Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

AN AXIOMATIC CHARACTERIZATION OF THE STONE DUALITY

GEORGI D. DIMOV

In this paper an axiomatic description of the Stone duality functors for Boolean algebras and distributive lattices with zero and identity is given.

In [3] Iv. Prodanov proved that every duality functor $F:\mathscr{L}\mathsf{C}\to\mathscr{L}\mathsf{C}$, where $\mathscr{L}\mathsf{C}$ is the category of all locally compact Abelian topological groups and continuous homomorphisms, such that for any two objects $G_1, G_2\in \mathrm{Obj}\,\mathscr{L}\mathsf{C}$ and for any two morphisms $f, g\in \mathrm{Mor}_{\mathscr{L}\mathsf{C}}(G_1, G_2)$ holds $F(f+g)=F(f)+F(g)$, is natural equivalent to the classical Pontrjagin duality functor, i. e. he gave an axiomatic description of the Pontrjagin duality. Iv. Prodanov asked if there is an axiomatic description of the Stone duality functors for Boolean algebras and for distributive lattices with zero and identity. The present article answers these questions in the affirmative. The results were announced without proofs in [6].

Let us first recall the definitions of the Stone duality functors (see [4; 5; 2; 1]).

A subset Y of a partially ordered set $(X,\le)$ is increasing (decreasing) if $x\in X$, $y\in Y$ and $y\le x$ $(x\le y)$ imply $x\in Y$.

An ordered space $(X,\mathscr{T},\le)$ (i. e. a set X, with a topology $\mathscr{T}$, endowed with a partial order $\le$) is said to be totally order disconnected (or briefly T. O. D.) if, given $x, y\in X$ with $y\not\le x$, then there exists a clopen increasing set U such that $y\in U$, $x\notin U$. It is easy to see that totally order disconnectedness of $(X,\mathscr{T},\le)$ implies that $\mathscr{T}$ is T_2, and reduces to total disconnectedness when $\le$ is the trivial order: $x\le y$ iff $x=y$.

A map $f:X\to Y$, where X, Y are partially ordered sets, is increasing if $x_1\le x_2$ in X implies $f(x_1)\le f(x_2)$ in Y. Obviously, f is increasing iff $f^{-1}(A)$ is increasing for every increasing subset A of Y.

We shall denote:

by $\mathscr{B}$ — the category of all Boolean algebras and Boolean homomorphisms;

by C — the category of all totally disconnected compact Hausdorff topological spaces and continuous maps;

by $\mathscr{R}$ — the category of all distributive lattices with zero and identity and lattice homomorphisms;

by $\mathscr{P}$ — the category of all compact T. O. D. ordered spaces and increasing maps;

by $T:\mathscr{B}\to\mathsf{C}$ and by $S:\mathsf{C}\to\mathscr{B}$ — the classical Stone duality functors;

by $K:\mathscr{R}\to\mathscr{P}$ and by $L:\mathscr{P}\to\mathscr{R}$ — the classical Stone-Priestley duality functors;

by 2 — the simplest Boolean algebra, which contains only 0 (zero) and 1 (identity);

by P — the one-point topological space; we shall write $P=\{p\}$;

by D — the two-point discrete topological space, which points we shall denote by $\bar{0}$ and $\bar{1}$;

by $\hat{D}$ — the space D, ordered by the relation $\leq : \bar{0}<\bar{1}$;

by $|X|$ — the cardinality of the set X.

If $B \in \operatorname{Obj} \mathscr{B}$, then TB is, by definition, the set $\operatorname{Mor}_{\mathscr{B}}(B, 2)$, endowed with the topology $\mathscr{T}$ induced by the Tychonoff topology of 2^B, where the set 2 is considered with the discrete topology. If A, $B \in \operatorname{Obj} \mathscr{B}$ and $f \in \operatorname{Mor}_{\mathscr{B}}(A, B)$, then $Tf : TB \to TA$ is defined by $Tf(\alpha)=\alpha \circ f$ for any $\alpha \in \operatorname{Mor}_{\mathscr{B}}(B, 2)=TB$.

If $X \in \operatorname{Obj} \mathsf{C}$, then SX is, by definition, the set of all clopen subsets of X with the obvious Boolean operations, zero and identity. If X, $Y \in \operatorname{Obj} \mathsf{C}$ and $f \in \operatorname{Mor}_{\mathsf{C}}(X, Y)$, then $Sf : SY \to SX$ is defined by $Sf(U)=f^{-1}(U)$ for any $U \in SY$.

If $A \in \operatorname{Obj} \mathscr{R}$, then KA is, by definition, the set $\operatorname{Mor}_{\mathscr{R}}(A, 2)$ with the topology $\mathscr{T}$ induced by the Tychnoff topology of 2^A (where the set 2 is considered with the discrete topology) and with the order $\leq$ defined with: $\alpha \leq \beta$ iff $\alpha(a) \leq \beta(a)$ for any $a \in A$ (α, $\beta \in KA$). If A, $B \in \operatorname{Obj} \mathscr{R}$ and $f \in \operatorname{Mor}_{\mathscr{R}}(A, B)$, then $Kf : KB \to KA$ is defined by $Kf(\alpha)=\alpha \circ f$ for any $\alpha \in KB$.

If $\hat{X}=(X, \mathscr{T}, \leq) \in \operatorname{Obj} \mathscr{P}$, then $L\hat{X}$ is, by definition, the set of all clopen increasing subsets of X with the obvious lattice operations, zero and identity. If $\hat{X}=(X, \mathscr{T}, \leq) \in \operatorname{Obj} \mathscr{P}$ and $\hat{Y}=(Y, \tau, \leq_1) \in \operatorname{Obj} \mathscr{P}$ and if $f \in \operatorname{Mor}_{\mathscr{P}}(\hat{X}, \hat{Y})$, then $Lf : L\hat{Y} \to L\hat{X}$ is defined by $Lf(U)=f^{-1}(U)$ for any $U \in L\hat{Y}$.

Finally, we recall that:

(i) If $\mathscr{K}_1$ and $\mathscr{K}_2$ are two categories and $F_1 : \mathscr{K}_1 \to \mathscr{K}_2$, $F_2 : \mathscr{K}_1 \to \mathscr{K}_2$ are two contravariant (covariant) functors, then the functors F_1 and F_2 are natural equivalent iff there exists a function

$$s\begin{cases} \operatorname{Obj} \mathscr{K}_1 \longrightarrow \operatorname{Mor} \mathscr{K}_2 \\ X \longrightarrow sX \in \operatorname{Mor}_{\mathscr{K}_2}(F_1X, F_2X), \end{cases}$$

such that sX is isomorphism for any $X \in \operatorname{Obj} \mathscr{K}_1$ and $sX_1 \circ F_1 f = F_2 f \circ sX_2$ ($sX_2 \circ F_1 f = F_2 f \circ sX_1$) for any $f \in \operatorname{Mor}_{\mathscr{K}_1}(X_1, X_2)$ (we shall write briefly $F_1 \underset{s}{\approx} F_2$ or $F_1 \approx F_2$);

(ii) A contravariant (covariant) functor $F : \mathscr{K}_1 \to \mathscr{K}_2$ is a duality (equivalence) if there exists a contravariant (covariant) functor $G : \mathscr{K}_2 \to \mathscr{K}_1$ (which is called a inverse duality to F (inverse equivalence to F)), such that $G \circ F \approx 1_{\mathscr{K}_1}$ and $F \circ G \approx 1_{\mathscr{K}_2}$, where $1_{\mathscr{K}_1} : \mathscr{K}_1 \to \mathscr{K}_1$ and $1_{\mathscr{K}_2} : \mathscr{K}_2 \to \mathscr{K}_2$ are the identity functors. Let us note that if $F : \mathscr{K}_1 \to \mathscr{K}_2$ is a duality (equivalence), then the inverse duality (inverse equivalence) G to F is unique up to natural equivalence; therefore it will be denoted by F^{-1}. An equivalence $F : \mathscr{K}_1 \to \mathscr{K}_1$ is called an autoequivalence. If $\mathscr{K}$ is a category, then the class of all (up to natural equivalence) autoequivalence functors $F : \mathscr{K} \to \mathscr{K}$ will be denoted by $\operatorname{Aut} \mathscr{K}$. It is clear that if $\operatorname{Aut} \mathscr{K}$ is a set, then it is a group.

Now we may formulate our first result.

Theorem 1. *If $\mathscr{K}$ is a full subcategory of the category $\mathscr{D}$ of all zero-dimensional Hausdorff spaces and continuous maps and if there exists* *a duality $F:\mathscr{B}\to\mathscr{K}$, then the category $\mathscr{K}$ coincides with the category* C *and $F\approx T$.*

Corollary 1. Aut $\mathscr{B}$ *and* Aut C *are one-point sets (and hence, the groups* Aut $\mathscr{B}$ *and* Aut C *are trivial).*

We need some definitions in order to formulate our second result.

Definition 1. *Let $(X, \mathscr{T}, \leq)$ be an ordered topological space and let $\mathscr{U}(X)$ $(\mathscr{L}(X))$ be the family of all clopen increasing (decreasing) subsets of X. Then the ordered topological space $(X, \mathscr{T}, \leq)$ is called an order zero-dimensional space (briefly O. Z. D.) if the family $\mathscr{U}(X)\cup\mathscr{L}(X)$ is an open subbase of the topology $\mathscr{T}$ of X.*

Obviously, if $(X, \mathscr{T}, \leq)$ is an O. Z. D. space and $\leq$ is the trivial order, then $(X, \mathscr{T})$ is a zero-dimensional space. Thus a T. O. D. space needn't be an O. Z. D. space. It is not difficult to construct an ordered space $(X, \mathscr{T}, \leq)$, which is Hausdorff and O. Z. D., but not T. O. D. Consequently, there is no including connection between the classes of all T. O. D. and all O. Z. D. ordered topological spaces, but it is easy to prove the following lemma.

Lemma 1. *Every T. O. D. compact ordered space is an O. Z. D. space.*

Let us define a covariant functor $F:\mathscr{P}\to\mathscr{P}$ by $E\widehat{X}=(X, \mathscr{T}, \leq_1)$ for every $\widehat{X}=(X, \mathscr{T}, \leq)\in \mathrm{Obj}\,\mathscr{P}$, where $x_1\leq_1 x_2$ iff $x_2\leq x_1$ $(x_1, x_2\in X)$, and by $Ef=f$ for every $f\in \mathrm{Mor}_{\mathscr{P}}(\widehat{X}, \widehat{Y})$ (where $\widehat{X}, \widehat{Y}\in \mathrm{Obj}\,\mathscr{P}$). It is obvious that this definition is correct and that $E\circ E=1_{\mathscr{P}}$; hence, E is an equivalence. It is easy to see that E is not natural equivalent to $1_{\mathscr{P}}$.

Now we may formulate the following theorem.

Theorem 2. *If $\mathscr{H}$ is a full subcategory of the category $\mathscr{F}$ of all T. O. D., O. Z. D. ordered topological spaces and continuous increasing maps and if there exists a duality $\Phi:\mathscr{R}\to\mathscr{H}$, then the category $\mathscr{H}$ coincides with the category $\mathscr{P}$ and either $\Phi\approx K$ or $\Phi\approx E\circ K$.*

Corollary 2. Aut $\mathscr{R}$ *and* Aut $\mathscr{P}$ *are sets and they are isomorphic as groups to* $\mathbf{Z}_2$.

We need some lemmas and propositions in order to prove Theorem 1 and Theorem 2.

The following proposition is well known.

Proposition 1. *If $\mathscr{K}_1$ and $\mathscr{K}_2$ are two categories and $G:\mathscr{K}_1\to\mathscr{K}_2$ is a duality, then the map*

$$G_{X,Y}\begin{cases}\mathrm{Mor}_{\mathscr{K}_1}(X, Y) \longrightarrow \mathrm{Mor}_{\mathscr{K}_2}(GY, GX)\\ \varphi \longrightarrow G\varphi\end{cases}$$

is a bijection for any $X, Y\in \mathrm{Obj}\,\mathscr{K}_1$.

Using Proposition 1 it is easy to prove the following lemma.

Lemma 2. a) $\Phi 2=P$ *and* $F2=P$ (see Theorems 1, 2);

b) *If $R_0=\{0, j, 1\}$, where $0<j<1$, then $\Phi R_0=\widehat{D}$ and if $B_0=\{0, b, b', 1\}$, where with b' we denote the Boolean complement of b, then $FB_0=D$.*

Corollary 3. a) $\Phi^{-1}P=2$ *and* $F^{-1}P=2$;

b) $\Phi^{-1}\widehat{D}=R_0$ *and* $F^{-1}D=B_0$.

Lemma 3. *For each $A\in \mathrm{Obj}\,\mathscr{R}$ $(B\in \mathrm{Obj}\,\mathscr{B})$ there exists a bijection $\mu_A:\mathrm{Mor}_{\mathscr{R}}(A, 2)\to\Phi A$ $(\mu_B:\mathrm{Mor}_{\mathscr{B}}(B, 2)\to FB)$, such that for each $f\in \mathrm{Mor}_{\mathscr{R}}(A_1, A_2)$ $(g\in \mathrm{Mor}_{\mathscr{B}}(B_1, B_2))$ the following diagram commutes:*

$$\begin{array}{ccccccc}\mathrm{Mor}_{\mathscr{R}}(A_2, 2) & \xrightarrow{h_2 f} & \mathrm{Mor}_{\mathscr{R}}(A_1, 2) & (\mathrm{Mor}_{\mathscr{B}}(B_2, 2) & \xrightarrow{h_2 g} & \mathrm{Mor}_{\mathscr{B}}(B_1, 2) \\ \mu_{A_2}\downarrow & & \downarrow\mu_{A_1} & \bar{\mu}_{B_2}\downarrow & & \downarrow\bar{\mu}_{B_1} \\ \Phi A_2 & \xrightarrow{\Phi f} & \Phi A_1 & FB_2 & \xrightarrow{Fg} & FB_1),\end{array}$$

where $h_2 f(\varphi)=\varphi\circ f$ $(h_2 g(\psi)=\psi\circ g)$ *for any* $\varphi\in\mathrm{Mor}_{\mathscr{R}}(A_2, 2)$ $(\psi\in\mathrm{Mor}_{\mathscr{B}}(B_2, 2))$ $(A_1, A_2\in\mathrm{Obj}\,\mathscr{R})$.

Proof. If $A\in\mathrm{Obj}\,\mathscr{R}$, then Proposition 1 implies that the map

$$\Phi_{A,2}\begin{cases}\mathrm{Mor}_{\mathscr{R}}(A, 2) & \longrightarrow & \mathrm{Mor}_{\mathscr{H}}(\Phi 2, \Phi A) \\ \varphi & \longrightarrow & \Phi\varphi\end{cases}$$

is a bijection.

From Lemma 2a) it follows that $\Phi 2=P$. Using this fact we may define a map

$$\varkappa_A\begin{cases}\mathrm{Mor}_{\mathscr{H}}(P, \Phi A) & \longrightarrow & \Phi A \\ t & \longrightarrow & t(p).\end{cases}$$

It is obvious that the map $\varkappa_A$ is a bijection.

We put $\mu_A=\varkappa_A\circ\Phi_{A,2}$. Then the map μ_A is a bijection and it is easy to see, that the corresponding diagram commutes.

The map $\bar{\mu}_B$ is constructed analogously.

Lemma 4. *Let* $A\in\mathrm{Obj}\,\mathscr{R}$ $(B\in\mathrm{Obj}\,\mathscr{B})$. *If we consider the set* $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ $(\mathrm{Mor}_{\mathscr{B}}(B, 2)$ *as a topological subspace of the topological space* 2^A (2^B) *where the set* 2 *is endowed with the discrete topology and the topology in* 2^A (2^B) *is the Tychonoff topology, then the map* μ_A $(\bar{\mu}_B)$, *which was defined in Lemma 3, is continuous.*

Proof. Let us prove first that the map $\mu_A:\mathrm{Mor}_{\mathscr{R}}(A, 2)\to\Phi A$ is continuous, where $\mu_A=\varkappa_A\circ\Phi_{A,2}$ (for the notations see the proof of Lemma 3).

Let $\varphi_0\in\mathrm{Mor}_{\mathscr{R}}(A, 2)$ and $x_0=\mu_A(\varphi_0)$. Since $\Phi A\in\mathrm{Obj}\,\mathscr{H}$, the space ΦA is an O. Z. D. space and hence, it has an open base $\mathfrak{A}$ consisting of the sets of the form $U\cap V$, where $U\in\mathscr{U}(\Phi A)$, $V\in\mathscr{Z}(\Phi A)$ (see Definition 1). Let $W_0=U_0\cap V_0$ belongs to $\mathfrak{A}$ (where $U_0\in\mathscr{U}(\Phi A)$, $V_0\in\mathscr{Z}(\Phi A)$) and let $x_0\in W_0$. We shall prove that there exists a neighbourhood N_0 of the point φ_0, such that $\mu_A(N_0)\subset W_0$.

Let $f_0, g_0:\Phi A\to\widehat{D}$ are two maps, such that $f_0^{-1}(\bar{1})=U_0$ and $g_0^{-1}(\bar{0})=V_0$. Then, $f_0, g_0\in\mathrm{Mor}_{\mathscr{H}}(\Phi A, \widehat{D})$ and hence we may consider the morphisms $\Phi^{-1}f_0:\Phi^{-1}\widehat{D}\to\Phi^{-1}\Phi A$ and $\Phi^{-1}g_0:\Phi^{-1}\widehat{D}\to\Phi^{-1}\Phi A$. From Corollary 3b) it follows that $\Phi^{-1}\widehat{D}=R_0=\{0, j, 1\}$ $(0<j<1)$. Let us fix a natural equivalence e, such that $1_{\mathscr{R}}\underset{e}{\approx}\Phi^{-1}\circ\Phi$. Now we put $t_{f_0}=((eA)^{-1}\circ\Phi^{-1}f_0)(j)$ and $t_{g_0}=((eA)^{-1}\circ\Phi^{-1}g_0)(j)$. If we denote by π_a the projection of $2^A=\Pi\{2_a=2 : a\in A\}$ onto 2_a, then the set $N_0=\mathrm{Mor}_{\mathscr{R}}(A, 2)\cap\pi_{t_{f_0}}^{-1}(\varphi_0(t_{f_0}))\cap\pi_{t_{g_0}}^{-1}(\varphi_0(t_{g_0}))$ is the requiring neighbourhood of the point φ_0.

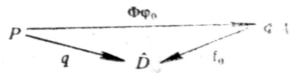
Indeed, it is obvious that N_0 is an open subset of $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ and φ_0 belongs to N_0. It remains to show that $\mu_A(N_0) \subset W_0$. Evidently, it is sufficient to prove that $\mu_A(N_{f_0}) \subset U_0$ and $\mu_A(N_{g_0}) \subset V_0$, where $N_{f_0} = \mathrm{Mor}_{\mathscr{R}}(A, 2) \cap \pi_{t_{f_0}}^{-1}(\varphi_0(t_{f_0}))$ and $N_{g_0} = \mathrm{Mor}_{\mathscr{R}}(A, 2) \cap \pi_{t_{g_0}}^{-1}(\varphi_0(t_{g_0}))$.

Let $f \in N_{f_0}$, i. e. $f \in \mathrm{Mor}_{\mathscr{R}}(A, 2)$ and $f(t_{f_0}) = \varphi_0(t_{f_0})$. By definition $\mu_A(f) = \Phi f(p)$. We must show that $\mu_A(f) \in U_0$, i. e. that $f_0(\Phi f(p)) = \bar{1}$.

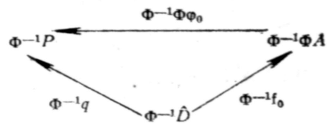
Let us consider the following map

$$q \begin{cases} P \longrightarrow \hat{D} \\ p \longrightarrow \bar{1}. \end{cases}$$

Then $q \in \mathrm{Mor}_{\mathscr{H}}(P, \hat{D})$ and the diagram

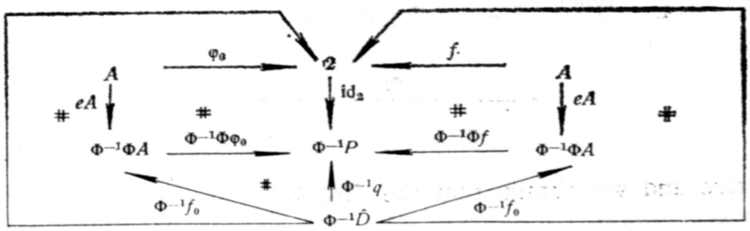


commutes (since, by Lemma 2a), $\Phi 2 = P$). Hence, the following diagram commutes:



We must show that $f_0 \circ \Phi f = q$.

Let us consider the following diagram, where in every subdiagram, which is commutative, lies the symbol $\#$:



(here $\alpha = f \circ (eA)^{-1} \circ \Phi^{-1} f_0$ and $\beta = \varphi_0 \circ (eA)^{-1} \circ \Phi^{-1} f_0$; $e2 = \mathrm{id}_2$, since $\Phi^{-1} P = 2$ and the equalities $e2(0) = 0$, $e2(1) = 1$ are obligatory).

We shall prove that $\Phi^{-1} f_0 \circ \Phi^{-1} \Phi f = \Phi^{-1} q$.

Indeed, since $\Phi^{-1} \hat{D} = R_0 = \{0, j, 1\}$ and $f(t_{f_0}) = \varphi_0(t_{f_0})$, $t_{f_0} = ((eA)^{-1} \circ \Phi^{-1} f_0)(j)$, we have $\alpha(j) = (f \circ (eA)^{-1} \circ \Phi^{-1} f_0)(j) = f(t_{f_0}) = \varphi_0(t_{f_0}) = (\varphi_0 \circ (eA)^{-1} \circ \Phi^{-1} f_0)(j) = \beta(j)$. But $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$. Hence, we obtain that $\alpha = \beta$. Now from $\Phi^{-1} q = \Phi^{-1} \Phi \varphi_0 \circ \Phi^{-1} f_0 = (\mathrm{id}_2 \circ \varphi_0 \circ (eA)^{-1}) \circ \Phi^{-1} f_0 = \mathrm{id}_2 \circ \beta = \beta$ and $\Phi^{-1} \Phi f \circ \Phi^{-1} f_0 = \mathrm{id}_2 \circ f \circ (eA)^{-1} \circ \Phi^{-1} f_0 = \mathrm{id}_2 \circ \alpha = \alpha$ it follows that $\Phi^{-1} q = \Phi^{-1} \Phi f \circ \Phi^{-1} f_0$. It is evident that this equality implies $q = f_0 \circ \Phi f$.

Therefore, we proved that $\mu_A(N_{f_0}) \subset U_0$.

If we consider now instead of the map q the map

$$r \begin{cases} P \longrightarrow \widehat{D} \\ p \longrightarrow \overline{0}, \end{cases}$$

we shall prove analogously that $r = g_0 \circ \Phi f$, for every $f \in N_{g_0}$, and hence, $\mu_A(N_{g_0}) \subset V_0$.

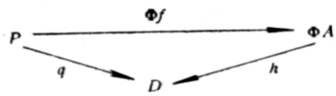
Consequently, the map μ_A is continuous.

The proof of the fact, that the map $\overline{\mu}_B$ is continuous, is analogous.

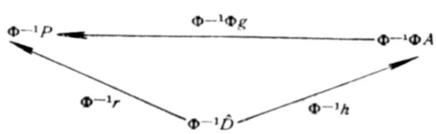
P r o o f of Theorem 1. From the definition of the functor T we have that for every $B \in \mathrm{Obj}\,\mathscr{B}$ the topological space TB coincides with the set $\mathrm{Mor}_{\mathscr{B}}(B, 2)$, endowed with a topology as in Lemma 4. From Lemmas 3, 4 it follows that the map $\mu_B : TB \to FB$ is a homeomorphism, since the space TB is compact. Hence, the space FB is compact for every $B \in \mathrm{Obj}\,\mathscr{B}$ and therefore, $\mathscr{K} = \mathbf{C}$. Now, from Lemma 3, we obtain that the functors T and F are natural equivalent, since $h_2 f = Tf$ for every $f \in \mathrm{Mor}_{\mathscr{B}}(B_1, B_2)$ (where $B_1, B_2 \in \mathrm{Obj}\,\mathscr{B}$).

L e m m a 5. *Let* $A \in \mathrm{Obj}\,\mathscr{R}$. *If* $\Phi^{-1}q(j) = 1$ (*where* $j \in R_0$ (*see Lemma* 2) and $q : P \to \widehat{D}$ *is the map, defined in the proof of Lemma 4 by* $q(p) = \overline{1}$), *then we order the set* $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ *by* $f \leq g$ *iff* $f(a) \leq g(a)$ *for every* $a \in A$, *and if* $\Phi^{-1}q(j) = 0$, *then we order the set* $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ *by* $f \leq g$ *iff* $g(a) \leq f(a)$ *for every* $a \in A$. *Then the map* $\mu_A : \mathrm{Mor}_{\mathscr{R}}(A, 2) \to \Phi A$, *defined in Lemma 3, is increasing.*

P r o o f. Let $f, g \in \mathrm{Mor}_{\mathscr{R}}(A, 2)$ and $f \leq g$, $f \neq g$. Let us assume that $\mu_A(f) \leq \mu_A(g)$, i. e. $\Phi f(p) \leq \Phi g(p)$. Since $\Phi A \in \mathscr{H}$, the space ΦA is T. O. D. Then there exists a clopen increasing subset U of the space ΦA, such that $\Phi f(p) \in U$ and $\Phi g(p) \notin U$. Let the map $h : \Phi A \to \widehat{D}$ is defined by $h^{-1}(\overline{1}) = U$. Then $h \in \mathrm{Mor}_{\mathscr{H}}(\Phi A, \widehat{D})$ and $h(\Phi f(p)) = \overline{1}$, $h(\Phi g(p)) = \overline{0}$. Therefore the diagram



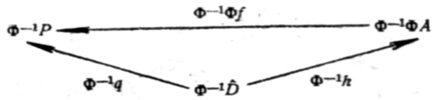
is commutative and we obtain, that the diagram



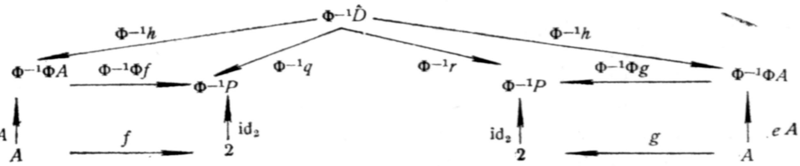
commutes. Analogously, if we consider the map

$$r \begin{cases} P \longrightarrow \widehat{D} \\ p \longrightarrow \overline{0} \end{cases}$$

then we shall have that the diagram



is commutative and we shall obtain that the following diagram



where e is the natural equivalence $1_{\mathscr{R}} \underset{e}{\approx} \Phi^{-1}\circ\Phi$, commutes. Since $q\neq r$, then $\Phi^{-1}q\neq\Phi^{-1}r$. Therefore $\Phi^{-1}q(j)\neq\Phi^{-1}r(j)$.

Now we must consider two cases:

(i) Let $\Phi^{-1}q(j)=1$. Then $\Phi^{-1}r(j)=0$.

We have $1=\Phi^{-1}q(j)=(\Phi^{-1}\Phi f\circ\Phi^{-1}h)(j)=(\mathrm{id}_2\circ f\circ(eA)^{-1}\circ\Phi^{-1}h)(j)$. Let $a_0=((eA)^{-1}\circ\Phi^{-1}h)(j)$. Then $a_0\in A$ and $f(a_0)=1$. But $0=\Phi^{-1}r(j)=(\Phi^{-1}\Phi g\circ\Phi^{-1}h)(j)=(\mathrm{id}_2\circ g\circ(eA)^{-1}\circ\Phi^{-1}h)(j)=(\mathrm{id}_2\circ g)(a_0)=g(a_0)$ and hence, $g(a_0)<f(a_0)$. Since $f\leq g$, i. e. $f(a)\leq g(a)$ for every $a\in A$, we get a contradiction. Therefore $\mu_A(f)\leq\mu_A(g)$.

(ii) Let $\Phi^{-1}q(j)=0$. Then $\Phi^{-1}r(j)=1$.

We put again $a_0=((eA)^{-1}\circ\Phi^{-1}h)(j)$ and we have, as in case (i), that $f(a_0)=\Phi^{-1}q(j)$ and $g(a_0)=\Phi^{-1}r(j)$. Hence, $f(a_0)<g(a_0)$. But $f\leq g$, i. e. $g(a)\leq f(a)$ for every $a\in A$. The contradiction shows, that $\mu_A(f)\leq\mu_A(g)$.

Lemma 6. *Let $A\in\mathrm{Obj}\,\mathscr{R}$ and the set* $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ *is ordered as in Lemma 5. Then the map* $(\mu_A)^{-1}:\Phi A\to\mathrm{Mor}_{\mathscr{R}}(A, 2)$ *is increasing.*

Proof. Let $x_1, x_2\in\Phi A$ and $x_1\leq x_2$, $x_1\neq x_2$. Let $f_i=(\mu_A)^{-1}(x_i)$, $i=1, 2$. Then $f_i\in\mathrm{Mor}_{\mathscr{R}}(A, 2)$, $i=1, 2$. We must consider two cases:

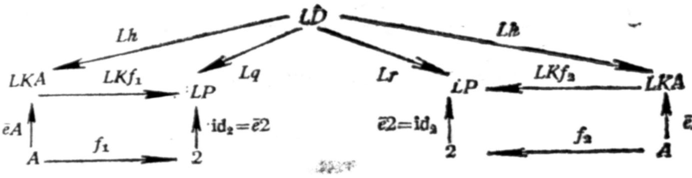
(i) Let $\Phi^{-1}q(j)=1$. Let us assume, that $f_1\leq f_2$.

Since the set $\mathrm{Mor}_{\mathscr{R}}(A, 2)$ in this case is ordered by $f\leq g$ iff $f(a)\leq g(a)$ for every $a\in A$, if we endow this set with a topology as in Lemma 4, then we shall obtain, that it coincides with the ordered topological space KA (see the definition of KA). Hence, it is a compact T. O. D. space. It follows that there exists a clopen increasing subset U of the space KA, such that $f_1\in U$, $f_2\notin U$. Let $h:KA\to\hat{D}$ is defined by $h^{-1}(\bar{1})=U$. Then $h\in\mathrm{Mor}_{\mathscr{H}}(KA, \hat{D})=\mathrm{Mor}_{\mathscr{P}}(KA, \hat{D})$ and $h(f_1)=\bar{1}$, $h(f_2)=\bar{0}$.

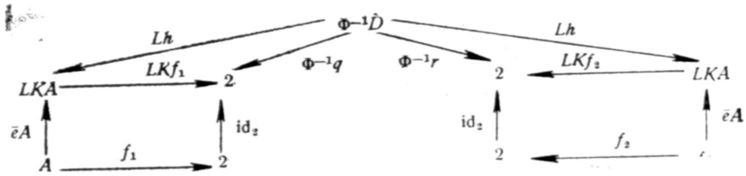
Since $K2=P$, we get maps $Kf_i:P\to KA$, $i=1, 2$. From the definition of the functor K we obtain, that $Kf_i(p)=Kf_i(\mathrm{id}_2)=f_i$, $i=1, 2$. Hence, $(h\circ Kf_1)(p)=\bar{1}$ and $(h\circ Kf_2)(p)=\bar{0}$. Now, if we define the map r by

$$r\begin{cases} P\longrightarrow\hat{D}\\ p\longrightarrow\bar{0},\end{cases}$$

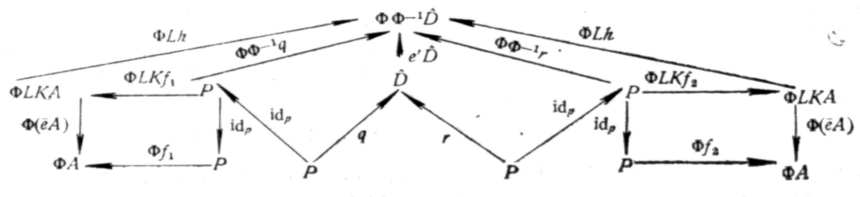
then the following diagram commutes:



Here by $\bar{e}$ is denoted the natural equivalence $1_{\mathscr{R}} \approx L \circ K$. From the definition of L we have that $L\hat{D}=\{\{\varnothing\}, \{\bar{1}\}, \{\hat{D}\}\}=R_0=\{0, j, 1\}$, where 0 corresponds to $\{\varnothing\}$, 1 — to $\{\hat{D}\}$ and j — to $\{\bar{1}\}$. Since $LP=\{\{\varnothing\}, \{P\}\}$, i. e. $LP=2=\{0, 1\}$, we obtain that $Lq(j)=Lq(\{\bar{1}\})=q^{-1}(\{\bar{1}\})=\{P\}=1=\Phi^{-1}q(j)$. Hence, $Lq=\Phi^{-1}q$. Analogously, $Lr(j)=Lr(\{\bar{1}\})=r^{-1}(\{\bar{1}\})=\{\varnothing\}=0=\Phi^{-1}r(j)$. Therefore $Lr=\Phi^{-1}r$ and we get the following commutative diagram



whence we obtain the following commutative diagram where by e' is denoted



the natural equivalence $1_{\mathscr{H}} \underset{e'}{\approx} \Phi \circ \Phi^{-1}$. We have that $x_i=\mu_A(f_i)=\Phi f_i(p)$, $i=1, 2$ and $x_1 \leq x_2$. The maps $(\Phi(\bar{e}A))^{-1}$, ΦLh and $(e'\hat{D})^{-1}$ are increasing, since they are morphisms in $\mathscr{H}$. Therefore we get, that $y_1=((e'\hat{D})^{-1} \circ \Phi Lh \circ (\Phi(\bar{e}A))^{-1} \circ \Phi f_1 \circ \mathrm{id}_p)(p)=((e'\hat{D})^{-1} \circ \Phi Lh \circ (\Phi(\bar{e}A))^{-1})(x_1) \leq ((e'\hat{D})^{-1} \circ \Phi Lh \circ (\Phi(\bar{e}A))^{-1})(x_2)$ $=((e'\hat{D})^{-1} \circ \Phi Lh \circ (\Phi(\bar{e}A))^{-1} \circ \Phi f_2 \circ \mathrm{id}_p)(p)=y_2$. But $y_1=((e'\hat{D})^{-1} \circ \Phi Lh \circ \Phi LKf_1)(p)=((e'\hat{D})^{-1} \circ \Phi\Phi^{-1}q)(p)=(q \circ (\mathrm{id}_p)^{-1})(p)=q(p)=\bar{1}$ and $y_2=((e'\hat{D})^{-1} \circ \Phi Lh \circ \Phi LKf_2)(p)=((e'\hat{D})^{-1} \circ \Phi\Phi^{-1}r)(p)=(r \circ (\mathrm{id}_p)^{-1})(p)=r(p)=\bar{0}$. Hence, we have that $y_1 > y_2$. The contradiction shows that $f_1 \leq f_2$.

(ii) Let $\Phi^{-1}q(j)=0$.

In this case the proof is analogous to the proof of the case (i). The only difference is that instead of the functors K and L we must use the functors $K'=E \circ K$ and $L'=L \circ E$, which are duality functors, too.

Proof of the Theorem 2. From the definition of the functor K ($K'=E\circ K$) we have that, for every $A\in \mathrm{Obj}\,\mathcal{R}$, the ordered topological space KA ($K'A$) coincides with the set $\mathrm{Mor}_{\mathcal{R}}(A, 2)$, endowed with the topology, described in Lemma 4, and with the order, described in Lemma 5 in the case when $\Phi^{-1}q(j)=1$ ($\Phi^{-1}q(j)=0$). From Lemmas 3, 4, 5, 6 it follows that, when $\Phi^{-1}q(j)=1$ ($\Phi^{-1}q(j)=0$), the map $\mu_A: KA\to \Phi A$ ($\mu_A: K'A\to \Phi A$) is an isomorphism in the category $\mathcal{H}$. Therefore ΦA is compact for every $A\in \mathrm{Obj}\,\mathcal{R}$ and hence, the category $\mathcal{H}$ coincides with the category $\mathcal{P}$ (from Lemma 1 it follows that $\mathcal{P}\subset\mathcal{H}$). Now, from Lemma 3, we obtain that either the functors K and Φ or the functors $K'=E\circ K$ and Φ are natural equivalent, since $h_2 f=Kf=K'f$ for every $f\in \mathrm{Mor}_{\mathcal{R}}(A_1\ A_2)$ (where $A_1, A_2\in \mathrm{Obj}\,\mathcal{R}$).

REFERENCES

1. P. R. Halmos. Lectures on Boolean Algebras. Berlin, 1974.
2. H. A. Priestley. Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, **2**, 1970, 186-190.
3. Iv. Prodanov. An axiomatic characterization of the Pontrjagin duality (to appear).
4. M. H. Stone. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.*, **41**, 1937, 321-364.
5. M. H. Stone. Topological representation of distributive lattices and Brouwerian logics. *Cas. Mat. Fys.*, **67**, 1937, 1-25.
6. G. D. Dimov. On the Stone duality. In: *General Topology and Its Relations to Modern Analysis and Algebra V*. Proc. Fifth Prague Topol. Symp. 1981, 145-146.

Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

Received 8. 3. 1982