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WAVE FRONTS OF SOLUTIONS TO BOUNDARY PROBLEMS FOR SYMMETRIC DISSIPATIVE SYSTEMS

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The work studies the wave fronts of solutions for first order symmetric strictly hyperbolic systems with dissipative boundary conditions. Using a suitable modification of the thechniques, developed by V. Ivrii, we obtain a variant of his well-known results on the propagation of wave fronts. Moreover, the finite speed of propagation of singularities is obtained for a large class of boundary problems, used in the scattering theory.

1. Introduction. In this work we study the singularities of solutions to the boundary problems for first order symmetric systems with dissipative boundary conditions. Our approach was suggested by the ideas, developed by V. Ivrii in [1]. Applying the main theorem in [1], treating the systems with strictly dissipative boundary conditions, V. Petkov obtains in [2] the finite speed of propagation of singularities for the transmission problem connected with the wave equation. In order to cover general dissipative boundary conditions, Ivrii used an estimate in suitable Sobolev spaces. The check of this estimate becomes complicated for the dissipative boundary conditions, which appear in the scattering theory for systems (see [2; 4]). Our goal is to obtain a result, similar to that given in [1], and applying this result to conclude that the speed of propagation of singularities for the systems, used in [2; 4] is finite.

We shall examine the singularities of solutions for first order systems, having the form

$$P(u)=(AD_{x_1}+Q)(u)=0 \tag{1.1}$$

in the domain $\mathbb{R}^+\times\mathbb{R}^{n-1}=\{x=(x_1, x'); x>0, x'\in\mathbb{R}^{n-1}\}$. Here $A=A(x_1, x', D_{x'})$ and $Q=Q(x_1, x', D_{x'})$ are $(d\times d)$ matrix-valued pseudo-differential operators of orders 0 and 1 respectively, smoothly depending on $x_1\in\overline{\mathbb{R}^+}=\{s\in\mathbb{R}; s\geqq 0\}$.

Throughout this work the space of pseudo-differential operators of order k with respect to x', depending smoothly on $x_1\in\overline{\mathbb{R}^+}$, is denoted by $\mathsf{L}^{k'}$. The principle symbol of the operator $b\in\mathsf{L}^{k'}$ is denoted by $b_k(\rho)$ with $\rho=(x_1, x', \xi')\in\overline{\mathbb{R}^+}\times\{T^*(\mathbb{R}^{n-1})\setminus 0\}$. We consider the following boundary condition:

$$(u|_{x_1=0})\in \operatorname{Ker} B(x', D_{x'}), \tag{1.2}$$

where $B=B(x', D_{x'})$ is a $(d\times d)$ matrix-valued pseudo-differential operator of order 0.

Let $(x'^*, \xi'^*)\in T^*(\mathbb{R}^{n-1})\setminus 0$ be fixed. We shall examine the singularities of solutions to the system, determined by the equation (1.1) and the boundary condition (1.2) in a small conic neighbourhood W of $\rho^*=(0, x'^*, \xi'^*)$ in $\overline{\mathbb{R}^+}\times\{T^*(\mathbb{R}^{n-1})\setminus 0\}$. More precisely, we shall investigate the singularities of a distribution $u(x_1, x')$, provided the following conditions are fulfilled:

$$(1.3)\qquad \begin{cases} \text{(a)} & \rho^* \notin WF'(P(u)), \\ \text{(b)} & (x'^*, \xi'^*) \notin WF(B(u(0, x'))), \\ \text{(c)} & WF'(u) \cap \{\varphi < 0\} \cap W = \emptyset. \end{cases}$$

Here $\varphi(x_1, x', \xi') \in C^\infty(W)$ is a real function, homogeneous of degree 0 with respect to ξ'. Recall the definition of wave front sets $WF'(u)$, used in [1], for the distribution

$$u(x_1, x') \in C(\overline{\mathbf{R}^+}, \mathcal{D}'(\mathbf{R}^{n-1})).$$

Definition. *Let $\rho = (x_1, x', \xi') \in W$. We say that $\rho \notin WF'(u)$ if there exists an operator $a \in \mathbf{L}^0$, such that $a_0(\rho) \neq 0$ and $a(u) \in C^\infty(\overline{\mathbf{R}^+} \times \mathbf{R}^{n-1})$.*

Our assumptions are close to those, given in [1]. First we set the following conditions:

$$(\mathrm{S})\qquad \begin{cases} A_0(\rho) = A_0^*(\rho),\ Q_1(\rho) = Q_1^*(\rho), \text{ where } \rho \in W \text{ and} \\ a^* \text{ denotes the complex adjoint matrix of } a. \end{cases}$$

$$(\mathrm{E})\qquad \begin{cases} \text{For } \rho \in W \text{ the matrix } A_0(\rho) \text{ is invertible and} \\ \qquad \operatorname{sgn}(A_0(\rho)) = 0. \end{cases}$$

It follows from (E) that the number d is even.

$$(\mathrm{Dm})\qquad \begin{cases} \text{(a)} & \dim(\operatorname{Ker}(B_0(\rho))) = d/2 \text{ for } \rho \in W \cap \{x_1 = 0\}, \\ \text{(b)} & \langle A_0(\rho)v, v\rangle \leq 0 \text{ for } v \in \operatorname{Ker}(B_0(\rho)),\ \rho \in W \cap \{x_1 = 0\}. \end{cases}$$

Here $\langle\ ,\ \rangle$ is the scalar product in $\mathbf{C}^d$. The boundary conditions with the property (Dm) are called maximally dissipative ones.

Our last assumption is connected with the function φ and the constant matrix $\tilde{A} = A_0(\rho^*)$:

$$(\mathrm{A})\qquad \begin{array}{ll} \text{(a)} & \varphi(\rho^*) = 0, \\ \text{(b)} & \operatorname{Re}(\langle\{\tilde{A}\xi_1 + Q_1(\rho), \varphi\} v, v\rangle(\rho^*)) > 0, \end{array}$$

where $v \in \mathbf{C}^d$ and $\{a, b\}$ is the Poisson bracket of the smooth functions $a = a(x, \xi)$, $b = b(x, \xi)$.

Our assumption (A) is stronger than that, given in [1], where the inequality (A) (b) is required to be fulfilled only for $v \in \operatorname{Ker}(\tilde{A}\xi_1 + Q_1(\rho^*))$. On the other hand, in [1] the following estimate is assumed:

$$(1.4)\quad |(I - \pi)u|_0 \leq \delta |u|_0 + C_\delta(|Pu|_{\sigma-1/2,L} + \|u\|_{Y,\sigma-1/2} + |Bu|_{Y,\sigma} + |u|_{\sigma-1/2} + |\mu_\delta u|_\delta),$$

where π is the projection on $\operatorname{Ker}(\tilde{A}\xi_1 + Q_1(\rho^*))$, $|\cdot|_0$ is the norm in the Hilbert space $[L_2(\mathbf{R}^+ \times \mathbf{R}^{n-1})]^d$ (for more details see Lemma 3.1 in [1]). The estimate (1.4) plays a crucial role in the investigation of singularities of the distribution $u(x_1, x')$ with the properties (1.3). As we mentioned above the verification of this estimate makes some troubles for non-strictly dissipative boundary conditions. The assumption (A) enables us to avoid the application of the estimate (1.4) in the scheme, proposed by Ivrii.

Our main result is the following

Theorem 1. *Suppose the assumptions* (S), (E), (Dm) *and* (A) are *fulfilled. Let the distribution* $u(x_1, x') \in C(\overline{R^+}, \mathscr{D}'(R^{n-1}))$ *satisfy the conditions* (1.3). *Then we have*

$$\rho^* \notin WF'(u). \tag{1.5}$$

In Section 2 we prove two preliminary lemmas, needed for the proof of Theorem 1. In Section 3 following the techniques, used by Ivrii, we obtain Theorem 1. In Section 4 we discuss the finite speed of propagation of singularities.

In conclusion we would like to thank V. Petkov for the help and abvices in the preparation of the work.

2. Preliminaries. The scalar product in the Sobolev space

$$H^s = H^s(R^{n-1}) \times \cdots \times H^s(R^{n-1}) \ (d \text{ times})$$

will be denoted by $(\ ,\ )_{s,0}$. The scalar product in the Hilbert space $\mathscr{H}^s = L_2(R^+, H^s)$ will be denoted by $(\ ,\ )_s$. Set $\|v\|_{s,0} = ((v, v)_{s,0})^{1/2}$ for $v \in H^s$ and $\|w\|_s = ((w, w)_s)^{1/2}$ for $w \in \mathscr{H}^s$. In the proof of Theorem 1 we shall need the following

Lemma 2.1. *Let the assumptions* (S) *and* (A) *be fulfilled. Given a number* $\varkappa > 0$ *consider the operator*

$$P_\varkappa = \exp(-\varkappa\psi)(\tilde{A}D_{x_1} + Q)\exp(\varkappa.\psi),$$

where $\tilde{A} = A_0(\rho^*)$, $\psi \in L^0$ *and* $\psi_0(\rho) = \varphi(\rho) + (x - x^*)^2 + (\xi'/|\xi'| - \xi'^*/|\xi'^*|)^2$. *Then for some constants* $a_0 > 0$, $b_0 > 0$, *independent of* $\varkappa$, *the inequality*

$$(a_0\varkappa - b_0)\|v\|_s^2 \leq \operatorname{Re}(i(P_\varkappa v, v)_s) + C\varkappa(\|\omega v\|_s^2 + \|v\|_{s-1/2}^2) + 1/2(\tilde{A}v, v)_{s,0} \tag{2.1}$$

holds for an operator $\omega \in L^0$ *with* $\rho^* \notin \operatorname{cone\,supp}(\omega)$ *and* $v \in \mathscr{H}^{s+1}$, $D_{x_1}(v) \in \mathscr{H}^{s+1}$.

Proof. The principle symbol of the operator $i[\tilde{A}D_{x_1} + Q, \ \psi]$ is $\{\tilde{A}\xi_1 + Q_1(\rho), \ \psi_0(\rho)\}$. Here $[S, T] = ST - TS$ is the commutator of the operators S and T. Applying the assumption (A), we can take the conic neighbourhood W so small that for some $a_0 > 0$ the inequality

$$\operatorname{Re}(\langle\{\tilde{A}\xi_1 + Q_1(\rho), \ \psi_0(\rho)\} w, w\rangle) \geq 2a_0\langle w, w\rangle$$

holds for $\rho \in W$, $w \in C^d$. Applying the sharp Garding's inequality for pseudodifferential systems, we are going to

$$a_0\|v\|_s^2 \leq \operatorname{Re}(i([\tilde{A}D_{x_1} + Q, \ \psi]\, v, v)_s) + C(\|\omega v\|_s^2 + \|v\|_{s-1/2}^2). \tag{2.2}$$

From the equality

$$P_\varkappa \equiv AD_{x_1} + Q + \varkappa[AD_{x_1} + Q, \ \psi] \quad (\operatorname{mod} L^{-1'})$$

we get

$$\operatorname{Re}(i(P_\varkappa v, v)_s) \geq \operatorname{Re}(i(Pv, v)_s) + a_0\varkappa\|v\|_s^2 - c_\varkappa(\|\omega v\|_s^2 + \|v\|_{s-1/2}^2). \tag{2.3}$$

On the other hand, the assumption (S) implies the following inequalities:

$$\operatorname{Re}(i(Qv, v)_s) \geq -b_0\|v\|_s^2, \tag{2.4}$$

$$\operatorname{Re}(i(\tilde{A}D_{x_1}v, v)_s) \geqq (-1/2)(\tilde{A}v, v)_{s,0}. \tag{2.5}$$

Now we bring together (2.2) – (2.5) and obtain the inequality (2.1). This proves the lemma.

A suitable decomposition of the unity $(d \times d)$ matrix I is obtained in the following

Lemma 2.2. *Consider $(d \times d)$ matrix $B(\rho)$, depending smoothly on the parameter ρ. Suppose that the number d is even and the matrix $B(\rho)$ has constant rank $d/2$. Then we can find invertible $(d \times d)$ matrix $\beta(\rho)$, smoothly depending on ρ, such that the equality*

$$I = \pi(\rho) + \beta(\rho)B(\rho) \tag{2.6}$$

holds. Here $\pi(\rho)$ is the orthogonal projection on the linear space $\operatorname{Ker}(B(\rho))$.

Proof. Let $[\operatorname{Ker}(B(\rho))]^\perp$ be the orthogonal complement of the linear space $\operatorname{Ker}(B(\rho))$. Since the linear map $B(\rho): [\operatorname{Ker}(B(\rho))]^\perp \to \operatorname{Im}(B(\rho))$ is isomorphism, we can determine the operator

$$\beta(\rho): \operatorname{Im}(B(\rho)) \to [\operatorname{Ker}(B(\rho))]^\perp,$$

such that

$$\beta(\rho)B(\rho) = I \quad \text{on} \quad [\operatorname{Ker}(B(\rho))]^\perp. \tag{2.7}$$

Since the matrix $B(\rho)$ has constant rank $d/2$, we can find a smooth basis in the linear space $[\operatorname{Ker}(B(\rho))]^\perp$. Respectively, $\operatorname{Im}(B(\rho))$ has a smooth basis. Consequently, the operator $\beta(\rho)$ smoothly depends on ρ. Let us extend smoothly the operator $\beta(\rho)$ as invertible operator on $\mathbf{C}^d$ with the property (2.7).

Given any $v \in \mathbf{C}^d$, applying the fact that $\pi(\rho)$ is the orthogonal projection on $\operatorname{Ker}(B(\rho))$, we obtain the following properties

$$w = (v - \pi(\rho)v) \in [\operatorname{Ker}(B(\rho))]^\perp, \quad B(\rho)w = B(\rho)v.$$

Applying these properties and (2.7), we obtain the equality $v - \pi(\rho)v = \beta(\rho)B(\rho)v$. This proves the Lemma.

3. Proof of Theorem 1. We can assume that the distribution $u(x_1, x')$ has a compact support, since our analysis is in some small conic neighbourhood of ρ^*. Moreover, we can find a number s, such that $a(u) \in \mathscr{H}^{s-1/2}$ for some $a \in \mathsf{L}^{0\diagdown}$ with $a_0(\rho^*) \neq 0$ and cone $\operatorname{supp}(a)$ sufficiently close to ρ^*. Using the assumption $\rho^* \notin WF'((AD_{x_1} + Q)u)$ and (E), we obtain

$$D^j_{x_1}(a(u)) \in \mathscr{H}^{s-1/2-j} \quad \text{for} \quad j = 1, 2, \ldots \tag{3.1}$$

On the other hand, without loss of generality we can assume that $A_0(\rho)$ is a constant $(d \times d)$ matrix. Indeed, we have the following equality

$$A_0(\rho) = A_0(\rho^*)(I + \Delta(\rho)) = (I + \tilde{\Delta}(\rho))A_0(\rho^*),$$

where $I + \Delta(\rho)$ and $I + \tilde{\Delta}(\rho)$ are positive matrices and $A_0(\rho^*)\Delta(\rho) = \tilde{\Delta}(\rho)A_0(\rho^*)$. Then we are going to

$$A_0(\rho) = (I + \tilde{\Delta}(\rho))^{1/2} A_0(\rho^*)(I + \Delta(\rho))^{1/2}.$$

Set $u = (I + \Delta(\rho))^{-1/2} v$, where v solves the equation

$$((A_0(\rho^*) + A_{-1})D_{x_1} + (I + \tilde{\Delta})^{-1/2} Q (I + \Delta)^{-1/2})v = 0, \quad A_{-1} \in \mathsf{L}^{-1\diagdown},$$

For ρ sufficiently close to ρ^* the assumptions (S), (E), (Dm) and (A) are fulfilled for $\widetilde{A}=A_0(\rho^*)$, $\widetilde{Q}(\rho)=(I+\widetilde{\Delta})^{-1/2}$, $Q(I+\Delta)^{-1/2}$. This observation shows, that we can assume $A_0(\rho)=A_0(\rho^*)$. Lemma 2.2 together with the condition $WF(B(u(0, x')))$ imply $\rho^*\notin WF((I-\pi)(u(0, x')))$.

These preliminary notes show that Theorem 1 follows from the following

Proposition 1. *Let the assumptions* (S), (E), (Dm), (A) *and the condition* (3.1) *be fulfilled. Suppose* $A_0(\rho)=A_0(\rho^*)$. *If* $\rho^*\notin WF'(Pu)$, $\rho^*\notin WF((I-\pi)(u(0, x')))$ *and* $WF'(u)\cap\{\varphi<0\}\cap W=\Phi$, *then we can find an operator* $q\in L^{0\searrow}$, *elliptic at* ρ^*, *so that* $q(u)\in\mathscr{H}^{s-\sigma}$ *for* $0<\sigma<1/2$.

Proof. The main step is the application of the inequality

$$(3.2)\qquad (a_0\varkappa-b_0)\|v\|_s^2\leqq\operatorname{Re}(i(P_\varkappa v, v)_s)$$
$$+C_\varkappa(\|\omega v\|_s^2+\|v\|_{s-1/2}^2)+(1/2)(A_0(\rho^*)v, v)_{s,0},$$

established in Lemma 2.1. Set $v=J_\varepsilon cb\exp(-\varkappa\psi)(u)$. Here $J_\varepsilon=(1-\varepsilon\Delta_{x'})^{-1}$, $\Delta_{x'}=\partial_{x_2}+\cdots+\partial_{x_n}$, $b, c\in L^{0\searrow}$, $b_0(\rho)=\chi(\psi(\rho)-\delta)$, $c_0(\rho)=\chi(-\varphi(\rho)-\delta)$ and

$$\chi(s)=\begin{cases}\exp(-s^{-2}) & \text{for}\quad s<0,\\ 0 & \text{for}\quad s\geqq 0.\end{cases}$$

Our aim is to obtain the inequality

$$(3.3)\qquad \|J_\varepsilon bc\exp(-\varkappa\psi)(u)\|_s\leqq c_\varkappa$$

with constant $c_\varkappa$ independent of $\varepsilon>0$. Then Lemma 2.2 in [1] can be applied and we shall obtain $bc\exp(-\varkappa\psi)(u)\in\mathscr{H}^{s-\sigma}$ for $0<\sigma<1/2$. On the other hand, conn supp $(bc)\subset W_\delta$, where the sets $W_\delta=\{\rho\in W;\ \psi(\rho)<\delta,\ \varphi(\rho)>-\delta\}$ are close to ρ^*, provided δ close to 0. The inequality (3.3) will be proved by estimating the right hand side of (3.2).

Since the property (3.1) holds, we have

$$(3.4)\qquad \|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_{s-1/2}\leqq C_\varkappa.$$

Choosing $\delta>0$ sufficiently small, we may assume $W_\delta\cap\text{cone supp}(\omega)=\Phi$. Thus we get the estimate

$$(3.5)\qquad \|\omega J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_s\leqq C_\varkappa.$$

Consider the term $\operatorname{Re}(i(P_\varkappa v, v)_s)$ in the right hand side of (3.2). Since

$$P_\varkappa v=P_\varkappa J_\varepsilon cb\exp(-\varkappa\psi)(u)=J_\varepsilon cb\exp(-\varkappa\psi)(u)+[P_\varkappa,\ J_\varepsilon cb]\exp(-\varkappa\psi)(u)$$

and

$$[P_\varkappa,\ J_\varepsilon cb]=[P, J_\varepsilon]cb+J_\varepsilon[P, c]b+J_\varepsilon c[P, b]\quad(\operatorname{mod} L^{-1\prime}),$$

we have

$$(3.6)\qquad \operatorname{Re}(i(P_\varkappa v, v)_s)\leqq C_\varkappa+\operatorname{Re}(i(J_\varepsilon cbP_\varkappa\exp(-\varkappa\psi)(u), v)_s)$$
$$+\operatorname{Re}(i[P, J_\varepsilon]cb\exp(-\varkappa\psi)(u), v)_s)+\operatorname{Re}(i\,J_\varepsilon[P, c]\,b\exp(-\varkappa\psi)(u), v)_s)$$
$$+\operatorname{Re}(i(J_\varepsilon c[P, b]\exp(-\varkappa\psi)(u), v)_s).$$

The second term in the right hand side of (3.6) is $0(1)$ because $\rho^*\notin WF'(P_\varkappa\exp(-\varkappa\psi)(u)$. For the third term in (3.6) we apply the equality $[P, J_\varepsilon)=A^\varepsilon J_\varepsilon$

$+R^{\varepsilon}$ with $A^{\varepsilon}\in L^{0\prime}$, $R^{\varepsilon}\in L^{-3\prime}$. Moreover, we have $|A^{\varepsilon}_0(\rho)|\leq C$, $|R^{\varepsilon}_{-3}(\rho)|\leq C/(1+|\xi'|)$ with constant $C>0$ independent of ε, $\varkappa$. Thus we have

$$\mathrm{Re}\,(i[P, J_\varepsilon]\,cb\exp(-\varkappa\psi)(u), v)_s)\leq C_\varkappa C\,\|J_\varepsilon cb\exp(-\varkappa\psi)u\|_s^2.$$

Let b^* be the operator, adjoint to $b\in L^{0\backslash}$ with respect to the scalar product $(\ ,\ )_{s,0}$. The principle symbol of the operator $ib^*[P, b]$ is $b_0(\rho)b_0'(\rho)\{\tilde{A}\xi_1+Q_1, \psi_0\}$, where $b_0'(\rho)=\chi'(\psi_0(\rho)-\delta)$. Set $\tilde{b}_0(\rho)=(-b_0(\rho)b_0'(\rho))^{1/2}$. Since the matrix $\{\tilde{A}\xi_1+Q_1, \psi_0\}-a_0I$ is positive, the sharp Garding's inequality implies that

$$\mathrm{Re}\,(iJ_\varepsilon c\,[P, b]\exp(-\varkappa\psi)(u), v)_s)\leq C_\varkappa - a_0\,\|J_\varepsilon c\tilde{b}\exp(-\varkappa\psi)(u)\|_s^2.$$

In a similar manner we obtain the estimate

$$\mathrm{Re}\,(i(J_\varepsilon[P, c]\,b\exp(-\varkappa\psi)(u), v)_s)\leq C_\varkappa + C\,\|J_\varepsilon\tilde{c}\,b\exp(-\varkappa\psi)(u)\|_s^2,$$

where $\tilde{c}_0(\rho)=(-\chi(-\varphi-\delta)\chi'(-\varphi-\delta))^{1/2}$ and C is independent of ε and $\varkappa$. These estimates, connected with the right hand side of (3.6), lead to the inequality

$$(3.7)\quad \mathrm{Re}\,(i(P_\varkappa v, v)_s)\leq C_\varkappa + C\,\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_s^2 + C\,\|J_\varepsilon\tilde{c}b\exp(-\varkappa\psi)(u)\|_s^2$$
$$-a_0\,\|J_\varepsilon c\,\tilde{b}\exp(-\varkappa\psi)u\|_s^2.$$

The last term in the right hand side of (3.2) is $(1/2)(\tilde{A}v, v)_{s,0}$. By using the fact that $\rho^*\notin WF((I-\pi)(u(0, x')))$ and the assumption (Dm), we are going to

$$(\tilde{A}v, v)_{s,0}\leq C_\varkappa + (\tilde{A}\pi(v), [(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)](u))_{s,0}$$
$$+(\tilde{A}[(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)](u), (v))_{s,0}$$
$$+(\tilde{A}[(I-\pi), Jcb\exp(-\varkappa\psi)](u), [(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)]u)_{s,0}.$$

The inequality $(u, v)_{s,0}\leq\lambda\,\|u\|^2_{s+1/2,0}+(4/\lambda)\,\|v\|^2_{s-1/2,0}$ for $\lambda>0$ shows that $(\tilde{A}v, v)_{s,0}\leq C_{\varkappa,\lambda}+\lambda\,\|[(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)](u)\|^2_{s+1/2,0}$. The commutator $[(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)]$ can be estimated in the same manner as $[P, J_\varepsilon cb]$. Consequently, we conclude that the inequality

$$\|[(I-\pi), J_\varepsilon cb\exp(-\varkappa\psi)](u)\|^2_{s+1/2,0}$$
$$\leq C_{\varkappa,\lambda}+C(\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|^2_{s-1/2,0}$$
$$+\|J_\varepsilon\tilde{c}\,b\exp(-\varkappa\psi)(u)\|^2_{s-1/2,0}+\|J_\varepsilon c\tilde{b}\exp(-\varkappa\psi)(u)\|^2_{s-1/2,0})$$

holds. Estimating the norm $\|\cdot\|^2_{s-1/2,0}$ by the norm $\|\cdot\|_s^2+\|D_{x_1}(\cdot)\|^2_{s-1}$ and applying the fact that $\rho^*\notin WF((D_{x_1}+A^{-1}Q)u)$, we get

$$\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|^2_{s-1/2,0}\leq C_\varkappa + C\,\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_s^2.$$

This estimate and the inequalities (3.2), (3.4), (3.5), (3.7) show that the following estimate is fulfilled:

$$(a_0\varkappa-b_0)\,\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_s^2$$
$$\leq C_{\varkappa,\lambda}+(\lambda C-d)\,\|J_\varepsilon\tilde{c}b\exp(-\varkappa\psi)(u)\|_s^2+C_\lambda\,\|J_\varepsilon c\tilde{b}\exp(-\varkappa\psi)(u)\|_s^2.$$

Choosing $\lambda>0$, such that $\lambda\dot{C}-d\leqq 0$, we get

$$(a_0\varkappa-b_0)\|J_\varepsilon cb\exp(-\varkappa\psi)u\|_s^2\leqq C_\varkappa+C\|J_\varepsilon cb\exp(-\varkappa\psi)u\|_s^2. \tag{3.8}$$

Let $\xi(s)\in C_0^\infty(\mathrm{R})$ be a function with the properties:

$$\begin{aligned} &\text{i)}\quad \xi(-\delta)=1,\quad \xi(s)\leqq 1,\\ &\text{ii)}\quad \{\rho\in W_\delta;\ \xi(\varphi(\rho))\neq 0\}\cap WF'(u)=\Phi. \end{aligned} \tag{3.9}$$

This choice of $\xi(s)$ is possible, since the condition $WF'(u)\cap\{\varphi<0\}\cap W=\Phi$ holds for small $\delta>0$. On the other hand, we have the inequality

$$(1-\xi(\varphi(\rho)))\tilde{c}_0(\rho)\leqq Dc_0(\rho) \tag{3.10}$$

with some constant $D>0$. Applying (3.8), (3.9) and (3.10), we are going to

$$(a_0\varkappa-b_0)\|J_\varepsilon cb\exp(-\varkappa\psi)(u)\|_s^2\leqq C_\varkappa.$$

Finally, choosing $\varkappa>0$ in such a manner that $a_0\varkappa-b_0=1$, we get the inequality (3.3). This completes the proof of Proposition 1.

4. Finite speed of propagation of singularities. This section is devoted to an application of Theorem 1 for first order symmetric hyperbolic systems with maximal dissipative boundary conditions, used in [2; 4]. Let K be a compact set in R^n and $\Omega=\mathrm{R}^n\setminus K$. We suppose that Ω has smooth boundary $\partial\Omega$. Consider the following problem:

$$\begin{aligned} &((\sum_{j=1}^n A_j(x)\partial_{x_j})-\partial_t)(u)=0 && \text{on}\quad \Omega\times\mathrm{R}^+,\\ &B(x)(u)=f && \text{on}\quad \partial\Omega\times\mathrm{R}^+,\\ &u(t,x)=0 && \text{for}\quad t\leqq 0, \end{aligned} \tag{4.1}$$

where $A_1(x), A_2(x),\dots,A_n(x)$ and $B(x)$ are $(d\times d)$ matrices, smoothly depending on $x\in\bar{\Omega}$ We take the following assumptions:

(H_1) $\left\{\begin{array}{l} A_1(x),\dots,A_n(x) \text{ are symmetric and for } |x|>R \\ \text{we have } A_j(x)=A_j^0,\quad j=1,2,\dots,n.\end{array}\right.$

(H_2) $\left\{\begin{array}{l} \text{The matrix } A(x,\xi)=\sum_{j=1}^n A_j(x)\xi_j \text{ has simple} \\ \text{eigenvalues } \tau_k(x,\xi)\neq 0,\ k=1,\dots,d,\ x\in\bar{\Omega},\ \xi=(\xi_1,\dots,\xi_n)\neq 0.\end{array}\right.$

(H_3) $\left\{\begin{array}{ll} \text{(a)} & \langle A(x,\nu(x))v,v\rangle\leqq 0 \text{ for } v\in\operatorname{Ker}B(x),\ x\in\partial\Omega, \text{ where} \\ & \nu(x) \text{ is the unit inward normal at } x\in\partial\Omega. \\ \text{(b)} & \text{the space } \operatorname{Ker}B(x) \text{ is maximal with respect to (a).}\end{array}\right.$

Setting $c=\max_{k,x,\xi}|\nabla_\xi\tau_k(x,\xi)|$, we shall prove that the speed of propagation of singularities for the problem (4.1) is not greater than c. More precisely, we have the following

Theorem 2. *Suppose the hypotheses* (H_1), (H_2) *and* (H_3) *are fulfilled. Let* $u(t,x)$ *be a solution to* (4.1) *with* $f(t,x)\in\mathcal{E}'(\mathrm{R}\times\partial\Omega)$. *Then if* $(y,t)\in$ sing supp(u), *there exists* $(x_0,t_0)\in$ supp(f), *so that* $|y-x_0|\leqq c|t-t_0|$.

P r o o f. We say that $u(t,x)$ verifies the property (P_T), if for every $(y,t)\in \operatorname{sing\,supp}(u)$ with $t<T$ the inequality (4.2) holds for some $(x_0,t_0)\in \operatorname{supp}(f)$. Since $u(t,y)=0$ for $t\leq 0$, the property (P_T) is fulfilled for $T\leqq 0$. Therefore Theorem 2 follows from

L e m m a 4.1. *If the property* (P_T) *holds for* $T<T_0$, *there exists* $\varepsilon>0$, *such that* (P_T) *holds for* $T<T_0+\varepsilon$.

P r o o f. The singularities of the distribution $u(t,x)$ propagate along the bicharacteristic of the operator $\partial_t-\Sigma_{j=1}^n A(x)\partial_{x_j}$ in the region $\Omega\times\mathbf{R}$. Let $\gamma(s)=(t(s),x(s),\tau(s),\xi(s))$, $0\leqq s\leqq\sigma$ be the bicharacteristic, connected with the symbol $\tau-\tau_k(x,\xi)$. Then we have

$$|x(\sigma)-x(0)|\leqq\int_0^\sigma|\nabla_\xi\tau_k(x(s),\ \xi(s))|\,ds\leqq c\sigma=c\,|t(\sigma)-t(0)|.$$

This observation shows that it is sufficient to study the singularities of $u(t,x)$ only near $\partial\Omega\times\mathbf{R}$.

Let $y_0\in\partial\Omega$ be a fixed point with

(4.3) $$|y_0-\hat{x}|>c\,|T_0-\hat{t}|\quad\text{for}\quad(\hat{x},\hat{t})\in\operatorname{supp}(f).$$

Consider the function $\varphi(t,x,\tau,\xi)=-c(T_0-t)-\delta+((x-y_0)^2+\delta^2)^{1/2}$, where $\delta>0$. Making a change of variables, we can assume that the boundary $\partial\Omega$ is given locally by $x_1=0$. Thus, we are in situation of Theorem 1. The assumptions of Theorem 1 follow from the hypotheses described above. We shall verify only the condition $WF'(u)\cap\{\varphi<0\}\cap W=\Phi$. When $(x,t)\in\operatorname{sing\,supp}(u)\cap\{\varphi<0\}$, from the inequality $\varphi<0$ we get

(4.4) $$|y_0-x|<c\,(T_0-t)$$

and $t<T_0$. For $t<T_0$ we can apply the property (P_T) and obtain

(4.5) $$|x-\hat{x}|\leqq c\,|t-\hat{t}|\quad\text{for}\quad(\hat{x},\hat{t})\in\operatorname{supp}(f).$$

But the inequalities (4.4) and (4.5) contradict (4.3). Thus, we can apply Theorem 1 and conclude that $u(t,x)$ is smooth near (y_0,T_0). Since $y_0\in\partial\Omega$ varies in a compact set, we can find $\varepsilon>0$ in such a manner that for $(y,t)\in\operatorname{sing\,supp}(u)\cap\{\partial\Omega\times\mathbf{R}\}$ and $t<T_0+\varepsilon$ the inequality (4.2) holds for some $(\hat{x},\hat{t})\in\operatorname{supp}(f)$.

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