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A SHORT AND CONSTRUCTIVE APPROACH TO THE JORDAN CANONICAL FORM OF A MATRIX

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1. Introduction. The many publications on the Jordan canonical form (J. c. f.) which appeared during the last decades ([1—26], [10, 215—234], [12, 149—202], [16, 143—155]) show that there still exists a demand for a short, elementary, and constructive approach to the J. c. f. of a matrix. In the present paper a new and simple algorithm is described to construct a matrix which transforms a given matrix whose eigenvalues are known into its J. c. f. This algorithm again proves the existence of the J. c. f. and requires in practice only a comparatively small number of calculations. Also diagonalizability and differential equations are considered.

2. Notations. Let A be a (n, n)-matrix with elements in $\mathbb{C}$, whose eigenvalues are explicity known and denote by I the (n, n)-identity matrix. Put $c(\lambda) := \det(\lambda I - A) = \Pi_{\nu=1}^{k} (\lambda - \lambda_\nu)^{n_\nu}$ where the λ_ν are different, $n_\nu \geqq 1$, $\nu = 1, \ldots, k$ and $n_1 + \cdots + n_k = n$. For $j = 1, \ldots, k$ we then put $p_j(\lambda) := (\lambda - \lambda_j)^{n_j}$, $q_j(\lambda) = c(\lambda) / p_j(\lambda)$, $q(\lambda) = \Sigma_{j=1}^{k} q_j(\lambda)$, $B_j := q_j(A)$, $B := q(A)$, $V_j := \{x \in \mathbb{C}^n : p_j(A)x = 0\}$. All considerations remain valid if $\mathbb{C}$ is replaced by an arbitrary field in which $c(\lambda)$ splits into linear factors.

3. Properties of the characteristic polynomial. Lemma 1. *B is nonsingular.* (See also [20, Lemma 2])

Proof. The Hamilton-Cayley-theorem for A and $q(\lambda) - q(\lambda_j) = (\lambda - \lambda_j) r_j(\lambda)$ imply $\Pi_{j=1}^{k} (q(A) - q(\lambda_j) I)^{n_j} = 0$, or $p(B) = 0$ where $p(\lambda) := \Pi_{j=1}^{k} (\lambda - q(\lambda_j))^{n_j}$ and $p(0) \neq 0$, since $q(\lambda_j) = q_j(\lambda_j) \neq 0$, $j = 1, \ldots, k$. Put $r(\lambda) := (p(0) - p(\lambda)) / p(0)\lambda$. Then $Br(B) = I$ and thus B is invertible.

Remark. After transforming A into a similar triangular matrix it follows that $q(A)$ has exactly the n_j-fold eigenvalues $q(\lambda_j)$, $j = 1, \ldots, k$. This again implies $\det B \neq 0$. Also for a triangular matrix the Hamilton-Cayley-theorem easily can be verified.

Theorem 1 (Rational decomposition theorem). *The columns of B_j span V_j, $j = 1, \ldots, k$, and $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_k$.*

Proof. Let V'_j be the subspace of $\mathbb{C}^n$ spanned by the columns of B_j. Since $p_j(A) B_j = 0$, $j = 1, \ldots, k$ (Hamilton-Cayley), $V'_j \subset V_j$ follows. Thus $B = \Sigma_{j=1}^{k} B_j$ and Lemma 1 imply $\mathbb{C}^n = V'_1 + \cdots + V'_k = V_1 + \cdots + V_k$. Assume that $x_1 + \cdots + x_k = 0$, where $x_j \in V_j$. Then $0 = B_j(x_1 + \cdots + x_k) = B_j x_j = (B_1 + \cdots + B_k) x_j = B x_j$ since for $i \neq j$ B_i contains the factor $p_j(A)$. Thus, $x_j = 0$, $j = 1, \ldots, k$, and $\mathbb{C}^n = V'_1 \oplus \cdots \oplus V'_k = V_1 \oplus \cdots \oplus V_k$. This and $V'_j \subset V_j$ imply $V'_j = V_j$ and hence $\operatorname{rank} B_j = \dim V_j$.

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4. The J. c. f. of A. For any $0 \neq x \in V_j$ $x^{(r)} := (A-\lambda_j I)^{r-1}x$, $r=1, 2, \ldots$ is called the λ_j-chain of x, and m its length, if $x^{(m)} \neq 0$ and $x^{(m+1)}=0$. Then $x^{(r)} \in V_j$ for all r and $1 \leqq m \leqq n_j$. Observe that $x^{(r+1)}=(A-\lambda_j I)x^{(r)}$, $r=1, 2, \ldots$

Starting with the non-zero columns $x_1, \ldots, x_{t_j}$ of B_j (or with any other set of non-zero vectors which span V_j) we construct a special (Jordan-) basis of V_j according to the following algorithm. (See Theorem 3 for dim $V_j=n_j$.)

a) Arrange the λ_j-chains of x_ν having length m_ν, $\nu=1, \ldots, t_j$, according to their lengths in decreasing order, and assume (w. l. o. g.) that $m_1 \geqq \cdots \geqq m_{t_j}$ holds. Next, assume that for some $s \geqq 1$ $x_\nu^{(m_\nu)}$, $\nu=1, \ldots, s$, are lineary independent. Then $x_\nu^{(r)}$, $r=1, \ldots, m$, $\nu=1, \ldots, s$, altogether are linearly independent. This can be seen by successively applying $(A-\lambda_j I)^{m_1-r}$, $r=1, \ldots m_1$, to a vanishing linear combination of these vectors which always yields , vanishing linear combination of $x_\nu^{(m_\nu)}$, $\nu=1, \ldots, s$. Therefore, eventually, all coefficients have to be zero (see [19, Lemma 1]).

b) Assume, next, that $x_\nu^{(m_\nu)}$, $\nu=1, \ldots, s+1$, are linearly dependent. Then $x_{s+1}^{(m_{s+1})}=\sum_{\nu=1}^{s} a_\nu x_\nu^{(m_\nu)}$ with certain coefficients a_ν and we put $y := x_{s+1} - \sum_{\nu=1}^{s} a_\nu x_\nu^{(m_\nu - m_{s+1}+1)}$. Observe that $m_\nu \geqq m_{s+1}$. If $y=0$, then the chain of x_{s+1} is deleted. If $y \neq 0$, then, again, the chain of x_{s+1} is deleted, and the chain of y, having length $< m_{s+1}$, is placed between the remaining chains such that all chains again are arranged according to their lengths in decreasing order. This elementary operation does not alter the total number of linearly independent vectors among all chains, and at the start $x_\nu^{(1)}=x_\nu$, $\nu=1, \ldots, t_j$, span V_j by Theorem 1.

The procedure described in b) can be repeated until, finally, a set of $s_j \geqq 1$ λ_j-chains $z_\nu^{(r)}$, $r=1, \ldots, n_{j\nu}$ (of length $n_{j\nu} \leqq n_j$), $\nu=1, \ldots, s_j$, remains with the property that now *all* $z^{(n_{j\nu})}$, $\nu=1, \ldots, s_j$, are linearly independent. Therefore, all $z_\nu^{(r)}$, $r=1, \ldots n_{j\nu}$, $\nu=1, \ldots, s_j$, together form a basis of V_j and dim $V_j=n_{j1}+\cdots+n_{js_j}$. Then $Az_\nu^{(r)}=(A-\lambda_j I)z_\nu^{(r)}+\lambda_j z_\nu^{(r)}=z_\nu^{(r+1)}+\lambda_j z_\nu^{(r)}$ for $r=1, \ldots, n_{j\nu}$ with $z_\nu^{(n_{j\nu}+1)}=0$ and, as usual, $A(z_\nu^{(1)}, z_\nu^{(2)}, \ldots, z_\nu^{(n_{j\nu})})=(z_\nu^{(1)}, z_\nu^{(2)}, \ldots, z_\nu^{(n_{j\nu})})J_{j\nu}$, where the $(n_{j\nu}, n_{j\nu})$-matrix $J_{j\nu}=\begin{pmatrix} \lambda_j & & 0 \\ 1 & \ddots & \\ 0 & \ddots 1 & \lambda_j \end{pmatrix}$ is called the λ_j-Jordan block corresponding to the λ_j-chain of z_ν. These considerations and Theorem 1 yield

Theorem 2 (J. c. f. of A). *If for $j=1, \ldots, k$, all λ_j—chains $z_\nu^{(1)}, \ldots, z_\nu^{(n_{j\nu})}$, $(n_{j\nu} \leqq n_j)$, $\nu=1, \ldots, s_j$, which were constructed above, are written next to each other and combined in the (n, n)—(chain) matrix C, then C is nonsingular. Furthermore, $AC=CJ$ or $C^{-1}AC=J$ holds with a (n, n)-(Jordan) matrix* $J=\begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_t \end{pmatrix}$, *where $J_1, \ldots, J_t$ $(t=s_1+\cdots+s_k)$ are all λ_j-Jordan blocks corresponding to the λ_j-chains in C $(j=1, \ldots, k)$.*

Next, put $r_{j\varrho} :=$ rank $(A-\lambda_j I)^\varrho$, $\varrho=0, 1, \ldots$ and let $N_{j\varrho}$ denote the number of λ_j-Jordan blocks in J which are (ϱ, ϱ)-matrices, $\varrho=1, 2, \ldots$.

Theorem 3. $r_{jn_j}=$ rank $p_j(A)=n-n_j$ *and therefore rank* $B_j=$ dim $V_j=n_j$. *Furthermore* $N_{j\varrho}=r_{j,\varrho-1}-2r_{j\varrho}+r_{j,\varrho+1}$ *for* $\varrho=1, 2, \ldots$ *and thus J is uniquely* *determined by A except for the order of succession of the Jordan blocks on the diagonal of J.*

Proof. Let $I_{j\nu}$ denote the $(n_{j\nu}, n_{j\nu})$-identity matrix. Theorem 2 implies $r_{j\varrho} = \text{rank } (J-\lambda_j I)^\varrho = \Sigma_{i=1}^{k} \Sigma_{\nu=1}^{s_i} \text{ rank} (J_{i\nu}-\lambda_j I_{i\nu})^\varrho = n - n_j + \Sigma_{\nu=1}^{s_i} \text{ rank } (J_{j\nu} - \lambda I_{j\nu})^\varrho$ $= n - n_j + \Sigma_{\nu=1}^{s_i} \max (n_{j\nu} - \varrho, 0)$. Since all $n_{j\nu} \leq n_j$, $r_{jn_j} = n - n_j$ follows and, hence rank $B_j = \dim V_j = n_j$ by Theorem 1. Next, $d^j_\varrho := r_{j,\varrho-1} - r_{j\varrho} = \Sigma_{n_{j\nu} \geq \varrho} 1$, and, hence $d_{j\varrho} - d_{j,\varrho+1} = \Sigma_{n_{j\nu}=\varrho} 1 = N_{j\varrho}$ follow.

5. Concluding remarks. Remark 1. Observe that the algorithm in 4 automatically yields a *longest* λ_j-chain. As soon as $s_j \lambda_j$-chains $y_\nu^{(r)}$, $r = 1, \ldots, n_{j\nu}$, $\nu = 1, \ldots, s_j$, are found such that $n_{j1} + \cdots + n_{js_j} = n_j$ and $y_\nu^{(n_{j\nu})}$, $\nu = 1, \ldots, s_j$ are linearly independent, the procedure b) of the algorithm can be stopped. If, in particular, A is nilpotent, then the chains consist of columns of $A^0, A, \ldots, A^n$.

Remark 2. By theorem 1 V_j coincides with the λ_j-eigenspace of A iff $(A-\lambda_j I)B_j = 0$.

Remark 3. Put $p(\lambda) := \Pi_{j=1}^{k} (\lambda - \lambda_j)$. Then by Theorem 1 A is diagonalizable iff $p(A) = 0$. (Observe that in $\mathbb{C}$ $p(\lambda) = c(\lambda)/d(\lambda)$, where $d(\lambda)$ is the gcd of $c(\lambda)$ and $c'(\lambda)$.) Since $p(A) = 0$ implies $(A-\lambda_j I)B_j = 0$ for $j = 1, \ldots, k$, by Remark 2 and Theorem 1 A altogether has n linearly independent eigenvectors. Therefore, A is diagonalizable and the converse is trivial.

Remark 4. By Lemma 1 $A = \Sigma_{j=1}^{k} AB_j B^{-1} = D + N$ where $N := \Sigma_{j=1}^{k} (A - \lambda_j I) B_j B^{-1}$ is nilpotent since $B_i B_j = 0$ for $i \neq j$, and $D := \Sigma_{j=1}^{k} \lambda_j B_j B^{-1}$ is diagonalizable by Remark 3, since $p(D) = 0$. Also N, D are commuting polynomials in A (see [5]).

Remark 5. $Y(t) := \Sigma_{j=1}^{k} e^{\lambda_j(t-t_0)} \Sigma_{\nu=0}^{n_j-1} \frac{(t-t_0)^\nu}{\nu!} (A-\lambda_j I)^\nu B_j$ satisfies $Y'(t) = AY(t)$ and $Y(t_0) = B$. Hence, by Lemma 1, $Y(t)$ is a fundamental matrix (rea for real A) of the differential equation $y' = Ay$ and $e^{A(t-t_0)} = Y(t)B^{-1}$ (see [20]).

Remark 6. As an example assume that $A = \begin{pmatrix} T_1 & & * \\ & \ddots & \\ O & & T_k \end{pmatrix}$ where for $j = 1, \ldots, k$, $T_j = \begin{pmatrix} \lambda_j & & * \\ & \ddots & \\ O & & \lambda_j \end{pmatrix}$ is a (n_j, n_j)-triangular matrix. Considering the general solution vector $x \in \mathbb{C}^n$ of $(A - \lambda_j I)^\varrho x = 0$ yields $n - r_{j\varrho} = n_j\text{-rank } (T_j - \lambda_j I_j)^\varrho$ for all j, ϱ where I_j is the (n_j, n_j)-identity matrix. Hence by Theorem 3 J only depends on the blocks $T_1, \ldots, T_k$ and A is diagonalizable by Remark 3 iff $T_1, \ldots, T_k$ are diagonal matrices. Assume, finally, that for some j $T_j = \begin{pmatrix} T_{j1} & & O \\ & \ddots & \\ O & & T_{js_j} \end{pmatrix}$ where for $\nu = 1, \ldots, s_j$ $(1 \leq s_j \leq n_j)$, $T_{j\nu} = \begin{pmatrix} \lambda_j & & * \\ & \ddots \ddots & \\ O & & \lambda_j \end{pmatrix}$ is a $(n_{j\nu}, n_{j\nu})$ triangular matrix with *non-zero* (if $n_{i\nu} > 1$) super diagonal elements. For $\nu = 1, \ldots, s_j$ let x_ν be the $(n_1 + \cdots + n_{j-1}) + (n_{j1} + \cdots + n_{j\nu})$-th column of B_j. Then, by considering $(A - \lambda_j I)^{r-1} B_j$, $r = 1, 2, \ldots$, the λ_j-chain of x_ν has length $n_{j\nu}$ and for $\nu = 1, \ldots, s_j$ all vectors of these chains form a Jordan basis of V_j. In particular, J contains exactly the λ_j-Jordan blocks $J_{j\nu}$ of type $(n_{j\nu}, n_{j\nu})$, $\nu = 1, \ldots, s_j$.

Example 1

$$A=\begin{pmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 3 & -4 \\ 15 & -10 & 11 & -11 \end{pmatrix}.$$ Here $\lambda_1=-1$, $n_1=4$. Hence $B_1=I$,

$$(A+I)=\begin{pmatrix} 2 & -1 & 1 & -1 \\ -3 & 4 & -5 & 4 \\ 8 & -4 & 4 & -4 \\ 15 & -10 & 11 & -10 \end{pmatrix},\ (A+I)^2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & -1 & 1 \end{pmatrix},\ (A+I)^3=0.$$

Thus, there are 4 λ_1-chains of length 3. We choose

$$x_1=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ x_2=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ with } x_1^{(1)}=x_1,\ x_1^{(2)}=\begin{pmatrix} -1 \\ 4 \\ -4 \\ -10 \end{pmatrix},\ x_1^{(3)}=\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \text{ and}$$

$$x_2^{(1)}=x_2,\ x_2^{(2)}=\begin{pmatrix} 1 \\ -5 \\ 4 \\ 11 \end{pmatrix},\ x_2^{(3)}=\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$ Since $x_2^{(3)}=-x_1^{(3)}$ we replace the chain of x_2 by the chain of $y:=x_1+x_2^{(1)}$. Then $$y^{(1)}=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ y^{(2)}=\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},\ y^{(3)}=0.$$

Since $y^{(2)}=x_1^{(3)}$, we finally replace the chain of y by the chain of $z:=y-x_1^{(3-2+1)}=y-x_1^{(2)}$. Then $$z^{(1)}=\begin{pmatrix} 1 \\ -3 \\ 5 \\ 10 \end{pmatrix},\ z^{(2)}=0 \text{ and}$$

$$C=(x_1^{(1)}, x_1^{(2)}, x_1^{(3)}, z)=\begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 4 & -1 & -3 \\ 0 & -4 & 0 & 5 \\ 0 & -10 & 1 & 10 \end{pmatrix},\ C^{-1}=\begin{pmatrix} -2 & 1 & -1 & 1 \\ -5 & 0 & 1 & 0 \\ -10 & 0 & 0 & 1 \\ -4 & 0 & 1 & 0 \end{pmatrix}$$

yield $C^{-1}AC=J=\begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix}$ with $J_1=\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$, $J_2=(-1)$.

Example 2

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$ Here $\lambda_1=1$, $n_1=2$ and $\lambda_2=-1$, $n_2=1$.

$$B_1=(A+I)=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix},\ B_2=(A-I)^2=\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Since $(A+I)B_2=0$ $y=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is λ_2-eigenvector.

Since $(A-I)B_1=\begin{pmatrix}-1&0&1\\-1&0&1\\1&0&1\end{pmatrix}$, we can choose as λ_1-chain:

$$x_1^{(1)}=\begin{pmatrix}1\\0\\-1\end{pmatrix},\quad x_1^{(2)}=-\begin{pmatrix}1\\1\\1\end{pmatrix}.\ \text{Then } C=(-x_1^{(1)},\ -x_1^{(2)},\ y)=\begin{pmatrix}-1&1&1\\0&1&-1\\1&1&1\end{pmatrix},$$

$$C^{-1}=\frac{1}{4}\begin{pmatrix}-2&0&2\\1&2&1\\1&-2&1\end{pmatrix}$$

yields $C^{-1}AC=J=\begin{pmatrix}J_1&O\\O&J_2\end{pmatrix}$ with $J_1=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $J_2=(-1)$.

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