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SOME PROPERTIES OF RIGHT INVERSES

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Let R_1 and R_2 be right inverses of commutative linear operators D_1 and D_2 . A necessary and sufficient condition for R_1 and R_2 to be commutative is given.

Let X be a linear space (over a field \mathcal{F} of scalars). We consider a linear (i. e. additive and homogeneous) operator A defined in a linear subset $\mathfrak{D}_A \subset X$, called the domain of A, and mapping \mathfrak{D}_A into X. We denote by L(X) the collection of all such operators. Write: $L_0(X) = \{A \in L(X): \mathfrak{D}_A = X\}$. Z_A will stand

for the kernel of A, i. e. $Z_A = \{x \in \mathfrak{D}_A : Ax = 0\}$. Definition 1. An operator $D \in L(X)$ is said to be right invertible if there exists an operator $R \in L_0(X)$ such that (1) $RX \subset \mathfrak{D}_D$, (2) DR = I, where

I denotes the identity operator.

The operator R is called a right inverse of D. The set of all right invertible operators belonging to L(X) will be denoted by R(X). The set of all right inverses for an operator $D \in R(X)$ will be denoted by \mathcal{R}_D .

Let $D \in \mathbf{R}(X)$. The kernel Z_D is called the space of constants for D and

every element $z \in Z_D$ is called a constant. Definition 2. An operator $F \in L(X)$ is said to be an initial operator for an operator $D \in R(X)$, corresponding to a right inverse R of D if (i) FX $=Z_D$, $F^2=F$. (ii) FR=0 on X.

The definition immediately implies that DF = 0 on X.

One can prove the following facts;

1) Let R be a right inverse of $D \in \mathbf{R}(X)$. Then $F \in L(X)$ is an initial operator for D (corresponding to R) if and only if the following identity F = I - RD holds on \mathfrak{D}_D (cf. Theorem 2.1. of [1]).

2) Suppose that we are given $D \in \mathbf{R}(X)$ and an operator $F \in L(X)$ such that $F^2 = F$ and $FX = Z_D$. Then F is an initial operator for D corresponding to the right inverse R = R - FR, where R is uniquely defined independently of the choice of a right inverse R of D (cf. Theorem 2.4. of [1]).

3) Let $D \in R(X)$ and let R and R_1 be two right inverses of D which are

commutative: $R_1R = RR_1$. Then $R_1 = R$ (cf. Proposition 2.3. of [1]). 4) Let $D \in R(X)$ and let F_1 , F be two commutative initial operators for

 $D: F_1F = FF_1$. Then $F_1 = F$ (cf. Proposition 2.4. of [1]).

I. H. Dimovski posed the following question: Suppose, we are given two commutative right invertible operators. Do right inverses exist for these operators which also commute?

The following theorem gives some answers to this question. Theorem 1. Suppose that $D_i \in \mathbf{R}(X)$, $R_i \in \mathfrak{D}_{D_i}(i=1, 2)$ and

 $D_1D_2=D_2D_1$ on $\mathfrak{D}_{D_1}\cap\mathfrak{D}_{D_2}$. (1)

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A necessary and sufficient condition for the operators R_1 and R_2 to be commutative is that there exists an operator $A \in L_0(X)$ such that $F_1A=0$, $F_2D_1A=0$,

where F_i is an initial operator for D_i corresponding to R_i (i=1, 2).

Proof. Write: $\widehat{D} = D_1 D_2$. By assumption (1) we have also $\widehat{D} = D_2 D_1$. It is clear that the operator \widehat{D} is right invertible and then has two different right $\widehat{R_1} = R_1 R_2, \ \widehat{R_2} = R_2 R_1. \ \text{Indeed}, \ \widehat{D} \ \widehat{R_1} = D_1 D_2 R_1 R_2 = D_2 D_1 R_1 R_2 = D_2 R_2$ inverses: =I, $\widehat{D}R_2=D_1D_2R_2R_1=D_1R_1=I$.

Denote by \widehat{F}_1 an initial operator for \widehat{D} corresponding to \widehat{R}_1 . We have

on $\mathfrak{T}_{\widehat{D}}$

$$\widehat{F}_{1} = I - \widehat{R}_{1}\widehat{D} = I - R_{1}R_{2}D_{1}D_{2} = I - R_{1}R_{2}D_{2}D_{1} = I - R_{1}(I - F_{2})D_{1}$$

$$= I - R_{1}D_{1} - R_{1}F_{2}D_{1} = F_{1} - R_{1}F_{2}D_{1}.$$

By Theorem 2 in [2] there exists an operator $A \in L_0(X)$ such that $\widehat{R}_2 = \widehat{R}_1$ $+\widehat{F}_1A$, i. e.

$$R_2R_1 = R_1R_2 + (F_1 - R_1F_2D_1)A.$$

We are looking for an operator A such that the component $(F_1 - R_1F_2D_1)A$ in (3) disappears. Sufficiency. If $F_1A=0$, $F_2D_1A=0$ then we have

 $R_2R_1 = R_1R_2 + (F_1 - R_1F_2D_1)A - R_1R_2 + F_1A - R_1F_2D_1A = R_1R_2,$

i. e. the operators R_1 and R_2 commute. Necessity. Suppose that $R_1R_2=R_2R_1$. Write $U=F_1A-R_1F_2D_1A$, i. e $U=R_1R_2-R_2R_1=0$. We have to show that $F_1A=F_2D_1A=0$. But, by definition, $F_1R_1=0$ and $F_1^2=F_1$. Thus $0=F_1U=F_1^2A-F_1R_1(F_2D_1A)=F_1A$, and we have $0=U=-R_1F_2D_1A$. Acting on both sides of the last equality by the operator D_1 , we find $0=D_1U=-D_1R_1F_2D_1A=-F_2D_1A$ what was to be proved. Remark 1. It follows from the proof of Theorem 2 in [2] that we can

put $A = \widehat{R}_1 + \widehat{R}_2 = R_1 R_2 - R_2 R_1$. Remark 2. If $D_1 = D_2 = D$ and R_1 , $R_2 \in \mathcal{R}_D$ then the condition $R_1 R_2 = R_2 R_1$ implies $R_2 = R_1$ (cf. Proposition 2.3 in [1], as we have mentioned at the beginning). In this case we have A=0.

Now we shall give some conditions for an operator to be a right inverse-Theorem 2. Suppose that $A \in L(X)$. If there exists an operator $B \in L_0(X)$

such that $BX \subset \mathfrak{D}_A$

(i) $\ker B = \{0\}.$ (ii) the operator P=I-BA (defined on \mathfrak{D}_A) is a projection into ker A.

(iii) PB = 0then the operator A is right invertible, B is a right inverse of A and P is

an initial operator for B corresponding to R.

Proof. By definition, $F^2 = P$ and $P\mathfrak{D}_A \subset \ker A$. Observe that the operator P is a projection onto $\ker A$. Indeed, if $x \in \ker A$ then Ax = 0. Hence

Px = x - BAx = x and P is a mapping onto.

Suppose that $x \in \mathfrak{D}_A$ is arbitrarily fixed. Then $Px = x - BAx \in \ker A$. Thus A(Px) = 0 and Ax - ABAx = AI - BAx = APx = 0. The arbitrariness of x implies that A = ABA on \mathfrak{D}_A .

Hence AB = ABAB. Writing U = AB we obtain $U^2 = U$, which implies

that the operator U=AB is a projection. Moreover, $\mathfrak{D}_U=\mathfrak{D}_{AB}=\mathfrak{D}_B=X$. Suppose that $U \neq I$ on X. Then there exists $y \in X$ such that $y \neq 0$ and $v = Uy - y \neq 0$. Since $U^2 = U$, we conclude that $ABv = Uv = U(Uy - y) = U^2y$ -Uy = Uy - Uy = 0. Thus

(4)ABv=0

and BA(Bv) = B(ABv) = 0, which implies that

PBv = Bv - BA(Bv) = Bv.

On the other hand, (4) implies that $Bv \in \ker A$. Thus, the condition (iii) and (5) together imply that Bv = P(Bv) = PBv = 0. Since Bv = 0, the condition (i) implies that v=0 which contradicts to our assumption. Thus ABy=y for all $y \in X$, i. e. AB = I on X. We therefore conclude that B is a right inverse for A and that P is an initial operator for A corresponding to B.

The following question arises: Will conditions (i), (ii), (iii) be all essential

for the proof of Theorem 2?

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