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UNIFORM CONVERGENCE OF PADE APPROXIMANTS. GENERAL CASE

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Let $f(z)=\Sigma f_v z^v$ be a formal power series. For each pair (n, m) of positive integers, $n, m \in \mathcal{N}$, we denote by $\pi_{n,m}(f)$ the Pade approximant to f of order (n, m). All the theorems about convergence of sequences of Pade approximants to f in its domain of holomorphy $\Omega(f)$ (on compact subsets) refer to convergence in capacity. The question about the uniform convergence is studied in this paper. Holomorphic functions are constructed for which there exist sequences of Pade approximants which converge to f in $\Omega(f)$ only in capacity. A function is constructed, holomorphic in the unit disk, for which there is a sequence $\{\pi_{n,n}\}$, $n \in \Lambda \subset \mathcal{N}$, such that $\pi_{n,n} \underset{n \in \Lambda}{\rightrightarrows} f$ only in the disk $D' = \{z, |z| \sqrt{0,85}\}$.

Let

$$f(z) = \sum_{v=0}^{\infty} f_v z^v \tag{1}$$

be a formal power series. For each fixed pair (n, m) of positive integers, $n, m \in \mathrm{N}$ we denote by $\mathcal{R}\ n,m$ the class of all the rational functions of order (n, m): $\mathcal{R}_{n,m}=\{p_n/p_m,\ p_m \neq 0\}$, where p_n and p_m are polynomials of degree $\leq n$ and $\leq m$, respectively. Let's denote by $\pi_{n,m}=\pi_{n,m}(f)$ the Pade approximant to f of order (n, m). It is well known that $\pi_{n,m}$ is a rational function in the class $\mathcal{R}_{n,m}$ which gives the best order of approximation to (1) in this class at the point $z=0$. Another definition of $\pi_{n,m}$: the numerator p and the denominator q are determined so that the following conditions are satisfied:

$$(f.q-p)(z)=A_{n,m}z^{n+m+1}+\text{higher degree terms}; \tag{2}$$

$A_{n,m}$ is a constant. Both definitions are equivalent (see [1; 2]). The function $\pi_{n,m}$ always exists and is unique in spite of the fact that p and q are not. We set $\pi_{n,m}=P_{n,m}/Q_{n,m}$ where $P_{n,m}$ and $Q_{n,m}$ have no common divisor and $Q_{n,m}$ is monic. The zeros of $Q_{n,m}$ are called free poles of $\pi_{n,m}$. It is well known that if

$$(f-\pi_{n,m})(z)=\tilde{A}_{n,m}z^{n+m+1-k} + \ldots \text{ with } k>0,\ \tilde{A}_{n,m}=\text{Const}\neq 0,$$

then $\pi_{n,m} \in \mathcal{R}_{n-k,m-k}$. If $\pi_{n,m}$ gives an order of approximation to (1) at $z=0$ not less than $n+m+1$, then (2) holds with $p=P_{n,m}$ and $q=Q_{n,m}$. If r is a rational function in the class $\mathcal{R}_{n,m}$ which interpolates f at the point $z=0$ not less than $n+m+1$ times then $r \equiv \pi_{n,m}$.

If we order the Pade approximants in a square scheme, we obtain the Pade-table to (1).

$$\begin{array}{llllll} \pi_{0,0}, & \pi_{1,0}, & \pi_{2,0}, & \pi_{3,0}, & \ldots, & \pi_{n,0}, \ldots \\ \pi_{0,1}, & \pi_{1,1}, & \pi_{2,1}, & \pi_{3,1}, & \ldots, & \pi_{n,1}, \ldots \\ \ldots & & & & & \\ \pi_{0,m}, & \pi_{1,m}, & \pi_{2,m}, & \pi_{3,m}, & \ldots, & \pi_{n,m}, \ldots \\ \ldots & & & & & \\ \pi_{0,n}, & \pi_{1,n}, & \pi_{2,n}, & \pi_{3,n}, & \ldots, & \pi_{n,n}, \ldots \\ \ldots & & & & & \end{array}$$

It is well known that the Pade-table is infinite, if and only if (1) doesn't represent a rational function. If $f=p_l/p_k$, $p_k \neq 0$, $\deg p_k=k$, $\deg p_l=l$, then outside the rectangle with corners at $(0, 0)$, $(0, l)$, (l, k) and $(k, 0)$ there appears, in the Pade-table, only the function f.

The reader can find detailed explanations of all the questions connected with the existence and uniqueness of the Pade approximants and with the structure of the Pade-table in the monography of Perron [1] and in the one of Baker [2].

Suppose, now, the power series (1) can be extended as a function, holomorphic in the unit disk $D(f \in H(D))$.

In this paper we shall consider the uniform convergence of sequences of Pade approximants to f on compact subsets in D. To investigate this problem, it is enough to discuss two cases: sequences $\{\pi_{n,m}\}$ with a fixed number m of the free poles, $m \in \mathbf{N}$ (or sequences $\{\pi_{n,m_n}\}$ where m_n goes to infinity very slowly in comparison to the degree n of the numerator: $m_n/n \to 0$ as $n \to \infty$), and sequences $\{\pi_{n,n}\}$ in the diagonal in the Pade-table of f (or sequences, very closed to the diagonal: $\{\pi_{n1,n2}\}$, $n_1/n_2 \to 1$ as $n \to \infty$). A short calculation shows that $\pi_{n,m}(f)=1/\pi_{m,n}(1/f)$, so that the problem of convergence of columns in the Pade-table can be reduced to the same one with respect to the rows.

In the case of rows the following theorem is valid (see [3]).

Theorem A. *Let $f \in H(D)$ and $m \in \mathbf{N}$ be fixed. Then the sequence $\{\pi_{n,m}\}$ converges, as $n \to \infty$, in capacity to f on each closed disk $\bar{D}_\rho$, $\rho<1$, $D_\rho=\{z, |z|<\rho\}$.* (For each compact set K in $\mathbf{C}$ the capacity can be defined as (see [4])

$$\operatorname{Cap} K=\lim_{n\to\infty} \min_{h \in p_K} \|h\|_K,$$

where P_K is the class of all polynomials of the form $h(z)=z^K+\ldots$ and $\|x\|_K=\max|(x)(z)|v_K$; remember that convergence in capacity on the compact set K means that for each positive number ε we have $\operatorname{Cap}\{z \in K, |(x)_n(z)|>\epsilon\} \underset{n\to\infty}{\to} 0$).

Denote, now, by D_m the largest disk in which (1) can be extended as a meromorphic function with not more than m poles (multiplicities included). Then the sequence $\{\pi_{n,m}\}$ converges to f, as $n \to \infty$, in capacity on each closed disk D_R, $D_R \subset D_m$. If, further, D_∞ is the largest disk, in which (1) has a meromorphic extension and if $m_n=0(n)$, $n \to \infty$, then $\{\pi_{n,m_n}\}$ converges to f, as $n \to \infty$, in capacity on each compact subset of D_∞.

Both results are announced by Gončar in 1976 (see [3]). They generalize the classical theorem of Montessus de Ballore according the convergence of $\{\pi_{n,m}\}$ in D_m, if f has exactly m poles there.

In both theorems of Gončar the assertions refer to convergence in capacity. There are no statements about the location of the free poles in D (remember that $f\in H(D)$). The behaviour of the free poles of the Pade approximants inside D can be very "incorrect" so that we can't speak about uniform convergence to f in D (on compact subsets). We shall illustrate this by an example. We shall prove the following

Theorem 1. *Let* $\Lambda(n)=\{n_k\}$, $\Lambda(m)=\{m_k\}$, $k\in\mathrm{N}$, *be sequences of positive integers*, $m_k=0(n_k)$ *as* $k\in\mathrm{N}$, $n_k+m_k+m_{k+1}<n_{k+1}$ *for* $k\in\mathrm{N}$. *Then there exists a function* f, $f\in H(D)$, *such that the set of the free poles of* $\{\pi_{n_k}, m_k(f)\}$ *is dense in* D, $k\in\mathrm{N}$.

The proof of this theorem is an application of the method of Rahmanov (see [7]). Suppose $\{a_k\}$, $k\in\mathrm{N}$, is a sequence of complex numbers which is dense in D. Each function g_k, determined by

$$g_k(z)=z^{n_k}a_k^{3m_k}(1-|a_k|)^{m_k}(z-a_k)^{-m_k}=\sum_{\nu=n_k}^{\infty} g_{k,\nu}z^{\nu}$$

is analytic in the disk $D_{|a_k|}=\{z,\ |z|<|a_k|\}$; for each $\nu\geq n_k$ we have

$$g_{k,\nu}=(1/2\pi i)\int_{|t|=|a_k|^2} t^{-\nu-1}g_k(t)dt.$$

We set

$$f(z)=1+\sum_{k=1}^{\infty}\sum_{\nu=n_k}^{n_k+m_k} g_{k,\nu}z^{\nu}=\sum_{\nu=0}^{\infty} f_{\nu}z^{\nu}.$$

It can be verified that this power series defines a function, analytic in D. Indeed, let $\nu\in\mathrm{N}$ be a fixed integer. If $n_k+m_k<\nu<n_{k+1}$ for some k, then $f_\nu=0$. Suppose $n_k\leq\nu\leq n_k+m_k$ for an integer k. Then $f_\nu=g_{k,\nu}$ and there follow the estimates $|f_\nu|\leq|a_k|^{n_k+m_k-k}\leq 1$. Consequently the inequality $\lim\limits_{\nu\in\mathrm{N}}|f_\nu|^{1/\nu}=1$ holds, this means that $f\in H(D)$. On the other hand, we have from the definition of $g_k(z)$ and $f(z)$ that

$$f(z)-\sum_{\nu=0}^{n_{k-1}+m_{k-1}} f_\nu z^\nu - z^{n_k}a_k^{3m_k}(1-|a_k|)^{m_k}(z-a_k)^{-m_k}$$

$$=\sum_{\nu=k}^{\infty}\sum_{l=n_\nu}^{(n_\nu+m_\nu)} g_{\nu,l}\,z^l-\sum_{l=n_k}^{\infty} g_{k,l}\,z^l=\sum_{\nu=k+1}^{\infty}\sum_{l=n_\nu}^{n_\nu+m_\nu} g_{\nu,l}\,z^l-\sum_{l=1+n_k+m_k}^{\infty} g_{k,l}\,z^l.$$

Consequently

$$\pi_{n_k}, m_k(z)=\sum_{\nu=0}^{n_{k-1}+m_{k-1}} f_\nu z^\nu+z^{n_k}a_k^{3m_k}(1-|a_k|)^{m_k}(z-a_k)^{-m_k}.$$

In this way we have constructed a function f, analytic in D, such that the set of the free poles of $\{\pi_{n_k}, m_k\}$, $k\in\mathrm{N}$, is dense in D. In this case the Pade-approximants $\{\pi_{n_k}, m_k\}$, $k\in\mathrm{N}$, don't converge uniformly on compact subsets of D, in other words this sequence doesn't reconstruct the corresponding function in its domain of holomorphy.

But, if it is previously known that all the functions π_{n,m_n}, $m_n=0(n)$, $n\to\infty$, are, for $n\geq n_0$, analytic in D, then the convergence in capacity in the theorem of Gončar means uniform convergence. Let's remind the reader of the following result of Gončar (see [3]):

Theorem B. *If Ω is a domain in $\mathbf{C}$ and if the sequence $\{\varphi_n\}$, $n\to\infty$ of analytic in Ω functions converges in capacity on each compact subset of Ω, then the convergence is uniform (and the limit function φ is analytic in Ω).*

Consequently, if $m\in\mathbf{N}$ is a fixed integer and if for all n, $n\geq n_0$, the functions $\pi_{n,m}$ are analytic in D, the sequence $\{\pi_{n,m}\}$ converges, as $n\to\infty$, uniformly on each closed disk $\bar{D}_\rho$, $\rho<1$.

Consider, now, the question about the uniform convergence of the diagonal $\pi_{n,n}(f)(=\pi_n(f))$, $n\in\mathbf{N}$, in the Pade-table of $f\,(f\in H(D))$. In this case there are no general results, except the theorem of Nutall (see [5]) and the one of Pommerenke (see [6]). Nutall has investigated entire and meromorphic functions, Pommerenke — functions analytic everywhere in the complex domain $\mathbf{C}$ except on a set of zero-capacity. In both cases $\pi_n(f)$ converges to f, as $n\in\mathbf{N}$, in capacity on each compact set in $\mathbf{C}$.

We can indicate functions, entire in $\mathbf{C}$, such that the sequence of the corresponding Pade approximants π_n, $n\to\infty$, converges uniformly on each closed disk $\bar{D}_R$, $R>0$. For instance, the function $f(z)=e^z$ has this property (see [1]).

As in the case of rows, the essence in the theorems of Nutall and Pommerenke lies in the convergence in capacity. Wallin (see [11]) has constructed an entire in $\mathbf{C}$ function for which there exists a sequence $\Lambda\subset\mathbf{N}$ and a compact set e such that $\pi_n(z)\to\infty$, as $n\in\Lambda$, for each $z\in e$. We shall prove the following

Theorem 2. *Let $\Lambda=\{n_k\}_{k=1}^\infty$ be a sequence of positive integers, $n_k>2n_{k-1}$, $n_k/n_{k+1}\to 0$, $k\in\mathbf{N}$, $n_0=0$. Then there exists an entire function f such that the set of the free poles of $\{\pi_{n_k}(f)\}_{n_k\in\Lambda}$ is dense in $\mathbf{C}$.*

Let $\{a_k\}_{k=1}^\infty$ be a sequence of complex numbers. We assume that $\{a_k\}$ is dense in $\mathbf{C}$ and that $\lim\limits_{k\to\infty}(1+|a_k|)^{1/n_k}\to 1$ as $k\in\mathbf{N}$. Each function, defined by

$$g_k(z)=2^{-n_k^2}(z-a_k)^{2n_{k-1}-n_k}z^{n_k}$$

is analytic in the disk $D_{|a_k|}$. For each $z\in D_{|a_k|}$ we have the expansion:

$$g_k(z)=\sum_{\nu=n_k}^{\infty} g_{k,\nu}z^\nu,$$

where

$$g_{k,\nu}=(1/2\pi i)\int_{|t|=1+|a_k|} t^{-\nu-1}g_k(t)dt.$$

We set

$$f(z)=1+\sum_{\nu=1}^{\infty}\sum_{l=n_\nu}^{2n_\nu} g_{\nu,l}\,z^l=\sum_{k=0}^{\infty} f_k z^k.$$

We shall estimate $\lim\limits_{\nu\to\infty}|f|^{1/\nu}$. Let ν be a fixed integer. If $2n_k<\nu<n_{k+1}$, then $f_\nu=0$. Suppose that $n_k\leq\nu\leq 2n_k$ for a integer k. Then

$$|f_\nu|^{1/\nu}\leq\left(\frac{(1+|a_k|)^{1/n_k}}{2}\right)^{n_k^2/n_k}\cdot\frac{1}{(1+|a_k|)^{2n_{k-1}/\nu}}<\left(\frac{(1+|a_k|)^{1/n_k}}{2}\right)^{n_k}.$$

This gives $\overline{\lim}_{\nu\to\infty}|f_\nu|^{1/\nu}=0$; consequently, f is an entire function.

We obtain further, for $k=0, 1, \ldots$ that

$$\pi_{n_k}(z)=\sum_{\nu=0}^{2n_{k-1}} f_\nu z^\nu + z^{n_k} 2^{-n_k^2}(z-a_k)^{2n_{k-1}-n_k}.$$

Really, $\{\Sigma_{\nu=0}^{2n_{k-1}} f_\nu z^\nu + z^{n_k} 2^{-n_k^2}(z-a_k)^{2n_{k-1}-n_k}\} \in R_{n_k}, m_k$ and

$$f(z)-\sum_{\nu=0}^{2n_{k-1}} f_\nu z^\nu - z^n k 2^{-n_k^2}(z-a_k)^{2n_{k-1}-n_k} = \sum_{\nu=k}^{\infty}\sum_{l=n_\nu}^{2n_\nu} g_{\nu,l}z^l -$$

$$-\sum_{l=n_k}^{\infty} g_{k,l}z^l = \sum_{\nu=k+1}^{\infty}\sum_{l=n_\nu}^{2n_\nu} g_{\nu,l}z^l - \sum_{l=2n_k+1}^{\infty} g_{k,l}z^l.$$

Let's return to the case of rows, $n\to\infty$, $m\in \mathbf{N}$ is fixed. If there is a sequence Λ, $\Lambda\subset \mathbf{N}$, such that $\pi_{n,m}\in H(D)$ for $n\in\Lambda$, then $\{\pi_{n,m}\}$ converges to f, as $n\in\Lambda$, uniformly on each compact subset of D. Consider, now, the same question in respect to the diagonal in the Pade-table: if $\pi_n\in H(D)$ for $n\in\Lambda$, $\Lambda\subset\mathbf{N}$, can we assert that $\pi_n\to f$, as $n\in\Lambda$, uniformly on each closed disk $\overline{D}_\rho$, $\rho<1$. Apart from the special case of entire and meromorphic functions (theorems of Nutall and Pommerenke) there aren't any general results in this connection. In 1973 Zinn-Justin proved that in the upper conditions π_n converges to f, as $n\to\infty$, on each closed disk $\overline{D}_\rho$ with $\rho<1/\sqrt{3}$ (see [8]). This result follows from the estimate of the interpolating formula of Hermite (we assume that $\deg Q_n=n$ and $f\in H(\overline{D})$)

$$\|f-\pi_n\|_{\overline{D}_\rho}^{1/n}=(1/2\pi)\|\int_{|t|=1}\frac{Q_n(t)}{Q_n(z)}\left(\frac{z}{t}\right)^{2n+1}\frac{T_n(t)}{T_n(z)}\;\frac{f(t)}{t-z}\|_{\overline{D}_\rho}^{1/n}$$

$$\leq C\| z^{2n+1}/Q_n(z)T_n(z)\|_{\overline{D}_\rho}/\min|t^{2n+1}/Q_n(t)T_n(t)|_{|t|=1};$$

T_n is an arbitrary polynomial of degree $\leq n$; C is a constant, $C=C(\rho)$. If we choose $T_n(z)=\Pi_{k=1}^n(z,\overline{\alpha_{n,k}}+1)$, where $\alpha_{n,k}$, $k=1,\ldots,n$ are the zeros of Q_n, we obtain that $\overline{\lim}_{n\to\infty}\|f-\pi_n\|_{\overline{D}_\rho}^{1/n}<1$ for each $\rho<1\sqrt{3}$. Gončar has investigated the upper estimate in the right side of the last inequality: there follows from his results in this connection that $\pi_n\rightrightarrows f$, as $n\to\infty$, on each closed disk $\overline{D}_\rho$ with $\rho<1/1,587\ldots$ (see [12]).

We can easily name functions, analytic in D, for which the corresponding Pade approximants π_n reconstruct them (in the sense that $\{\pi_n\}$ converges, $n\to\infty$, uniformly on compact subsets) in the whole domain of holomorphy. Let's remember, for instance, the functions of Markoff's type: let $\mu'(t)>0$ on the real segment $\Delta=[-1,1]$, the function $\widehat{\mu}(\varsigma)=\int_\Delta \mu(t)dt/(\varsigma-t)$ is analytic everywhere in $\mathbf{C}$ apart from Δ. We set $\widehat{\mu}(1/z)=f(z)$; it is well known (see for instance [10]) that $\pi_n\rightrightarrows f$, $n\to\infty$ uniformly (and even geometrically) on each compact set in $\mathbf{C}-(-\infty,-1]\cup[1,\infty)$.

It is more difficult to point at a function f, $f\in H(D)$, for which there is a sequence Λ, $\Lambda\subset\mathbf{N}$, such that the functions $\pi_n(f)$ don't reconstruct f, as $n\in\Lambda$, in the whole unit disk D. Rahmanov has given such an example. He has constructed a function f, $f \in H(D)$, for which there is $\Lambda \subset \mathbf{N}$ such that $\pi_n \in H(D)$, $n \in \Lambda$ $\pi_n \rightrightarrows f$, as $n \in \Lambda$, only on each closed disk $\bar{D}_\rho$ with $\rho < 0{,}8$ (see [7]).

Now we shall construct another example function g with the indicated properties. We shall prove the following

Theorem 3. *Let $\Lambda = \{n_k\}$ be a sequence of positive integers, $n_0 = 0$, $n_k > 2n_{k-1}$, $k = 1, 2 \ldots$, and $n_{k-1}/n_k \to 0$ as $k \to \infty$. Then there exists a function g, $g \in H(D)$, such that $\pi_n(g) \in H(D)$, $n \in \Lambda$, and $\pi_n \to \infty$ as $n \in \Lambda$, uniformly on each compact subset of the domain $\Omega = D \cap \{z, |z|^2 > 2e|z-1|\}$.*

Consequently, in this case the sequence $\pi_n (= \pi_n(g))$, $n \in \Lambda$, converges to g uniformly only on each disk $\bar{D}_\rho$ with $\rho < \sqrt{2e + e^2} - 2e = 0{,}85 \ldots$

The scheme of the construction of the function g is an application of a method of Wallin (see [11]).

Before beginning the proof, we formulate two lemmas.

Lemma 1. *Let $\varphi(z) = \Sigma_{\nu=0}^{\infty} \varphi_\nu z^\nu$ be a formal power sereis; for each $n \in \mathbf{N}$ we set*

$$[\varphi(z)]_{(n)} = \sum_{\nu \leq n} \varphi_\nu z^\nu, \quad (\varphi(z))_{(n)} = \sum_{\nu \geq n} \varphi_\nu z^\nu.$$

Then there holds the representation $\pi_n(\varphi)(z) = \pi_n([\varphi(z)]_{(2n)})$.

The proof of this lemma follows immediately from the definitions of Pade approximants (see [5]).

Assume, now, that $(\varphi - \pi_n(\varphi))(z) = 0(z^{2n+1})$. Then Lemma 1 gives

$$[\varphi(z)]_{(2n)} - \pi_n(\varphi)(z) = ([\varphi(z)]_{(2n)} Q_n(z))_{(2n+1)} / Q_n(z)$$

and

$$(\varphi - \pi_n(\varphi))(z) = ([\varphi(z)]_{(2n)} Q_n(z))_{(2n+1)} / Q_n(z) + (\varphi(z))_{(2n+1)}.$$

Lemma 2. *Let $n \in \mathbf{N}$. Then the system of equations*

$$x_0 = \text{Const} \neq 0, \quad x_1\binom{n}{0} + x_0\binom{n}{1} = 0$$

$$x_2\binom{n}{0} + x_1\binom{n}{1} + x_0\binom{n}{2} = 0$$

$$\vdots$$

$$x_n\binom{n}{0} + x_{n-1}\binom{n}{1} + \ldots + x_0\binom{n}{n} = 0$$

$$\ldots\ldots\ldots\ldots\ldots$$

has unique set of solutions $x_\nu = (-1)^\nu \binom{n+\nu-1}{\nu}$, $\nu = 1, 2, \ldots$

The proof of this lemma has been given in [11].

Let, now, $\Lambda = \{n_k\}$ be a sequence of integers, determined as in Theorem 3. We shall construct a function g with the required in the theorem properties.

We set

$$g(z) = \sum_{\nu=0}^{\infty} g_\nu z^\nu.$$

Let's fix an integer k. We shall describe the k-th step in the construction of g. Assume, the first $1+2n_{k-1}$ coefficients are already chosen, $g_0=1$, and assume that they satisfy the conditions

(3) $$|g_\nu|^{1/\nu}\leq 1, \quad \nu=0,1,\ldots, \quad 2n_{k-1}.$$

We shall require that (3) holds also for $g_{2n_{k-1}+1},\ldots, g_{2n_k}$.

We set, later, $\pi_{n_k}=P_{n_k}/Q_{n_k}$ and

$$P_{n_k}(z)=a_{k,0}z^{n_k}+\ \ldots\ +a_{k,n_k}$$
$$Q_{n_k}(z)=b_{k,0}z^{n_k}+\ \ldots\ +b_{k,n_k}.$$

We want, also, that

(4) $$(g-\pi_{nk})(z)=0(z^{2n_k+1}).$$

The last requirement gives (see (2)) the following system:

(5) $$\begin{aligned} &b_{k,n_k}=a_{k,nk}\\ &g_1b_{k,n_k}+b_{k,n_k-1}=a_{k,n_k-1}\\ &\vdots\\ &g_{n_k}b_{k,n_k}+g_{n_k-1}b_{k,n_k-1}+\ \ldots\ +b_{k,0}=a_{k,0}\end{aligned}$$

and

(6) $$\begin{aligned} &g_{n_k+1}\cdot b_{k,n_k}+\ \ldots\ +g_{2n_{k-1}+1}\cdot b_{k,2n_{k-1}}+\ \ldots\ +g_0b_{k,0}=0\\ &\vdots\\ &g_{2n_k}b_{k,n_k}+\ \ldots\ +g_{n_k+2n_{k-1}}\cdot b_{k,2n_{k-1}}+\ \ldots\ +g_{n_k}b_{k,0}=0.\end{aligned}$$

The system (6) consists of n_k homogeneous equations, the coefficients $g_{2n_{k-1}+1},\ldots, g_{2n_k}$ are unknown, we shall choose $b_{k,0},\ldots, b_{k,n_k}$ in a proper way.

Because of Lemma 2 we have $\pi_{n_k}=\pi_{n_k}([g]_{(2n_k)})$ and

(7) $$[g(z)]_{(2n_k)}-\pi_{n_k}(z)=(Q_{n_k}(z))\sum_{\nu=0}^{2n_k}g_\nu z^\nu)_{(2n_k+1)}/Q_{n_k}(z).$$

The calculation of $(Q_{n_k}(z)[g(z)]_{(2n_k)})_{(2n_k+1)}$ gives

$$((\sum_{\nu=0}^{2n_k}g_\nu z^\nu)(\sum_{l=0}^{n_k}b_{k,l}\,z^{n_k-l}))_{(2n_k+1)}=b_{k,0}z^{n_k}\sum_{\nu=n_k+1}^{2n_k}g_\nu z^\nu$$
$$+b_{k,1}z^{n_k-1}\sum_{\nu=n_k+2}^{2n_k}g_\nu z^\nu+\ \ldots\ +b_{k,n_k-1}g_{2n_k}z^{2n_k}.$$

We set $b_{k,0}=\ \ldots\ =b_{k,2n_{k-1}-1}=0$ and $g_{n_k+2n_{k-1}+2}=\ \ldots=g_{2n_k}=0.$

With this choice (7) gets a simple form, namely

(8) $$([g(z)]_{2n_k})\cdot Q_{n_k}(z))_{(2n_k+1)}/Q_{n_k}(z)=g_{n_k+2n_{k-1}+1}z^{2n_k+1}/Q_{n_k}(z).$$

Following the schema of Wallin, we set $Q_{n_k}(z)=(z-1)^{n_k-2n_{k-1}}$. Because of Lemma 2, we get the following set of solutions:

(9) $$g_{n_k+2n_{k-1}+1-\nu}=g_{n_k+2n_{k-1}+1}\binom{n_k-2n_{k-1}+\nu-1}{\nu},\quad \nu=1,\ldots,n_k.$$

Now we shall choose $g_{n_k+2n_{k-1}+1}$ in the way that (3) holds also for $g_{2n_{k-1}+1},\ldots,g_{n_k+2n_{k-1}+1}$.

For $\nu=1,\ldots,n_k$ we have the following estimates:

$$\binom{n_k-2n_{k-1}+\nu-2}{\nu}=\frac{(n_k-2n_{k-1}+\nu-1)\ldots(n_k-2n_{k-1})}{\nu!}<\frac{(2n_k-2n_{k-1})^\nu}{\nu}$$

$$<2^{n_k}\left(\frac{n_k-n_{k-1}}{\nu}\right)^\nu\frac{\nu}{\nu!}<2^{n_k}\left(\frac{e(n_k-n_{k-1})}{\nu}\right)^\nu.$$

In order to estimate the last expressions we have to remember that for each $s\in\mathbf{N}$ the function $\varphi_s(x)=(es/x)^x$, $x\geq 0$, has a maximum at the point $x=s$. Consequently

$$\binom{n_k-2n_{k-1}+\nu-1}{\nu}\leq 2^{n_k}e^{n_k-n_{k-1}}.$$

Combining (9) and the last inequality, we obtain

(10) $$|g_{2n_{k-1}+\nu}|\leq|g_{n_k+2n_{k-1}+1}|2^{n_k}.e^{n_k-n_{k-1}},\ \nu=1,\ldots,n_k.$$

Now we choose $g_{n_k+2n_{k-1}+1}$ so that $g_{n_k+2n_{k-1}+1}=(2e)^{-n_k}$. Because of this choice we get that (3) holds also for $g_{2n_{k-1}+1},\ldots,g_{2n_k}$.

Since $Q_{n_k}(1)=0$ the Padé approximant to $[g]_{(n_k)}$ of order n_k will be equal to $\pi_{n_k}=P_{n_k}/Q_{n_k}$ (see (5) and (6)), if $P_{n_k}(1)\neq 0$. Let's turn to the system (5). For $P_{n_k}(1)$ we get the following expression

$$P_{n_k}(1)=\sum_{\nu=0}^{2n_{k-1}}g_\nu\Big(\sum_{l=2n_{k-1}}^{n_k}b_{k,l}\Big)+\sum_{\nu=2n_{k-1}+1}^{n_k}g_\nu\Big(\sum_{l=\nu}^{n_k}b_{k,l}\Big)$$

(we note that $b_{k,0}=\ldots=b_{k,2n_{k-1}-1}=0$, $b_{k,l}=(-1)^l(n_k-2n_{k-1})$, $l=2n_{k-1},\ldots,n_k$ ...

The calculation shows that $P_{n_k}(1)$ depends on $g_{n_k+2n_{k-1}+1}$ as on a multiplicative factor; since $g_{n_k+2n_{k-1}+1}\neq 0$, we get that $P_{n_k}(1)\neq 0$. Consequently $\pi_{n_k}=P_{n_k}/Q_{n_k}$ where $Q_{n_k}(z)=(z-1)^{n_k-2n_{k-1}}$ and the coefficients $a_{k,0},\ldots,a_{k,n_k}$ are determined by system (5).

By this the kth step in the construction of the function g is finished.

Let's summarise: we have chosen the polynomial $Q_{n_k}(z)=(z-1)^{n_k-2n_{k-1}}$ and the coefficients $g_{n_k}+2_{n_{k-1}}+1,\ldots,g_{2n_k}$, ($g_{2n}=\ldots=g_{n_k+2n_{k-1}}+2=0$, $=g_{n_k+2n_{k-1}+1}=(2e)^{-n_k}$, so that $\pi_{n_k}=P_{n_k}/Q_{n_k}$, where P_{n_k} is defined by system (5), and there holds the inequality (see (10))

(11) $$|g_{2n_{k-1}+\nu}|\leq 1,\ \nu=1,\ldots,2n_k-2n_{k-1}.$$

There holds also the representation (8).

Combining (11) and (3), we get $|g_\nu|^{1/\nu}\leq 1$, $\nu=0,1,\ldots,2n_k$. But k is an arbitrary integer, $k\geq 1$. Consequently the function g determined by

$$g(z)=1+\sum_{k=1}^{\infty}\sum_{\nu=2n_{k-1}+1}^{2n_k} g_\nu z^\nu$$

is analytic in D and for each $n_k \in \Lambda$ there holds the representation

$$\Big|\Big(\sum_{\nu=0}^{2n_k} g_\nu z^\nu Q_{n_k}(z)\Big)_{(2n_k+1)}\Big| = |z|^{2n_k+1}/(2e)^{n_k}|Q_{n_k}(z)|.$$

To finish the proof of the theorem we have to estimate $|(g-\pi_{n_k}(z)|$. For each $z\in D$ we have

$$|(g-\pi_{n_k})(z)| \geq \Big|\Big(\sum_{\nu=0}^{2n_k} g_\nu z^\nu (z-1)^{n_k-2n_k-1}\Big)_{(2n_k+1)}\Big| - |(g(z))_{(2n_k+1)}|$$

$$\geq |z^{2n_k+1}/(2e)^{n_k}(z-1)^{n_k-2n_k-1}| - 0(|z|^{2n_k+1}).$$

Let K be a compact subset of the domain $\Omega=\{z,\ |z|^2>2e|z-1|\}\cap D$. For each $z\in\Omega$ we have

$$\varliminf_{n_k\in\Lambda} |z^{2n_k}(2e)^{-n_k}(z-1)^{2n_k-1-n_k}|^{1/n_k} \geq C>1$$

(we remember that $n_{k-1}/n_k\to 0$ as $k\to\infty$), C is a constant which depends only on K. This gives

$$\varliminf_{n_k\in\Lambda} \min_{z\in K} |(g-\pi_{n_k})(z)|^{1/n_k}>1$$

consequently, $\pi_{n_k}\to\infty$ as $k\to\infty$, uniformly on the compact set $K\subset\Omega$.

By this we have finished the proof of Theorem 3.

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