\qquad f_2'(x_1, x_2, x_3, x_4, x_5, x_6)=0;$$

$$(9)\qquad f_3'(x_1, x_2, x_3, x_4, x_5, x_6)=0$$

form a basis of the identities of the algebra of all matrices of second order over the field K. We call it the basis of Razmyslov [1]. In this work we denote by $M(2, K)$ the algebra of all matrices of second order over the field K, and by $\mathrm{Sl}(2, K)$ the Lie algebra of all matrices of second order with zero trace over the field K. Recall that f_i', $i=1, 2, 3$ are obtained from f_i, $i=1, 2, 3$

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as follows: if $f_i=\Sigma_j a_{ij}[u_{ij}, v_{ij}]$, $i=1, 2, 3$, where u_{ij} and v_{ij} are commutators and the weight of $u_{ij}\geqq 2$, then

$$f'_i=\sum_j a_{ij}(u_{ij}\circ[v_{ij}, x_6]), \qquad i=1, 2, 3$$

In [2] Leron asserts that Rosset, using a computer, had proved every polynomial identity of degree 5 of the algebra $M(2, K)$ being a consequence of the standard identity $S_4=0$ and the identity $H(x_1, x_2, \ldots, x_5)=0$.

In this work we shall prove this assertion without using a computer. This means that the three identities $f_1=0$, $f_2=0$ and $f_3=0$ can be removed from the basis of Razmyslov.

At last, using a result of Filippov [3], namely that from the Lie identity $\Phi=[y, z, [t, x], x]+[y, x, [z, x], t]=0$ follow all Lie identities of the Lie algebra $\mathrm{Sl}(2, K)$, we shall obtain a basis of the identities of $M(2, k)$, which contains four identities.

1. Representation modules for the symmetric group. A vector space over the field K, which is a module over the group algebra $K\Sigma_k$ is called a representation module for the symmetric group Σ_k.

Definition 1. *Let V be $K\Sigma_k$-module, $v\in V$, $1\leq i, j\leq k$ and let (i, j) be a transposition from Σ_k. Then $v\in V$ is called (i, j)-symmetric, if $(i, j)v=v$ and v is called (i, j)-skew symmetric, if $(i, j)v=-v$.*

Example. Let V_k be the set of the multilinear identities of the algebra $M(2, K)$ in the variables $x_1, x_2, \ldots, x_k$.

The action of Σ_k on V_k is defined as follows: If $f(x_1, x_2, \ldots, x_k)\in V_k$ and $\sigma\in\Sigma_k$, then $\sigma f(x_1, x_2, \ldots, x_k)=f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)})$. This action induces on V_k the structure of a $K\Sigma_k$-module. The element $f(x_1, \ldots, x_i, \ldots, x_j, \ldots)\in V_k$ is (i, j)-symmetric if and only if, $f(x_1, \ldots, x_i, \ldots, x_j, \ldots)=f(x_1, \ldots, x_j, \ldots, x_i, \ldots)$ and $f(x_1, \ldots, x_i, \ldots, x_j, \ldots)$ is (i, j)-skew symmetric if and only if

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots)=-f(x_1, \ldots, x_j, \ldots, x_i, \ldots).$$

Definition 2. *Let V be a $K\Sigma_k$-module, $v\in V$ and $r\geqq 0$ a given integer. The element v is called r-symmetric if for some distinct numbers $i_1, j_1, i_2, j_2, \ldots, i_r, j_r$ from among $1, 2, \ldots, k$, v is (i_t, j_t)-symmetric for all $t=1, 2, \ldots, r$. If in addition v is (p, q)-skew symmetric for all $1\leq p, q\leq k$, such that $\{p, q\}\cap\{i_1, j_1, \ldots, i_r, j_r\}$ is empty, then v is called r-perfect.*

Remark. Every element $v\in V$ is 0-symmetric, an element $v\in V$ is 0-perfect iff v is (p, q)-skew symmetric for all $1\leq p, q\leq k$. In particular a polynomial $f\in V_k$ is 0-perfect iff it is a scalar multiple of the standard polynomial S_k.

Theorem 1 (Leron [2]). *Let V be a $K\Sigma_k$-module and Q_r and P_r be the subspaces of V, spanned respectively by its r-symmetric and r-perfect elements. Then:*

a) $Q_r=P_r+Q_{r+1}$,

b) $V=P_0+P_1+\cdots+P_r+Q_{r+1}$, *for all* $r\geq 0$,

c) $V=P_0+P_1+\cdots+P_t$, *for some t, that is V is generated by its perfect elements.*

2. A basis of the identities of $M(2, K)$. V. Drenski brought to my attention, that all Lie identities of the algebra $M(2, K)$ are consequences of the standard identity $S_4=0$. To verify this assertion, by having the above result of Filippov [3], it is enough to prove:

Remark 1. The identity $\Phi=[y, z, [t, x], x]+[y, x, [z, x], t]=0$ of $M(2, K)$ is a consequence of the standard identity $S_4=0$.

Proof. Taking into account that the standard identity $S_4=0$ has another form:

$$S_4(x_1, \ldots, x_4)=([x_1, x_2]\circ[x_3, x_4])+([x_2, x_3]\circ[x_1, x_4])+([x_3, x_1]\circ[x_2, x_4])=0$$

and that $[ab, c]=a[b, c]+[a, c]b$ for all a, b, c, we can prove that

$$\Phi=[y, z, [t, x], x]+[y, x, [z, x], t]$$
$$=-xS_4(x, y, z, t)-S_4(xy, x, z, t)+S_4(xz, x, y, t)+S_4(tx, x, y, z).$$

This is the required equality.

Remark 2. It is easy to prove that the following equality is true: $f_2=[[z, y], [x, y], y]=S_4(z, x, y^2, y)$ which shows that the identity $f_2=0$ in the basis of Razmyslov is a consequence of the standard identity $S_4=0$.

We denote by V_5 the space of all multilinear identities of $M(2, K)$ in the variables x_1, x_2, x_3, x_4, x_5

Theorem 2. *Every element of V_5 follows from the three identities $S_4(x_1, x_2, x_3, x_4)=0$, $H(x_1, x_2, x_3, x_4, x_5)=0$ and $f_2(x_1, x_2, x_3, x_4, x_5)=0$.*

Proof. By Theorem 1 of Leron, we have $V_5=P_0+P_1+Q_2$. It is well known that every 0-perfect element of V_5 is a consequence of $S_4(x_1, x_2, x_3, x_4)$, i. e. every element of P_0 is a consequence of $S_4(x_1, x_2, x_3, x_4)$.

Proposition 1. *Every element of P_1 is a consequence of $S_4(x_1, x_2, x_3, x_4)$.*

It is enough to prove Proposition 1 for the elements of V_5, which are, say, (1, 2)-perfect.

Lemma 1. *The following four elements of V_5 are (1, 2)-perfect and linearly independent:*

$$p_1=x_1S_4(x_2, x_3, x_4, x_5)+x_2S_4(x_1, x_3, x_4, x_5)$$

$$p_2=S_4(x_1, x_3, x_4, x_5)x_2+S_4(x_2, x_3, x_4, x_5)x_1$$

$$p_3=S_4(x_1x_2, x_3, x_4, x_5)+S_4(x_2x_1, x_3, x_4, x_5)$$

$$p_4=S_4(x_2, x_3x_1, x_4, x_5)+S_4(x_1, x_3x_2, x_4, x_5)+S_4(x_2, x_3, x_4x_1, x_5)$$
$$+S_4(x_1, x_3, x_4x_2, x_5)+S_4(x_1, x_3, x_4, x_5x_2)+S_4(x_2, x_3, x_4, x_5x_1).$$

Proof. We have only to prove that p_i, $i=1, 2, 3, 4$ are linearly independent. Let a_1, a_2, a_3, a_4 be elements of K, such that

(11) $$a_1p_1+a_2p_2+a_3p_3+a_4p_4=0.$$

Equating to zero the coefficients of the following monomials:

(12) $$x_1x_2x_3x_4x_5,\ x_1x_3x_2x_4x_5,\ x_1x_3x_4x_2x_5,\ x_1x_3x_4x_5x_2$$

we get respectively the equations $a_1+a_3=0$, $-a_1+a_4=0$, $a_1+a_4=0$, $-a_1+a_2+a_4=0$ and obtain: $a_1=a_2=a_3=a_4=0$.

Lemma 2. *Let $f(x_1, x_2, x_3, x_4, x_5)\in V_5$ be (1, 2)-perfect element, in which the monomials (12) occur with zero coefficients. Then $f(x_1, x_2, x_3, x_4, x_5)$ is the zero polynomial.*

Proof. Let us introduce some notation. Let

$$f=\sum_{\sigma\in\Sigma_5} a_\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)};\quad \text{for}\quad \sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ i_1 & i_2 & i_3 & i_4 & i_5\end{pmatrix}\in\Sigma_5$$

we shall write $a_\sigma=a_{i_1i_2i_3i_4i_5}$.

Recall that f is $(1, 2)$-symmetric and (i, j)-skew symmetric for all $2 < i \neq j \leq 5$. Therefore if some monomial occurs in f with zero coefficient, then so do all the monomials, obtained from it by a permutation of $\{3, 4, 5\}$ or $\{1, 2\}$. Thus in order to prove that every monomial has zero coefficient in f we have only to consider the various possible positions of x_1 and x_2 in the monomial, without regard to the order of the other variables. We distinguish several cases:

a) Monomials in which x_2 immediately succeeds x_1. With

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ e_{i_1 j_1} & e_{i_2 j_2} & e_{i_3 j_3} & e_{i_4 j_4} & e_{i_5 j_5} \end{pmatrix} 1 \leq i_t,\ j_t \leq 2$$

we denote the substitution for the variables in f, the sequence $e_{i_1 j_1}$, $e_{i_2 j_2}$, $e_{i_3 j_3}$, $e_{i_4 j_4}$, $e_{i_5 j_5}$ of matrix units of $M(2, K)$, in which the matrix assigned to each variable is written directly under that variable. Since f is an identity of $M(2, K)$, after such a substitution we always obtain the zero matrix.

By the hypothesis of the lemma we have $\alpha_{12345} = 0$. Now consider the following substitution:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ e_{11} & e_{11} & e_{11} & e_{12} & e_{22} \end{pmatrix}.$$

Since we must obtain the zero matrix, the coefficient of the matrix unit e_{12} must be zero. Therefore we have $\alpha_{12345} + \alpha_{13245} + \alpha_{31245} = 0$, since $\alpha_{12345} = \alpha_{13245} = 0$, so that we obtain $\alpha_{31245} = 0$. Substituting

$$\begin{pmatrix} x_3 & x_1 & x_2 & x_4 & x_5 \\ e_{12} & e_{22} & e_{22} & e_{21} & e_{11} \end{pmatrix},$$

we obtain $\alpha_{31245} + \alpha_{53124} = 0$ and $\alpha_{53124} = 0$. The substitution

$$\begin{pmatrix} x_3 & x_4 & x_5 & x_1 & x_2 \\ e_{12} & e_{22} & e_{21} & e_{11} & e_{11} \end{pmatrix}$$

gives $\alpha_{34512} = 0$, which completes the proof of case a).

b) Monomials in which x_1 and x_2 are separated by a single variable, say x_3. Substituting

$$\begin{pmatrix} x_4 & x_1 & x_3 & x_2 & x_5 \\ e_{21} & e_{11} & e_{11} & e_{11} & e_{12} \end{pmatrix},$$

we have $\alpha_{41325} + \alpha_{41235} + \alpha_{43125} = 0$, and then $\alpha_{41325} = 0$. Now consider in g the substitution

$$\begin{pmatrix} x_4 & x_5 & x_1 & x_3 & x_2 \\ e_{22} & e_{21} & e_{11} & e_{11} & e_{11} \end{pmatrix},$$

we obtain $\alpha_{45132} = 0$.

c) Monomials in which x_1 and x_2 are separated by three variables. We have $\alpha_{13452} = 0$ by the hypothesis of the lemma.

d) Monomials in which x_1 and x_2 are separated by two variables. By the hypothesis of the lemma we have $\alpha_{13425}=0$. Substituting

$$\begin{pmatrix} x_5 & x_1 & x_3 & x_4 & x_2 \\ e_{11} & e_{11} & e_{12} & e_{22} & e_{22} \end{pmatrix}$$

we obtain $\alpha_{51342}=0$ and lemma 2 is proved.

Now let $f\in V_5$ be (1, 2)-perfect and the monomials (12) occur in f with the coefficients β_1, β_2, β_3, β_4. Then there exists an uniquely determined element $g\in V_5$, $g=\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3+\alpha_4 p_4$, $\alpha_i\in K$, such that the four monomials (12) occur in g also with the coefficients β_1, β_2, β_3, β_4, where α_1, α_2, α_3, α_4 are defined, as the solutions of the system of linear equations

$$\begin{cases} \alpha_1+\alpha_3=\beta_1 \\ -\alpha_1+\alpha_4=\beta_2 \\ \alpha_1+\alpha_4=\beta_3 \\ -\alpha_1+\alpha_2+\alpha_4=\beta_4. \end{cases}$$

Since $f-g\in V_5$ is (1,2)-perfect and the four monomials (12) occur in $f-g$ with zero coefficients, by Lemma 2 $f-g$ is the zero polynomial i. e. $f=g$. Thus the proof of Proposition 1 is completed.

Proposition 2. *Every element of Q_2 is consequence of S_4, H and $f_2(x_1, x_2, x_3, x_4, x_5)$.*

It is enough to prove the assertion for polynomials, which are, say, (1, 2) and (3, 4)-symmetric.

Lemma 3. *The following six elements of V_5 are consequences of $S_4(x_1, x_2, x_3, x_4)$, $H(x_1, x_2, x_3, x_4, x_5)$ and $f_2(x_1, x_2, x_3, x_4, x_5)$, (1, 2) and (3, 4)-symmetric and linearly independent:*

$$\begin{aligned} q_1 &= S_4(x_1x_3, x_2, x_4, x_5)+S_4(x_2x_3, x_1, x_4, x_5) \\ &\quad +S_4(x_2x_4, x_2, x_3, x_5)+S_4(x_2x_4, x_1, x_3, x_5), \\ q_2 &= S_4(x_3x_1, x_2, x_4, x_5)+S_4(x_3x_2, x_1, x_4, x_5) \\ &\quad +S_4(x_4x_1, x_2, x_3, x_5)+S_4(x_4x_2, x_1, x_3, x_5), \\ q_3 &= [[x_1, x_3]\circ[x_2, x_4], x_5]+[[x_2, x_3]\circ[x_1, x_4], x_5], \\ q_4 &= [[x_1, x_5]\circ[x_2, x_4], x_3]+[[x_1, x_5]\circ[x_2, x_3], x_4] \\ &\quad +[[x_2, x_5]\circ[x_1, x_4], x_3]+[[x_2, x_5]\circ[x_1, x_3], x_4], \\ q_5 &= [[x_3, x_5]\circ[x_2, x_4], x_1]+[[x_3, x_5]\circ[x_1, x_4], x_2] \\ &\quad +[[x_4, x_5]\circ[x_2, x_3], x_1]+[[x_4, x_5]\circ[x_1, x_3], x_2], \\ q_6 &= [[x_5, x_1], [x_4, x_2], x_3]+[[x_5, x_1], [x_3, x_2], x_4] \\ &\quad +[[x_5, x_2], [x_4, x_1], x_3]+[[x_5, x_2], [x_3, x_1], x_4] \\ &\quad +[[x_5, x_3], [x_4, x_1], x_2]+[[x_5, x_4], [x_3, x_1], x_2] \\ &\quad +[[x_5, x_3], [x_4, x_2], x_1]+[[x_5, x_4], [x_3, x_2], x_1]. \end{aligned}$$

Proof. It is evident that $q_1, q_2, q_3, q_4, q_5, q_6$ are consequences of S_4, H, f_2 and (1, 2) and (3, 4)-symmetric.

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ be elements of K, such that

$$\gamma_1 q_1+\gamma_2 q_2+\gamma_3 q_3+\gamma_4 q_4+\gamma_5 q_5+\gamma_6 q_6=0. \tag{13}$$

Equating to zero the coefficients of the following six monomials:

$$x_1x_2x_3x_4x_5,\quad x_1x_3x_2x_4x_5,\quad x_3x_1x_4x_2x_5,\quad x_5x_4x_3x_1x_2,\quad x_5x_1x_3x_2x_4,\quad x_1x_3x_2x_5x_4 \tag{14}$$

we get respectively the equations

$$\begin{aligned}
-\gamma_1-\gamma_5+\gamma_6 &=0,\\
\gamma_1-\gamma_2+\gamma_3+\gamma_5-\gamma_6&=0,\\
\gamma_1-\gamma_2+\gamma_3-\gamma_4+\gamma_6&=0,\\
-\gamma_2+\gamma_5+\gamma_6 &=0,\\
-\gamma_1-\gamma_3+\gamma_4+\gamma_6 &=0,\\
-\gamma_1+\gamma_2+\gamma_4-\gamma_5 &=0,
\end{aligned}$$

which imply that $\gamma_1=\gamma_2=\gamma_3=\gamma_4=\gamma_5=\gamma_6=0$, since the following determinant is different from zero

$$\begin{vmatrix}
-1 & 0 & 0 & 0 & -1 & 1\\
1 & -1 & 1 & 0 & 1 & -1\\
1 & -1 & 1 & -1 & 0 & 1\\
0 & -1 & 0 & 0 & 1 & 1\\
-1 & 0 & -1 & 1 & 0 & 1\\
-1 & 1 & 0 & 1 & -1 & 0
\end{vmatrix}. \tag{15}$$

Lemma 4. *Let* $f=\sum_{\sigma\in\Sigma_5}\alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(5)}\in V_5$ *be* (1, 2) *and* (3, 4)-*symmetric. Suppose that the six momomials* (14) *occur in* f *with zero coefficients. Then* f *is the zero polynomial.*

Proof. We wish to prove that $\alpha_\sigma=0$ for all $\sigma\in\Sigma_5$. By the hypothesis of the lemma we have: $\alpha_{12345}=\alpha_{13245}=\alpha_{31425}=\alpha_{54312}=\alpha_{51324}=\alpha_{13254}=0$. Substituting

$$\begin{pmatrix} x_1 & x_2 & x_5 & x_3 & x_4\\ e_{11} & e_{11} & e_{12} & e_{22} & e_{22}\end{pmatrix},$$

we obtain $\alpha_{12534}=0$, similarly we have $\alpha_{34512}=0$. We distinguish several cases:

a) Monomials of the form $x^{i_1}x_{i_2}x_{i_3}x_{i_4}x_5$. By the hypothesis of the lemma we have $\alpha_{12345}=\alpha_{13245}=\alpha_{31425}=0$. The substitutions

$$\begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5\\ e_{11} & e_{11} & e_{11} & e_{12} & e_{22}\end{pmatrix},\quad \begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5\\ e_{21} & e_{11} & e_{11} & e_{11} & e_{12}\end{pmatrix},\quad \begin{pmatrix} x_1 & x_3 & x_4 & x_2 & x_5\\ e_{11} & e_{11} & e_{11} & e_{12} & e_{22}\end{pmatrix}$$

lead respectively to $\alpha_{31245}=\alpha_{13425}=\alpha_{34125}=0$. Thus we have proved that $\alpha_{i_1i_2i_3i_4 5}=0$ where (i_1, i_2, i_3, i_4) is an arbitrary permutation of (1, 2, 3, 4).

b) Monomials of the form $x_5 x_{i_1} x_{i_2} x_{i_3} x_{i_4}$.
By the hypothesis of the lemma we have $\alpha_{54312}=\alpha_{51324}=0$. The substitutions

$$\begin{pmatrix} x_3 & x_1 & x_2 & x_4 & x_5 \\ e_{12} & e_{22} & e_{22} & e_{21} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_3 & x_4 & x_2 & x_5 \\ e_{12} & e_{22} & e_{22} & e_{21} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_5 & x_1 & x_3 & x_2 & x_4 \\ e_{22} & e_{21} & e_{11} & e_{11} & e_{11} \end{pmatrix}$$

lead respectively to $\alpha_{31245}+\alpha_{53124}=\alpha_{53124}=\alpha_{51342}=\alpha_{54231}=\alpha_{51234}=0$. Thus we have proved that $\alpha_{5 i_1 i_2 i_3 i_4}=0$.

c) Monomials of the form $x_{i_1} x_{i_2} x_{i_3} x_5 x_{i_4}$. By the hypothesis of the lemma we have $\alpha_{13254}=0$. The substitutions

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ e_{11} & e_{11} & e_{12} & e_{22} & e_{22} \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_5 & x_4 \\ e_{11} & e_{11} & e_{11} & e_{12} & e_{22} \end{pmatrix}, \quad \begin{pmatrix} x_3 & x_4 & x_1 & x_2 & x_5 \\ e_{11} & e_{11} & e_{12} & e_{22} & e_{22} \end{pmatrix},$$

$$\begin{pmatrix} x_3 & x_4 & x_1 & x_5 & x_2 \\ e_{12} & e_{22} & e_{22} & e_{21} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_3 & x_4 & x_1 & x_5 & x_2 \\ e_{11} & e_{11} & e_{11} & e_{12} & e_{22} \end{pmatrix}$$

lead to $\alpha_{12354}=\alpha_{31254}=\alpha_{34152}=\alpha_{31452}=\alpha_{13452}=0$.

Since f is (1,2) and (3,4)-symmetric, the proof of the case c) is completed.

d) Monomials of the form $x_{i_1} x_5 x_{i_2} x_{i_3} x_{i_4}$. It is easy to see that $\alpha_{45312}=0$. The substitutions

$$\begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5 \\ e_{12} & e_{22} & e_{21} & e_{11} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_4 & x_5 & x_3 & x_1 & x_2 \\ e_{22} & e_{21} & e_{11} & e_{11} & e_{11} \end{pmatrix}$$

lead to $\alpha_{45132}=\alpha_{45123}=0$. Now $\alpha_{52134}=0$ implies $\alpha_{25134}=0$. The substitutions

$$\begin{pmatrix} x_3 & x_1 & x_4 & x_2 & x_5 \\ e_{12} & e_{22} & e_{21} & e_{11} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_2 & x_5 & x_1 & x_3 & x_4 \\ e_{22} & e_{21} & e_{11} & e_{11} & e_{11} \end{pmatrix}$$

imply $\alpha_{25314}=\alpha_{25341}=0$, which completes the proof of the case.

e) Monomials of the form $x_{i_1} x_{i_2} x_5 x_{i_3} x_{i_4}$. We have shown that $\alpha_{12534}=0$ and $\alpha_{34512}=0$. The substitutions

$$\begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5 \\ e_{12} & e_{21} & e_{12} & e_{21} & e_{11} \end{pmatrix}, \quad \begin{pmatrix} x_3 & x_1 & x_4 & x_2 & x_5 \\ e_{12} & e_{21} & e_{12} & e_{21} & e_{11} \end{pmatrix},$$

$$\begin{pmatrix} x_1 & x_3 & x_2 & x_4 & x_5 \\ e_{12} & e_{21} & e_{12} & e_{22} & e_{22} \end{pmatrix}, \quad \begin{pmatrix} x_3 & x_1 & x_5 & x_2 & x_4 \\ e_{11} & e_{11} & e_{12} & e_{22} & e_{22} \end{pmatrix}$$

imply that $\alpha_{13524}=\alpha_{31542}=\alpha_{14532}=\alpha_{31524}=0$. So we have $\alpha_{i_1 i_2 5 i_3 i_4}=0$. Thus Lemma 4 is proved.

Now let $f \in V_5$ be $(1,2)$ and $(3,4)$-symmetric and let the six monomials (14) occur in f respectively with coefficients $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$. Then there exists an uniquely determined element $g \in V_5$, $g=\Sigma_{i=1}^{6} \gamma_i q_i$, $\gamma_i \in K$, such that the six monomials (14) occur in g also with the coefficients: $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ are uniquely defined, as the solution of the following system of linear equations:

$$\begin{aligned}
-\gamma_1-\gamma_5+\gamma_6 &= \delta_1,\\
\gamma_1-\gamma_2+\gamma_3+\gamma_5-\gamma_6 &= \delta_2,\\
\gamma_1-\gamma_2+\gamma_3-\gamma_4+\gamma_6 &= \delta_3,\\
-\gamma_2+\gamma_5+\gamma_6 &= \delta_4,\\
-\gamma_1-\gamma_3+\gamma_4+\gamma_6 &= \delta_5,\\
-\gamma_1+\gamma_2+\gamma_4-\gamma_5 &= \delta_6,
\end{aligned}$$

with determinant (15) is different from zero.

Since $f-g\in V_5$ is (1, 2) and (3, 4)-symmetric and the six monomials (14) occur in $f-g$ with zero coefficients so by Lemma 4, $f-g$ is the zero polynomial i. e. $f=g$ is a consequence of S_4, H, $f_2(x_1, x_2, \ldots, x_5)$. Thus proposition 2 is proved, which also completes the proof of Theorem 2.

By Remark 2, every element of V_5 is consequence of the two identities $S_4=0$ and $H(x_1, \ldots, x_5)=0$ and we obtain the result of Rosset, that all identities of degree 5 of the algebra $M(2, K)$ follow from the two identities $S_4=0$ and $H=0$, without using a computer.

This fact means that the three identities $f_1=0$, $f_2=0$ and $f_3=0$ can be removed from the basis of Razmyslov.

Denote by $\Phi(x_1, x_2, \ldots, x_5)=0$, the Lie identity, obtained by linearization the Filippov's identity $\Phi=[y, z, [t, x], x]+[y, x, [z, x], t]=0$.

Replacing the identity $f_2(x_1, x_2, \ldots, x_5)=0$ in the basis of Razmyslov by the identity $\Phi(x_1, x_2, \ldots x_5)=0$ and also the identity $f_2'(x_1, x_2, \ldots, x_6)=0$ by the identity $\Phi'(x_1, x_2, \ldots, x_6)=0$ it is easy to see, by the proof of Theorem 4 of Razmyslov [1], that the two identities $f_1'(x_1, x_2, \ldots, x_6)=0$ and $f_3'(x_1, x_2, \ldots, x_6)=0$ also can be removed from the basis of Razmyslov, so we have

Corollary 1. *The following four identities form a basis of the identities of the algebra $M(2, K)$:*

$$S_4(x_1, x_2, x_3, x_4)=0, \quad H(x_1, x_2, \ldots, x_5)=0, \quad \Phi'(x_1, x_2, \ldots, x_6)=0,$$

$$4[z, x](v_1\circ v_2)=[z, v_1, v_2, x]+[z, v_2, v_1, x]-[x, v_1, z, v_2]-[x, v_2, z, v_1],$$

where $v_1=[t_1, t_2]$, $v_2=[t_3, t_4]$, $v_1\circ v_2=v_1v_2+v_2v_1$.

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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