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SOME PROPERTIES OF RIGHT INVERSES

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Let R_1 and R_2 be right inverses of commutative linear operators D_1 and D_2. A necessary and sufficient condition for R_1 and R_2 to be commutative is given.

Let X be a linear space (over a field $\mathcal{F}$ of scalars). We consider a linear (i. e. additive and homogeneous) operator A defined in a linear subset $\mathfrak{D}_A \subset X$, called the domain of A, and mapping $\mathfrak{D}_A$ into X. We denote by $L(X)$ the collection of all such operators. Write: $L_0(X) = \{A \in L(X): \mathfrak{D}_A = X\}$. Z_A will stand for the kernel of A, i. e. $Z_A = \{x \in \mathfrak{D}_A : Ax = 0\}$.

Definition 1. *An operator $D \in L(X)$ is said to be right invertible if there exists an operator $R \in L_0(X)$ such that* (1) $RX \subset \mathfrak{D}_D$, (2) $DR = I$, *where I denotes the identity operator.*

The operator R is called a *right inverse* of D. The set of all right invertible operators belonging to $L(X)$ will be denoted by $\boldsymbol{R}(X)$. The set of all right inverses for an operator $D \in \boldsymbol{R}(X)$ will be denoted by $\mathcal{R}_D$.

Let $D \in \boldsymbol{R}(X)$. The kernel Z_D is called the *space of constants* for D and every element $z \in Z_D$ is called a *constant.*

Definition 2. *An operator $F \in L(X)$ is said to be an initial operator for an operator $D \in \boldsymbol{R}(X)$, corresponding to a right inverse R of D if* (*i*) $FX = Z_D$, $F^2 = F$. (*ii*) $FR = 0$ *on* X.

The definition immediately implies that $DF = 0$ on X.

One can prove the following facts;

1) Let R be a right inverse of $D \in \boldsymbol{R}(X)$. Then $F \in L(X)$ is an initial operator for D (corresponding to R) if and only if the following identity $F = I - RD$ holds on $\mathfrak{D}_D$ (cf. Theorem 2.1. of [1]).

2) Suppose that we are given $D \in \boldsymbol{R}(X)$ and an operator $F \in L(X)$ such that $F^2 = F$ and $FX = Z_D$. Then F is an initial operator for D corresponding to the right inverse $R = \widehat{R} - F\widehat{R}$, where R is uniquely defined independently of the choice of a right inverse $\widehat{R}$ of D (cf. Theorem 2.4. of [1]).

3) Let $D \in \boldsymbol{R}(X)$ and let R and R_1 be two right inverses of D which are commutative: $R_1R = RR_1$. Then $R_1 = R$ (cf. Proposition 2.3. of [1]).

4) Let $D \in \boldsymbol{R}(X)$ and let F_1, F be two commutative initial operators for D: $F_1F = FF_1$. Then $F_1 = F$ (cf. Proposition 2.4. of [1]).

I. H. Dimovski posed the following question: *Suppose, we are given two commutative right invertible operators. Do right inverses exist for these operators which also commute?*

The following theorem gives some answers to this question.

Theorem 1. *Suppose that $D_i \in \boldsymbol{R}(X)$, $R_i \in \mathcal{R}_{D_i}$ $(i = 1, 2)$ and*

$$D_1D_2 = D_2D_1 \quad on \quad \mathfrak{D}_{D_1} \cap \mathfrak{D}_{D_2}. \tag{1}$$

A necessary and sufficient condition for the operators R_1 and R_2 to be commutative is that there exists an operator $A \in L_0(X)$ such that

$$F_1 A = 0, \quad F_2 D_1 A = 0, \tag{2}$$

where F_i is an initial operator for D_i corresponding to R_i $(i=1,\ 2)$.

Proof. Write: $\widehat{D} = D_1 D_2$. By assumption (1) we have also $\widehat{D} = D_2 D_1$. It is clear that the operator $\widehat{D}$ is right invertible and then has two different right inverses: $\widehat{R_1} = R_1 R_2$, $\widehat{R_2} = R_2 R_1$. Indeed, $\widehat{D}\widehat{R_1} = D_1 D_2 R_1 R_2 = D_2 D_1 R_1 R_2 = D_2 R_2 = I$, $\widehat{D} R_2 = D_1 D_2 R_2 R_1 = D_1 R_1 = I$.

Denote by $\widehat{F_1}$ an initial operator for $\widehat{D}$ corresponding to $\widehat{R_1}$. We have on $\mathfrak{D}_{\widehat{D}}$

$$\widehat{F_1} = I - \widehat{R_1}\widehat{D} = I - R_1 R_2 D_1 D_2 = I - R_1 R_2 D_2 D_1 = I - R_1(I - F_2) D_1$$
$$= I - R_1 D_1 - R_1 F_2 D_1 = F_1 - R_1 F_2 D_1.$$

By Theorem 2 in [2] there exists an operator $A \in L_0(X)$ such that $\widehat{R_2} = \widehat{R_1} + \widehat{F_1} A$, i. e.

$$R_2 R_1 = R_1 R_2 + (F_1 - R_1 F_2 D_1) A. \tag{3}$$

We are looking for an operator A such that the component $(F_1 - R_1 F_2 D_1) A$ in (3) disappears.

Sufficiency. If $F_1 A = 0$, $F_2 D_1 A = 0$ then we have

$$R_2 R_1 = R_1 R_2 + (F_1 - R_1 F_2 D_1) A - R_1 R_2 + F_1 A - R_1 F_2 D_1 A = R_1 R_2,$$

i. e. the operators R_1 and R_2 commute.

Necessity. Suppose that $R_1 R_2 = R_2 R_1$. Write $U = F_1 A - R_1 F_2 D_1 A$, i. e $U = R_1 R_2 - R_2 R_1 = 0$. We have to show that $F_1 A = F_2 D_1 A = 0$. But, by definition, $F_1 R_1 = 0$ and $F_1^2 = F_1$. Thus $0 = F_1 U = F_1^2 A - F_1 R_1 (F_2 D_1 A) = F_1 A$, and we have $0 = U = -R_1 F_2 D_1 A$. Acting on both sides of the last equality by the operator D_1, we find $0 = D_1 U = -D_1 R_1 F_2 D_1 A = -F_2 D_1 A$ what was to be proved.

Remark 1. It follows from the proof of Theorem 2 in [2] that we can put $A = \widehat{R_1} - \widehat{R_2} = R_1 R_2 - R_2 R_1$.

Remark 2. If $D_1 = D_2 = D$ and R_1, $R_2 \in \mathcal{R}_D$ then the condition $R_1 R_2 = R_2 R_1$ implies $R_2 = R_1$ (cf. Proposition 2.3 in [1], as we have mentioned at the beginning). In this case we have $A = 0$.

Now we shall give some conditions for an operator to be a right inverse.

Theorem 2. *Suppose that $A \in L(X)$. If there exists an operator $B \in L_0(X)$ such that $BX \subset \mathfrak{D}_A$*

(i) $\ker B = \{0\}$.

(ii) *the operator $P = I - BA$ (defined on $\mathfrak{D}_A$) is a projection into* ker A.

(iii) $PB = 0$

then the operator A is right invertible, B is a right inverse of A and P is an initial operator for B corresponding to R.

Proof. By definition, $P^2 = P$ and $P\mathfrak{D}_A \subset \ker A$. Observe that the operator P is a projection *onto* ker A. Indeed, if $x \in \ker A$ then $Ax = 0$. Hence $Px = x - BAx = x$ and P is a mapping onto.

Suppose that $x \in \mathfrak{D}_A$ is arbitrarily fixed. Then $Px = x - BAx \in \ker A$. Thus $A(Px) = 0$ and $Ax - ABAx = AI - BAx = APx = 0$. The arbitrariness of x implies that $A = ABA$ on $\mathfrak{D}_A$.

Hence $AB=ABAB$. Writing $U=AB$ we obtain $U^2=U$, which implies that the operator $U=AB$ is a projection. Moreover, $\mathfrak{D}_U=\mathfrak{D}_{AB}=\mathfrak{D}_B=X$.

Suppose that $U\neq I$ on X. Then there exists $y\in X$ such that $y\neq 0$ and $v=Uy-y\neq 0$. Since $U^2=U$, we conclude that $ABv=Uv=U(Uy-y)=U^2y-Uy=Uy-Uy=0$. Thus

$$ABv=0 \tag{4}$$

and $BA(Bv)=B(ABv)=0$, which implies that

$$PBv=Bv-BA(Bv)=Bv. \tag{5}$$

On the other hand, (4) implies that $Bv\in \ker A$. Thus, the condition (iii) and (5) together imply that $Bv=P(Bv)=PBv=0$. Since $Bv=0$, the condition (i) implies that $v=0$ which contradicts to our assumption. Thus $ABy=y$ for all $y\in X$, i. e. $AB=I$ on X. We therefore conclude that B is a right inverse for A and that P is an initial operator for A corresponding to B.

The following question arises: *Will conditions* (i), (ii), (iii) *be all essential for the proof of Theorem* 2?

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