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ON THE DEPENDENCE OF THE DIFFERENTIAL PROPERTIES OF A FUNCTION ON ITS BEST ALGEBRAIC APPROXIMATION

KAMEN G. IVANOV

We investigate the following problem: if for some natural r the series $\Sigma_{\nu=1}^{\infty} \nu^{r-1} E_\nu(f)_p$ converges, where $E_\nu(f)_p$ is the best algebraic approximation of the function $f \in L_p[-1, 1]$ $1 \leq p \leq \infty$), then what can we say about the existence of the derivatives of f and some of their structural properties.

1. Introduction. The problem we consider is one of the so called inverse problems in approximation theory. The investigations of that sort for trigonometrical approximation have been started by Bernstein and have been put in an appropriate form by T i m a n (see [1, p. 346]) and the corresponding references. The algebraic version is more complicated because of the effect of the ends. Nevertheless numerous articles dealing with this problem have appeared. Let us mention the following three papers: in [2] F u k s m a n considers the uniform approximation and use the local moduli of continuity; in [3] P o t a p o v deals with the L_p approximation but uses structural characteristics based on the modified translation concept; in [4] S t e n s connects Fuksman's results with results in terms of the modified transformation and the modified derivative in uniform metric. In the present article we give an algebraic analog of Timan's theorem [1, p. 346] using as a structural characteristic moduli based on the usual translation. As a consequence we get an equivalent form of the inverse part of Fuksman's results. The paper contains the proofs of some results announced in [5].

2. Definitions and preliminaries. We shall consider functions belonging to the spaces $L_p = L_p[-1, 1]$ $(1 \leq p \leq \infty)$ with the norm $\| f \|_p = [\int_{-1}^{1} | f(x) |^p dx]^{1/p}$ for $1 \leq p < \infty$. As usual when such problems are considered, $L_\infty[-1, 1]$ denotes $C[-1, 1]$, with sup norm. Let H_n be the set of all algebraic polynomials of a degree at most n. If $w \in C[-1, 1]$, $w \geq 0$, then the best L_p approximation of the function $f \in L_p[-1, 1]$ by elements of H_n with the weight w is $E_n(w; f)_p = \inf\{ \| (f-Q)w \|_p : Q \in H_n\}$; $E_n(1; f)_p = E_n(f)_p$.

For $x \in [-1, 1]$; $t > 0$; $i, r, n \in \mathbf{N} = \{1, 2, \dots\}$ we set $\Delta(t, x) = t\sqrt{1-x^2} + t^2$; $\Delta_n(x) = \Delta(n^{-1}, x)$; $\psi_{r,i}(x) = \psi_i(x) = (1-x^2)^{i-r/2}$ for $[r/2] < i \leq r$ and $\psi_{r,i}(x) = 1$ for $1 \leq i \leq [r/2]$.

We shall need the following properties of $\Delta(t, x)$:

$$\text{If } \lambda > 0, \quad x, y \in [-1, 1], \quad | x-y | \leq \lambda \Delta(t, x), \text{ then} \tag{2.1}$$
$$\Delta(t, x)/(4\lambda+2) \leq \Delta(t, y) \leq (2\lambda+2)\Delta(t, x)$$

The property (2.1) is proved under the additional conditions $\lambda \geq 1$, $2\lambda t \leq 1$ in [6] — inequality (2.5). The proof of (2.1) is similar without these restrictions

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Further on $c(A, B, \dots)$ will denote a positive constant depending only on the parameters in brackets. These constants may differ at each occurrence.

As structural characteristics we shall use the moduli $(k \in \mathbb{N})$

$$\tau_k(f, w; \theta)_{q,p} = \| w(\cdot)\omega_k(f, \cdot; \theta(\cdot))_q \|_p, \tag{2.2}$$

where

$$\omega_k(f, x; \theta(x))_q = \Big[\frac{1}{2\theta(x)} \int_{-\theta(x)}^{\theta(x)} |\Delta_v^k f(x)|^q dv\Big]^{1/q} \tag{2.3}$$

for $1 \leq q < \infty$ and

$$\omega_k(f, x; \theta(x))_\infty = \sup\{|\Delta_v^k f(x))|: |v| \leq \theta(x)\}.$$

In (2.3) the finite difference $\Delta_v^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+iv)$ is defined as 0 if x or $x+kv$ are not in $[-1, 1]$. In (2.2): θ may depend on various parameters and is positive being a function of $x \in [-1, 1]$; the weight w is non-negative and continuous; $1 \leq p \leq \infty$; $1 \leq q \leq \infty$. For the properties of moduli (2.2) see [6]. Among thiese properties we shall use $(1 \leq p, q \leq \infty$, real α, $f, g \in L_p[-1, 1])$:

$$\tau_k(f+g, w; \theta)_{q,p} \leq \tau_k(f, w; \theta)_{q,p} + \tau_k(g, w; \theta)_{q,p}; \tag{2.4}$$

$$\tau_k(f, w_1; \theta)_{q,p} \leq \tau_k(f, w_2; \theta)_{q,p} \text{ if } w_1(x) \leq w_2(x) \tag{2.5}$$

for each $x \in [-1, 1]$;

$$\tau_k(f, (n\Delta_n)^\alpha; \Delta_n)_{p,p} \leq c(k, \alpha) \| f(n\Delta_n)^\alpha \|_p; \tag{2.6}$$

$$\tau_k(f, (n\Delta_n)^\alpha; \Delta_n)_{p,p} \leq c(k, \alpha) n^{-k} \| f^{(k)}(n\Delta_n)^{\alpha+k} \|_p \text{ for } f^{(k)} \in L_p. \tag{2.7}$$

Let us mention that inequalities (2.4) and (2.5) follow immediately from (2.2); inequalities (2.6) and (2.7) are consequences of Theorem 4.1 and Corollary 4.3 in [6]. We shall need the following three inequalities about polynomials

$$\| Q' \|_p \leq cn^2 \| Q \|_p \quad \text{for each } Q \in H_n; \tag{2.8}$$

$$\| (\sqrt{1-x^2})^{k+\mu} Q^{(k)}(x) \|_p \leq c(k, \mu)\; n^k \| (\sqrt{1-x^2})^\mu Q(x) \|_p \tag{2.9}$$

for every $Q \in H_n$, $\mu \geq 0$;

$$\| (n\Delta_n)^{k+\mu} Q^{(k)}(x) \|_p \leq c(k, \mu) m^k \| (n\Delta_n)^\mu Q \|_p \tag{2.10}$$

for each $m \leq n$ and each $Q \in H_m$, $\mu \geq 0$.

Inequalities (2.8) and (2.9) are given in [7] and [8] respectively. Inequality (2.10) is proved in Corollary 5 in [9]. We shall also use the following obvious inequality

$$(a+b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha) \text{ for each } \alpha, a, b > 0. \tag{2.11}$$

We finish this section with

Lemma 1. *Let* $1 \leq p \leq \infty$, $f \in L_p[-1, 1]$, $\psi(x) = \sqrt{1-x^2}$, μ *be real. Then*

$$|\tau_1(f\psi, (n\Delta_n)^\mu; \Delta_n)_{p,p} - \tau_1(f, \psi(n\Delta_n)^\mu; \Delta_n)_{p,p}| \leq c(\mu) n^{-1} \| (n\Delta_n)^\mu f \|_p.$$

Proof. Let $|x-y|\le\Delta_n(x)$, $x, y\in[-1, 1]$. Then

$$|\psi(x)-\psi(y)|=|\int_x^y \frac{udu}{\sqrt{1-u^2}}|\le|\int_x^y \frac{du}{\sqrt{1-u^2}}|.$$

Let $x=\cos t$ for some $t\in[0,\pi]$. In the above integral we set $u=\cos v$. Now Lemma 3 in [10] yields

$$|\psi(x)-\psi(y)|\le\max\{\int_t^{t+4n^{-1}} dt, \int_{t-4n^{-1}}^{t} dt\}=4n^{-1}.$$

This inequality, (2.3) and the identity

$$f(x)\psi(x)-f(y)\psi(y)=[f(x)-f(y)]\psi(x)+f(y)[\psi(x)-\psi(y)]$$

give

$$|w_1(f\psi, x; \Delta_n)_p-\psi(x)w_1(f, x; \Delta_n)_p|\le 4n^{-1}[\frac{1}{2\Delta_n(x)}\int_{J(x,\Delta_n(x))}|f(y)|^p dy]^{1/p},$$

where $J(x, g(x))=[x-g(x), x+g(x)]\cap[-1, 1]$. Now using Lemma 2.2 in [6] and (2.1) we get

$$|\tau_1(f\psi, (n\Delta_n)^\mu; \Delta_n)_{p,p}-\tau_1(f, \psi(n\Delta_n)^\mu; \Delta_n)_{p,p}|$$

$$\le\frac{4}{n}[\int_{-1}^{1}\frac{(n\Delta_n(x))^{\mu p}}{2\Delta_n(x)}\int_{J(x,\Delta_n(x))}|f(y)|^p dy\, dx]^{1/p}$$

$$\le\frac{4}{n}[\int_{-1}^{1}|f(y)|^p\int_{J(y,(2+\sqrt{3})\Delta_n(y))}\frac{(n\Delta_n(x))^{\mu p}}{2\Delta_n(x)}dx\, dy]^{1/p}$$

$$\le c(\mu)n^{-1}[\int_{-1}^{1}|(n\Delta_n(y))^\mu f(y)|^p dy]^{1/p}=c(\mu)n^{-1}\|(n\Delta_n)^\mu f\|_p.$$

3. The main theorem. Theorem 1. *Let $f\in L_p[-1, 1]$ $(1\le p\le\infty)$ and the series $\Sigma_{\nu=1}^{\infty}\nu^{r-1}E_\nu(f)_p$ converge for some $r\in\mathbb{N}$. Then $f=F$ almost everywhere, if $p<\infty$, or $f=F$, if $p=\infty$, F has $(r-1)$st locally absolutely continuous derivative in $(-1, 1)$, $F^{(i)}.\psi_{r,i}\in L_p[-1, 1]$ for $i=1, 2, \dots, r$ and*

$$\tau_k(F^{(i)}, (n\Delta_n)^i; \Delta_n)_{p,p} \tag{3.1}$$

$$\le c(k, r)[n^{-k}\sum_{\mu=0}^{n}(\mu+1)^{i+k-1}E_\mu(f)_p+n^{-i}\sum_{\mu=n+1}^{\infty}\mu^{2i-1}E_\mu(f)_p]$$

for $i=0, 1, \dots, \rho=[r/2]$;

$$\tau_k(F^{(i)}\psi_{r,i}, (n\Delta_n)^{r-i}; \Delta_n)_{p,p} \tag{3.2}$$

$$\le c(k, r)[n^{-k}\sum_{\mu=0}^{n}(\mu+1)^{i+k-1}E_\mu(f)_p+n^{i-r}\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_n(f)_p]$$

for $i=\rho+1, \rho+2, \dots, r$ for $k=1$, if r is odd, or for each k, if r is even.

Remark. If the function φ is not a constant in the interval $[-1, 1]$, then the orders of the moduli of a function g and the function $g\varphi$ are different in general. If φ has a continuous kth derivative for $k=1, 2, \dots$, or φ

satisfies certain weaker conditions for $k=1$, then the orders of the kth moduli of g and $g\varphi$ coincide in many cases (but not always, e. g. $g\in H_{k-1}$). Thus, the behaviour of the moduli of $g\varphi$ is not significant enough for the structure of g. But it is impossible to omit the factor $\psi_{r,i}$ in (3.2) because it prevents a possible fast increasing of $F^{(i)}$ at the end-points of the interval and ensures $F^{(i)}\psi_{r,i}\in L_p$. On the other hand, this increasing cannot influence the good approximation of F because of the properties of the algebraic polynomials as an approximating system. Let us mention that the factor $\psi_{r,i}$ also occurs in Fuksman's [2] and Stens' [4] articles.

Proof of Theorem 1. We follow the idea of [1, p. 346]. Let P_ν be the polynomial of the best L_p approximation of f of degree ν, i. e.

$$E_\nu(f)_p=\|f-p_\nu\|_p. \tag{3.3}$$

For $i=0, 1, \dots, r$ we consider the sequence $S_n^{(i)}=p_1^{(i)}+\sum_{\nu=0}^{n-1}\{p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)}\}$.

a) $i=0, 1, \dots, \rho$. Then (2.8) and (3.3) give

$$\|p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)}\|_p\leq c(i)2^{(\nu+1)2i}\|p_{2^{\nu+1}}-p_{2^\nu}\|_p$$

$$\leq c(i)2^{(\nu+1)r}\{\|p_{2^{\nu+1}}-f\|_p+\|p_{2^\nu}-f\|_p\}\leq c(i)2^{(\nu-1)(r-1)+2r+\nu-1}\,.\,2E_{2^\nu}(f)_p$$

$$\leq c(i)2^{2r+1}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}(2^{\nu-1})^{r-1}E_{2^\nu}(f)_p\leq c(i,r)\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{r-1}E_\mu(f)_p.$$

Hence for $m>n\geq N$ we have

$$\|S_n^{(i)}-S_m^{(i)}\|_p=\|\sum_{\nu=n}^{m-1}(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)})\|_p\leq\sum_{\nu=n}^{m-1}\|p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)}\|_p$$

$$\leq c(r,i)\sum_{\mu=2^{n-1}+1}^{2^{m-1}}\mu^{r-1}E_\mu(f)_p\leq c(r,i)\sum_{\mu=2^{N-1}+1}^{\infty}\mu^{r-1}E_\mu(f)_p.$$

Therefore, the sequence $\{S_n^{(i)}\}_{n=1}^\infty$ converges in $L_p[-1,1]$ to $f_i\in L_p$.

b) $i=\rho+1, \rho+2, \dots, r$. Then (2.8), (2.9) and (3.3) give

$$\|(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)})\psi_i\|_p=\|(p_{2^{\nu+1}}-p_{2^\nu})^{(i)}(x)(\sqrt{1-x^2})^{2i-r}\|_p$$

$$\leq c(r)(2^{\nu+1})^{2i-r}\|(p_{2^{\nu+1}}-p_{2^\nu})^{(i+r-2i)}\|_p\leq c(r)(2^{\nu+1})^{2i-r+2(r-i)}\|p_{2^{\nu+1}}-P_{2^\nu}\|_p$$

$$\leq c(r)(2^{\nu+1})^r E_{2^\nu}(f)_p.$$

Arguing as in a) we get that the sequence $\{S_n^{(i)}\psi_i\}_{n=1}^\infty$ converges in $L_p[-1,1]$ to $g_i=\psi_i f_i\in L_p$. Hence there is a subsequence $\{S_{n_m}\}_{m=1}^\infty$ such that for each $i=0, 1, \dots, r$ we have

$$S_{n_m}^{(i)}\psi_i\xrightarrow[m\to\infty]{}f_i\psi_i\ \text{(a. e.) and}\ \|(S_{n_m}^{(i)}-f_i)\psi_i\|_p\xrightarrow[m\to\infty]{}0. \tag{3.4}$$

Let $x_0\in(-1,1)$ be such that

$$S_{n_m}^{(i)}(x_0)\xrightarrow[m\to\infty]{}f_i(x_0)\ \text{for}\ i=0, 1, \dots, r. \tag{3.5}$$

Let $x,\ x_0\in[-y,\ y]$ for $y\in]1/2,\ 1)$. For $i=1,\ 2,\dots\ r$ we set $\Phi(x)=f_{i-1}(x)-f_{i-1}(x_0)-\int_{x_0}^{x}f_i(t)dt$. Then

$$|\Phi(x)|=|f_{i-1}(x)-S_{n_m}^{(i-1)}(x)-f_{i-1}(x_0)+S_{n_m}^{(i-1)}(x_0)-\int_{x_0}^{x}[f_i(t)-S_{n_m}^{(i)}(t)]dt|$$

$$\leq|f_{i-1}(x)-S_{n_m}^{(i-1)}(x)|+|f_{i-1}(x_0)-S_{n_m}^{(i-1)}(x_0)|+\int_{-y}^{y}|f_i(t)-S_{n_m}^{(i)}(t)|\,dt.$$

Hence

(3.6)
$$\|\Phi\|_{L_p[-y,y]}$$
$$\leq\|f_{i-1}-S_{n_m}^{(i-1)}\|_{L_p[-y,y]}+(2y)^{1/p}\{|f_{i-1}(x_0)-S_{n_m}^{(i-1)}(x_0)|+\int_{-y}^{y}|f_i(t)-S_{n_m}^{(i)}(t)|\,dt\}$$
$$\leq\|\psi_{i-1}(f_{i-1}-S_{n_m}^{(i-1)})\|_{L_p[-y,y]}/\psi_{i-1}(y)+2|f_{i-1}(x_0)-S_{n_m}^{(i-1)}(x_0)|$$
$$+2\|\psi_i(f_i-S_{n_m}^{(i)})\|_{L_1[-y,y]}/\psi_i(y)$$
$$\leq\|\psi_{i-1}(f_{i-1}-S_{n_m}^{(i-1)})\|_{L_p[-y,y]}/\psi_{i-1}(y)+2|f_{i-1}(x_0)-S_{n_m}^{i-1}(x_0)|$$
$$+4\|\psi_i(f_i-S_{n_m}^{(i)})\|_{L_p[-y,y]}/\psi_i(y).$$

Let m tend to infinity in the last line of (3.6). Then using (3.4) and (3.5) we get $\|\Phi\|_{L_p[-y,y]}=0$. Since y can be chosen arbitrarily closed to 1, we have $f_{i-1}(x)=f_{i-1}(x_0)+\int_{x_0}^{x}f_i(t)dt$ a. e. in $(-1,\ 1)$ for $1=1,\ 2,\dots,\ r$.

Remark 2. Here "a, e." means "for almost all x" if $p<\infty$ and "for all x" if $p=\infty$.

Therefore f coincides a. e. with a function F having $r-1$st locally absoutely continuous derivative in $(-1,\ 1)$ and $F^{(\rho)}\in L_p[-1,\ 1]$, $F^{(i)}\psi_i\in L_p[-1,1]$ for $i=\rho+1,\ \rho+2,\dots,\ r$.

We shall now prove (3.1), i. e. $i=0,\ 1,\dots,\ \rho$. We set $m=[\ln n/\ln 2]+1$. Then (2.4) give

(3.7) $\tau_k(f^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}\leq\tau_k(f^{(i)}-S_m^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}+\tau_k(S_m^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}.$

Using (2.4), (2.7) and (2.10) we get

(3.8)
$$\tau_k(S_m^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}\leq\tau_k(p_1^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}$$
$$+\sum_{\nu=0}^{m-1}\tau_k(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}$$
$$\leq c(k)\{\|n^i(\Delta_n)^{k+1}(p_1^{(i+k)}-p_0^{(i+k)})\|_p+\sum_{\nu=0}^{m-1}\|n^i(\Delta_n)^{k+i}(p_{2^{\nu+1}}^{(k+i)}-p_{2^\nu}^{(k+i)})\|_p$$
$$\leq c(k)\,.\,n^{-k}\{\|p_1-p_0\|_p+\sum_{\nu=0}^{m-1}2^{(\nu+1)(k+i)}\|p_{2^{\nu+1}}-p_{2^\nu}\|_p\}$$
$$\leq c(k)n^{-k}\sum_{\mu=0}^{n}(\mu+1)^{k+i-1}E_\mu(f)_p.$$

From (2.6), (2.11), (2.9) and (2.8) we obtain

$$(3.9)\qquad \tau_k(f^{(i)}-S_m^{(i)},\ (n\Delta_n)^i;\ \Delta_n)_{p,p}\leq c(k)\,\|\,(f^{(i)}-S_m^{(i)})(n\Delta_n)^i\,\|_p$$

$$\leq c(k,\ i)\sum_{\nu=m}^{\infty}\|\,(p_{2^{\nu+1}}(x)-p_{2^\nu}^{(i)}(x))(\sqrt{1-x^2})^i\,\|_p+n^{-i}\,\|\,p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)}\,\|_p\}$$

$$\leq c(k,\ i)\sum_{\nu=m}^{\infty}\{2^{(\nu+1)i}\,\|\,p_{2^{\nu+1}}-p_{2^\nu}\,\|_p+n^{-i}2^{(\nu+1)2i}\,\|\,p_{2^{\nu+1}}-p_{2^\nu}\,\|_p\}$$

$$\leq c(k,\ i)n^{-i}\sum_{\nu=m}^{\infty}2^{(\nu+1)2i}\,\|\,p_{2^{\nu+1}}-p_{2^\nu}\,\|_p\leq c(k,\ i)\,n^{-i}\sum_{\mu=n+1}^{\infty}\mu^{2i-1}E_\mu(f)_p.$$

Inequalities (3.7), (3.8) and (3.9) prove (3.1).

We shall now prove (3.2), i. e. $i=\rho+1,\ \rho+2,\ldots,r$. Let k, r be those from the condition of the theorem and $m=[\ln n/\ln 2]+1$. Then (2.4) give

$$(3{\cdot}10)\qquad \tau_k(F^{(i)}\psi_i,\ (n\Delta_n)^{r-i};\ \Delta_n)_{p,p}$$

$$\leq\tau_k((F^{(i)}-S_{m+1}^{(i)})\psi_i,\ (n\Delta_n)^{r-i};\ \Delta_n)_{p,p}+\tau_k(S_{m+1}^{(i)}\psi_i,\ (n\Delta_n)^{r-i},\ \Delta_n)_{p,p}.$$

Using (2.6), (2.11), (2.9), (2.8) and the inequality $n\,2^{-m}\leq 1$ we get

$$(3.11)\qquad \tau_k((F^{(i)}-S_{m+1}^{(i)})\psi_i,\ (n\Delta_n)^{r-i};\ \Delta_n)_{p,p}\leq c(k,\ r)\,\|\,(F^{(i)}-S_{m+1}^{(i)})\psi_i(n\Delta_n)^{r-i}\,\|_p$$

$$\leq c(k,\ r)n^{r-i}\sum_{\nu=m+1}^{\infty}\{\,\|\,(p_{2^{\nu+1}}(x)-p_{2^\nu}(x))^{(i)}(1-x^2)^{i/2}\,\|_p$$

$$+n^{i-r}\,\|\,(p_{2^{\nu+1}}(x)-p_{2^\nu}(x))^{(i)}(1-x^2)^{i-r/2}\,\|_p$$

$$\leq c(k,\ r)\sum_{\nu=m+1}^{\infty}\{2^{(\nu+1)i}\,\|p_{2^{\nu+1}}-p_{2^\nu}\,\|_p+n^{i-r}2^{(\nu+1)(2i-r+2(r-i))}\|\,p_{2^{\nu+1}}-p_{2^\nu}\,\|_p\}$$

$$\leq c(k,\ r)\sum_{\nu=m+1}^{\infty}[1+(n2^{-\nu-1})^{-i+r}]n^{i-r}2^{(\nu+1)r}(\,\|p_{2^{\nu+1}}-f\,\|_p+\|\,p_{2^\nu}-f\,\|_p)$$

$$\leq c(k,\ r)n^{i-r}\sum_{\nu=m+1}^{\infty}2^{(\nu+1)r}E_{2^\nu}(f)_p\leq c(k,\ r)n^{i-r}\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_\mu(f)_p.$$

When proving (3.11) we do not pay attention whether r is even or odd because we do not use the differential properties of modulus τ_k.

Let first r be even ($r=2\rho$). Then $\psi_{r,i}(x)=(1-x^2)^{i-\rho}$ is a polynomial of degree $2(i-\rho)$, i. e. the function $\psi_{r,\,i}$ has not any singularities at the end-points of the interval. Therefore we can prove (3.2) for arbitrary k.

Using (2.7) and (2.10) we get

$$(3.12)\qquad \tau_k(S_{m+1}^{(i)}\,\psi_i,\ (n\Delta_n)^{r-i},\ \Delta_n)_{p,p}\leq c(k,\ r)n^{-k}\,\|\,(n\Delta_n)^{k+r-i}\,(S_{m+1}^{(i)}\,\psi_i)^{(k)}\,\|_p$$

$$\leq c(k,\ r)n^{-k}\{\,\|\,(n\Delta_n)^{k+r-i}(\,p_1^{(i)}\,\psi_i)^{(k)}\,\|_p+\sum_{\nu=0}^{m}\|\,(n\Delta_n)^{k+r-i}[(\,p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)}\,)\psi_i]^{(k)}\,\|_p\}$$

$$\leq c(k,\ r)n^{-k}\{(2(i-\rho))^k\,\|\,(n\Delta_n)^{r-i}\,p_1^{(i)}\,\psi_i\,\|_p+\sum_{\nu=0}^{m}(2(i-\rho)2^{\nu+1})^k\,\|\,(n\Delta_n)^{r-i}(\,p_{2^{\nu+1}}^{(i)}$$

$$-p_{2^\nu}^{(i)})\psi_i\|_p\}\le c(k,r)n^{-k}\{\|(n\Delta_n)^i(p_1^{(i)}-p_0^{(i)})\|_p+\sum_{\nu=0}^{m}2^{(\nu+1)k}\|(n\Delta_n)^i(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)})\|_p\}$$

$$\le c(k,r)n^{-k}\{\|p_1-p_0\|_p+\sum_{\nu=0}^{m}2^{(\nu+1)(k+i)}\|p_{2^{\nu+1}}-p_{2^\nu}\|_p\}$$

$$\le c(k,r)n^{-k}\{E_0(f)_p+\sum_{\nu=0}^{m}2^{(\nu+1)(k+1)}E_{2^\nu}(f)_p\}\le c(k,r)n^{-k}\sum_{\mu=0}^{n}(\mu+1)^{k+i-1}E_\mu(f)_p.$$

Now (3.10), (3.11) and (3.12) prove (3.2) for even r.

Let r be odd ($r=2\rho+1$). Therefore, $\psi_{r,i}$ is not a polynomial and has singularities at the end-points of the interval. Using Lemma 1, (2.7), (2.10) and (2.5) we get

$$\tau_1(S_{m+1}^{(i)}\psi_i,(n\Delta_n)^{r-i};\Delta_n)_{p,p} \tag{3.13}$$

$$\le\tau_1(S_{m+1}^{(i)}(1-x^2)^{i-\rho-1},\sqrt{1-x^2}(n\Delta_n)^{r-1};\Delta_n)_{p,p}+c(r)n^{-1}\|S_{m+1}^{(i)}\psi_i(n\Delta_n)^{r-i}\|_p$$

$$\le\tau_1(S_{m+1}^{(i)}(1-x^2)^{i-\rho-1},(n\Delta_n)^{r-i+1};\Delta_n)_{p,p}+c(r)n^{-1}\|S_{m+1}^{(i)}(n\Delta_n)^i\|_p$$

$$\le c(r)n^{-1}\|(n\Delta_n)^{r-i+2}(S_{m+1}^{(i)}(1-x^2)^{i-\rho-1})'\|_p+c(r)n^{-1}\|S_{m+1}^{(i)}(n\Delta_n)^i\|_p$$

$$\le c(r)n^{-1}\{\|(n\Delta_n)^{r-i+2}(p_1^{(i)}(x)(1-x^2)^{i-\rho-1})'\|_p+\|p_1^{(i)}(n\Delta_n)^i\|_p$$

$$+\sum_{\nu=0}^{n}[\|n\Delta_n)^{r-i+2}((p_{2^{\nu+1}}^{(i)}(x)-p_{2^\nu}^{(i)}(x))(1-x^2)^{i-\rho-1})'\|_p+\|(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)})(n\Delta_n)^i\|_p]\}$$

$$\le c(r)n^{-1}\{\|(n\Delta_n)^{r-i+1}p_1^{(i)}(x)(\sqrt{1-x^2})^{2i-r-1}\|_p+\|p_1^{(i)}(n\Delta_n)^i\|_p$$

$$+\sum_{\nu=0}^{m}[2^{\nu+1}\|(n\Delta_n)^{r-i+1}(p_{2^{\nu+1}}^{(i)}(x)-p_{2^\nu}^{(i)}(x))(\sqrt{1+x^2})^{2i-r-1}\|_p$$

$$+\|(p_{2^{\nu+1}}^{(i)}-p_{2^\nu}^{(i)})(n\Delta_n)^i\|_p)\}\le c(r)n^{-1}\{\|(n\Delta_n)^i(p_1^{(i)}-p_0^{(i)})\|_p$$

$$+\sum_{\nu=0}^{m}2^{\nu+1}\|(n\Delta_n)^i(p_{2^{\nu+1}}-p_{2^\nu})^{(i)}\|_p\}\le c(r)n^{-1}\sum_{\mu=0}^{n}(\mu+1)^iE_\mu(f)_p.$$

We complete the proof of the theorem by applying (3.11) and (3.13) in (3.10).

4. Corollaries. 1. Under the conditions of Theorem 1 for every $k\in\mathbf{N}$ and $\alpha>\rho$ we have

$$E_{n+k}((n\Delta_n)^\rho;F^{(\rho)})_p\le c(k,r)[n^{-k}\sum_{\mu=0}^{u}(\mu+1)^{\rho+k-1}E_\mu(f)_p+n^{-\rho}\sum_{\mu=n+1}^{\infty}\mu^{2\rho-1}E_\mu(f)_p]. \tag{4.1}$$

We get (4.1) as a consequence of Theorem 1 and Stečkin's type assertion ($g\in L_p[-1,1]$)

$$E_{n+k}((n\Delta_n)^\rho;g)_p\le c(\rho,k)\ \tau_k(g,(n\Delta_n)^\rho;\Delta_n)_{p,p}$$

proved in [11].

2. Under the conditions of Theorem 1 we have for $\alpha>\rho$

(4.2) $$E_n(f)_p=0\ (n^{-\rho-\alpha}) \Leftrightarrow E_n((n\Delta_n)^\rho;\ F^{(\rho)})_p=0\ (n^{-\alpha}).$$

(4.2) follows from Corollary 1 and the inequality

(4.3) $$E_{n+\rho}(f)_p \leqq c(\rho)n^{-\rho}E_n((n\Delta_n)^\rho;\ F^{(\rho)})_p$$

proved in [9]. (4.1) and (4.3) show that $E_n(f)_p$ is naturally connected with the weighted approximation $E_n((n\Delta_n)^\rho;\ F^{(\rho)})_p$ of the derivative $F^{(\rho)}$ but not with $E_n(F^{(\rho)})_p$.

3. Using the simple fact that

$$\frac{1}{1+\sqrt{2}}\,t \leqq \frac{\varepsilon}{\sqrt{1-x^2}+\sqrt{\varepsilon}} \leqq t$$

for $\varepsilon=\Delta(t,\ x)$, we get the inverse part of Fuksman's result in [2] as a consequence of Theorem 1.

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Centre for Mathematics and Mechanics
Sofia 1090 P. O. Box 373

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