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A NOTE ON BERMAN'S PHENOMENON IN INTERPOLATION THEORY

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In 1975, D. L. Berman introduced a sequence of polynomials which interpolate a fixed, but arbitrary, continuous function, f. Even though the nth polynomial in the sequence interpolates the function at $n+1$ points, the polynomials do not necessarily converge to the function. In this paper we determine necessary and sufficient conditions on f for Berman's polynomials to converge uniformly to f.

If $-1 \leqq x_n < x_{n-1} < \dots < x_2 < x_1 \leqq +1$ and $f: [-1, 1] \to (-\infty, \infty)$ then there is a unique polynomial $H_{2n-1}(f, x)$ such that

(a) the degree of $H_{2n-1}(f, x)$ does not exceed $2n-1$,

(b) $H_{2n-1}(f, x_k) = f(x_k)$, $k=1, 2, \dots, n$,

(c) $H'_{2n-1}(f, x_k) = 0$, $k=1, 2, \dots, n$.

In 1916, L. Fejér [3] gave a proof of K. Weierstrass' approximation theorem using these polynomials. From now on, $x_k = x_{k,n} = \cos((2k-1)\pi/(2n))$ for $k=1, 2, \dots, n$ where $n \geqq 1$.

Theorem 1 (L. Fejér) *If $f \in C([-1, 1])$, then $\lim_{n\to\infty} \| H_{2n-1}(f) - f \|_\infty = 0$, where $\|\cdot\|_\infty$ denotes the uniform norm on $C([-1, 1])$.*

Notice that when n is even $x_{k,n} \neq 0$.

In 1975, D. L. Berman [1] considered the effect of adding the single node $x_{0,n}=0$ to the point system when n was even.

Specifically, for $n=2m$, let $R_{2n+1}(f, x)$ be the unique polynomial such that:

(a) the degree of $R_{2n+1}(f, x)$ does not exceed $2n+1$,

(b) (i) $R_{2n+1}(f, x_k) = f(x_k)$, $k=1, 2, \dots, n$,

(ii) $R_{2n+1}(f, 0) = f(0)$,

(c) (i) $R'_{2n+1}(f, x_k) = 0$, $k=1, 2, \dots, n$,

(ii) $R'_{2n+1}(f, 0) = 0$.

By the condition (b) (ii) we are guaranteed that $\lim_{n\to\infty} |R_{2n+1}(f, 0) - f(0)| = 0$. Berman showed that this is *all* that we are guaranteed:

Theorem 2 (D. L. Berman). *If $f(t) \equiv t$ and $0 < |x| < 1$ then the sequence $\{|R_{2n+1}(f, x) - f(x)| : n=2, 4, 6, \dots\}$ is divergent.*

Hence, by adding a single point to the system of nodes, one can annihilate the approximation properties of Hermite-Fejér interpolation polynomials. This type of situation has been called Berman's phenomenon in a paper by Cook and Mills [2].

One may now ask the question, *If $R_{2n+1}(f, x)$ does not converge to $f(x)$ for $f(t) \equiv t$, what are necessary and sufficient conditions on f for the sequence $\{\| R_{2n+1}(f) - f \|_\infty : n=2, 4, 6, \dots\}$ to converge to 0?*

In this paper we shall answer this question by very elementary methods and present a simple proof of Berman's theorem.

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First, we represenf Berman's polynomials in terms of Fejér's polynomials. If $T_n(x)=T_n(\cos\theta)=\cos n\theta$ denotes the Chebyshev polynomial of degree n then one can check that

$$(1)\quad R_{2n+1}(f,\ x)=H_{2n-1}(f,\ x)+T_n(x)^2(f(0)-H_{2n-1}(f,\ 0))-xT_n(x)^2H'_{2n-1}(f,\ 0)$$

by using the defining conditions for $H_{2n-1}(f,\ x)$ and $R_{2n+1}(f,\ x)$.

Then, from (1) and Theorem 1 it follows that $\lim_{n\to\infty}\|R_{2n+1}(t)-f\|_\infty=0$ is equivalent to $\lim_{n\to\infty}H_{2n-1}'(f,\ 0)=0$,

From L. Fejér's work [3], p. 66, formula (3)] we know that

$$H_{2n-1}(f,\ x)=\sum_{k=1}^{n} f(k_k)\frac{T_n(x)^2(1-xx_k)}{n^2(x-x_k)^2}$$

and therefore

$$H'_{2n}(f,\ 0)=\frac{1}{n^2}\sum_{k=1}^{n} f(x_k)\left(\frac{2-x_k^2}{x_k^3}\right).$$

Thus we have shown

Theorem 3. $\lim_{n\to\infty}\|R_{2n+1}(f)-f\|_\infty=0$ *if and only if*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} f(x_k)\left(\frac{2-x_k^2}{x_k^3}\right)=0.$$

The representation (1) also gives us an elementary derivation of Berman's Theorem. By Theorem 1 and formula (1), if $0<|x|<1$ then $\{R_{2n+1}(f,\ x):n=2$ $4,\ 6,\dots\}$ divereges if and only if $\{H'_{2n-1}(f,\ 0):n=2,\ 4,\ 6,\dots\}$ does not converge to 0. In the case $f(t)\equiv t$, we have

$$H'_{2n-1}(f,\ 0)=n^{-2}\sum_{k=1}^{n}(2-x_k^2)x_k^{-2}=n^{-2}\sum_{k=1}^{n}(2x_k^{-2})-n^{-1}=2-n^{-1}.$$

(Here we have used formula (12) from [2].) Theorem 2 follows immediately.

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