

Provided for non-commercial research and educational use.  
Not for reproduction, distribution or commercial use.

# Serdica

## Bulgariacae mathematicae publicationes

---

# Сердика

## Българско математическо списание

---

The attached copy is furnished for non-commercial research and education use only.  
Authors are permitted to post this version of the article to their personal websites or  
institutional repositories and to share with other researchers in the form of electronic reprints.

Other uses, including reproduction and distribution, or selling or  
licensing copies, or posting to third party websites are prohibited.

For further information on  
Serdica Bulgaricae Mathematicae Publicationes  
and its new series Serdica Mathematical Journal  
visit the website of the journal <http://www.math.bas.bg/~serdica>  
or contact: Editorial Office  
Serdica Mathematical Journal  
Institute of Mathematics and Informatics  
Bulgarian Academy of Sciences  
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49  
e-mail: [serdica@math.bas.bg](mailto:serdica@math.bas.bg)

## A NOTE ON BERMAN'S PHENOMENON IN INTERPOLATION THEORY

S. J. GOODENOUGH, T. M. MILLS

In 1975, D. L. Berman introduced a sequence of polynomials which interpolate a fixed, but arbitrary, continuous function,  $f$ . Even though the  $n$ th polynomial in the sequence interpolates the function at  $n+1$  points, the polynomials do not necessarily converge to the function. In this paper we determine necessary and sufficient conditions on  $f$  for Berman's polynomials to converge uniformly to  $f$ .

If  $-1 \leq x_n < x_{n-1} < \dots < x_2 < x_1 \leq +1$  and  $f: [-1, 1] \rightarrow (-\infty, \infty)$  then there is a unique polynomial  $H_{2n-1}(f, x)$  such that

- (a) the degree of  $H_{2n-1}(f, x)$  does not exceed  $2n-1$ ,
- (b)  $H_{2n-1}(f, x_k) = f(x_k)$ ,  $k=1, 2, \dots, n$ ,
- (c)  $H'_{2n-1}(f, x_k) = 0$ ,  $k=1, 2, \dots, n$ .

In 1916, L. Fejér [3] gave a proof of K. Weierstrass' approximation theorem using these polynomials. From now on,  $x_k = x_{k,n} = \cos((2k-1)\pi/(2n))$  for  $k=1, 2, \dots, n$  where  $n \geq 1$ .

**Theorem 1** (L. Fejér) *If  $f \in C([-1, 1])$ , then  $\lim_{n \rightarrow \infty} \|H_{2n-1}(f) - f\|_\infty = 0$ , where  $\|\cdot\|_\infty$  denotes the uniform norm on  $C([-1, 1])$ .*

Notice that when  $n$  is even  $x_{k,n} \neq 0$ .

In 1975, D. L. Berman [1] considered the effect of adding the single node  $x_{0,n} = 0$  to the point system when  $n$  was even.

Specifically, for  $n=2m$ , let  $R_{2n+1}(f, x)$  be the unique polynomial such that:

- (a) the degree of  $R_{2n+1}(f, x)$  does not exceed  $2n+1$ ,
- (b) (i)  $R_{2n+1}(f, x_k) = f(x_k)$ ,  $k=1, 2, \dots, n$ ,  
(ii)  $R_{2n+1}(f, 0) = f(0)$ ,
- (c) (i)  $R'_{2n+1}(f, x_k) = 0$ ,  $k=1, 2, \dots, n$ ,  
(ii)  $R'_{2n+1}(f, 0) = 0$ .

By the condition (b) (ii) we are guaranteed that  $\lim_{n \rightarrow \infty} |R_{2n+1}(f, 0) - f(0)| = 0$ . Berman showed that this is *all* that we are guaranteed:

**Theorem 2** (D. L. Berman). *If  $f(t) \equiv t$  and  $0 < |x| < 1$  then the sequence  $\{|R_{2n+1}(f, x) - f(x)| : n=2, 4, 6, \dots\}$  is divergent.*

Hence, by adding a single point to the system of nodes, one can annihilate the approximation properties of Hermite-Fejér interpolation polynomials. This type of situation has been called Berman's phenomenon in a paper by Cook and Mills [2].

One may now ask the question, *If  $R_{2n+1}(f, x)$  does not converge to  $f(x)$  for  $f(t) \equiv t$ , what are necessary and sufficient conditions on  $f$  for the sequence  $\{|R_{2n+1}(f) - f\|_\infty : n=2, 4, 6, \dots\}$  to converge to 0?*

In this paper we shall answer this question by very elementary methods and present a simple proof of Berman's theorem.

First, we represent Berman's polynomials in terms of Fejér's polynomials. If  $T_n(x) = T_n(\cos \theta) = \cos n\theta$  denotes the Chebyshev polynomial of degree  $n$  then one can check that

$$(1) \quad R_{2n+1}(f, x) = H_{2n-1}(f, x) + T_n(x)^2(f(0) - H_{2n-1}(f, 0)) - xT_n(x)^2H'_{2n-1}(f, 0)$$

by using the defining conditions for  $H_{2n-1}(f, x)$  and  $R_{2n+1}(f, x)$ .

Then, from (1) and Theorem 1 it follows that  $\lim_{n \rightarrow \infty} \|R_{2n+1}(f) - f\|_{\infty} = 0$  is equivalent to  $\lim_{n \rightarrow \infty} H'_{2n-1}(f, 0) = 0$ ,

From L. Fejér's work [3], p. 66, formula (3)] we know that

$$H_{2n-1}(f, x) = \sum_{k=1}^n f(x_k) \frac{T_n(x)^2(1 - xx_k)}{n^2(x - x_k)^2}$$

and therefore

$$H'_{2n}(f, 0) = \frac{1}{n^2} \sum_{k=1}^n f(x_k) \left( \frac{2 - x_k^2}{x_k^3} \right).$$

Thus we have shown

**Theorem 3.**  $\lim_{n \rightarrow \infty} \|R_{2n+1}(f) - f\|_{\infty} = 0$  if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n f(x_k) \left( \frac{2 - x_k^2}{x_k^3} \right) = 0.$$

The representation (1) also gives us an elementary derivation of Berman's Theorem. By Theorem 1 and formula (1), if  $0 < |x| < 1$  then  $\{R_{2n+1}(f, x) : n = 2, 4, 6, \dots\}$  diverges if and only if  $\{H'_{2n-1}(f, 0) : n = 2, 4, 6, \dots\}$  does not converge to 0. In the case  $f(t) \equiv t$ , we have

$$H'_{2n-1}(f, 0) = n^{-2} \sum_{k=1}^n (2 - x_k^2)x_k^{-2} = n^{-2} \sum_{k=1}^n (2x_k^{-2}) - n^{-1} = 2 - n^{-1}.$$

(Here we have used formula (12) from [2].) Theorem 2 follows immediately.

**Acknowledgement.** We are pleased to have this opportunity to thank Miss R. Myers for her technical assistance in the preparation of this paper.

#### REFERENCES

1. Д. Л. Берман. Расширенный интерполяционный процесс Эрмита—Фейера. *Известия ВУЗ, Математика*, 1975, № 1, 93—96.
2. W. Lyle Cook, T. M. Mills. On Berman's phenomenon in interpolation theory. *Bull. Austral. Math. Soc.*, 12, 1975, 457—465.
3. L. Fejér. Über Interpolation. *Nachr. K. Ges. Wiss. Göttingen Math.-Phys. Kl.*, 1916, 66—91.

Bendigo College of Advanced Education  
Bendigo, Victoria, 3550 Australia

Received 26. 3. 1980