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ONESIDED APPROXIMATION WITH ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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The purpose of this paper is to give Jackson's type theorem for the best onesided approximation in $L_1(-\infty, +\infty)$ with entire functions of an exponential type by means of the moduli $\tau_1(f, \delta)_{L_1}$.

Onesided approximation of functions was first cosidered by G. Freud and T. Ganelius in [1] and [2]. They give the first nontrivial direct estimates for the best onesided polynomial and spline approximation.

We shall use the following modulus for the function $f(x)$:

$$\tau_1(f, \delta)_{L_1} = \| \omega_1(f, x, \delta) \|_{L_1}; \quad \omega_1(f, x, \delta) = \sup\{|f(t+h)-f(t)| : t, t+h \in [x-\delta, x+\delta]\}$$

calling it an average modulus. Moduli of this type were first considered by Bl. Sendov [3] and P. P. Korovkin [4]. Many properties of these moduli are given in [5] by Dolgenko and Sevastianov. This modulus possesses the following property: $\tau_1(f, \lambda\delta)_{L_1} \leqq (c_1 \lambda + 1)^{c_2} \tau_1(f, \delta)_{L_1}$, where c_1 and c_2 are constants, discribed in [7]. We can set $c_1 = 4$ and $c_2 = 4$ but these aren't the best possible constants.

V. Popov, A. Andreev and Bl. Sendov used this modulus and obtained [6] Jackson's type theorems for the onesided polynomial and spline approximation. V. Popov and A. Andreev used moduli of this type and obtained [7] Steckin's type theorems for onesided trigonometrical and spline approximation. V. Popov in [8], using these moduli gives the converse theorem for the onesided trigonometrical approximation.

Definition. *We say that $f(z)$ is an entire function of exponential type of order $\sigma \geqq 0$, if for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such, that for every z, $|z| > R_\varepsilon$, the inequality $|f(z)| \leqq e^{(\sigma+\varepsilon)|z|}$ holds.*

We denote by E_σ the set of all entire functions of exponential type of order σ, which are bounded above the real line.

Definition. *The best onesided approximation of $f(x)$ in $L_1(-\infty, +\infty)$ with the set E_σ is:*

$$\widetilde{\mathrm{E}}\mathrm{xp}(\sigma)(f)_{L_1} = \inf\{\| u_1 - u_2 \|; \quad u_1(x) \geqq f(x) \geqq u_2(x); \quad u_1, u_2 \in E_\sigma\}.$$

In this paper we shall prove the following theorem:

Theorem 1. *If $f(x)$ is a function such that*

i) $$\tau_1(f, \sigma^{-1}) < \infty,$$

ii) $$f(x) \in L_1(-\infty, +\infty)$$

then $\widetilde{\mathrm{E}}\mathrm{xp}(\sigma)(f)_{L_1} \leqq c\tau_1(f, \sigma^{-1})_{L_1}$ *where c is an absolute constant, and* $\sigma \geqq \mathrm{const} > 1$.

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To this end we shall prove some lemmas. Let's consider the function $F_{m,r}(t)=[t^{-1}\sin(mt/2)]^{2r}$. Evidently this function belongs to E_{mr}.

Our first aim is to approximate onesidedly the following stepfunction:

$$f(x)=\begin{cases} M, & x\in[-1,1],\\ 0, & x\notin[-1,1].\end{cases}$$

Lemma 1. *Let's determine $C_{m,r}$ as a constant depending on the parameters m and r such that $C_{m,r}\int_{-\infty}^{+\infty}F_{m,r}(t)dt=1$. If $m>0$ and $r>0$ then $C_{m,r}\leqq 0{,}5(\pi/m)^{2r-1}$ holds true.*

Proof. In the interval $[0,\pi/2]$ the function $x^{-1}\sin x$ is a decreasing one and therefore $x^{-1}\sin x\geqq 2/\pi$ for every $x\in[0,\pi/2]$. Then $(mt/2)^{-1}\sin mt/2\geqq 2/\pi$ where for t we have $0\leqq mt/2\leqq\pi/2$ or $0\leqq t\leqq\pi/m$, i. e. $t^{-1}\sin mt/2\geqq m/\pi$ holds $\forall\, t\in[0,\pi/m]$. Then we can write the following chain

$$C_{m,r}^{-1}=\int_{-\infty}^{+\infty}[t^{-1}\sin mt/2]^{2r}dt=2\int_0^{\infty}[t^{-1}\sin mt/2]^{2r}dt$$

$$\geqq 2\int_0^{\pi/m}[t^{-1}\sin mt/2]^{2r}dt\geqq 2\int_0^{\pi/m}(m/\pi)^{2r}dt=2(m/\pi)^{2r-1},$$

whereof it follows, that $C_{m,r}\leqq 0{,}5\,(\pi/m)^{2r-1}$.

Lemma 2. *Let $\tilde f(x)$ be a non-negative function. We denote*

$$\tilde\psi(x)=\int_{-\infty}^{+\infty}\tilde f(x+t)\,D_{m,r}(t)dt,$$

where $D_{m,r}(t)=C_{m,r}[t^{-1}\sin mt/2]^{2r}$ and $r>0{,}5$. Then the following inequality

$$|\tilde f(x)-\tilde\psi(x)|\leqq(\sup_{|x|<\infty}\tilde f(x)).(2r-1)^{-1}[\pi/(m\delta_x)]^{2r-1}+\omega_1(\tilde f,x,\delta_x)$$

holds, where $\delta_x>0$ we can choose depending on x.

Proof. We have

$$\tilde\psi(x)=\int_{-\infty}^{+\infty}\tilde f(x+t)\,D_{m,r}(t)\,dt=\int_{-\infty}^{+\infty}[\tilde f(x+t)-\tilde f(x)]D_{m,r}(t)dt+\tilde f(x).$$

Hereby

$$|\tilde\psi(x)-\tilde f(x)|\leqq\Big|\int_{-\infty}^{-\delta_x}[\tilde f(x+t)-\tilde f(x)]\,D_{m,r}(t)dt+\int_{\delta_x}^{+\infty}[\tilde f(x+t)-\tilde f(x)]D_{m,r}(t)dt\Big|$$

$$+\Big|\int_{-\delta_x}^{\delta_x}[\tilde f(x+t)-\tilde f(x)]\,D_{m,r}(t)dt\Big|$$

$$\leqq[\sup_{|x|<\infty}\tilde f(x)]\int_{\delta_x\leqq|x|<\infty}D_{m,r}(t)dt+\omega_1(\tilde f,x,\delta_x)\int_{-\delta_x}^{\delta_x}D_{m,r}(t)\,dt.$$

But on the other hand

$$C_{m,r}\int_{\delta_x\leqq|x|<\infty}[t^{-1}\sin mt/2]^{2r}dt=2C_{m,r}\int_{\delta_x}^{\infty}[t^{-1}\sin mt/2]^{2r}dt$$

$$\leqq 2C_{m,r}\int_{\delta_x}^{\infty}t^{-2r}\,dt=2C_{m,r}\,(2r-1)^{-1}\,\delta_x^{-2r+1}.$$

Applying Lemma 1 we obtain $2(2r-1)^{-1}C_{m,r}\,\delta_x^{-2r+1}\leq 0,52\,(2r-1)^{-1}\,[\pi/(m\delta_x)]^{2r-1}$ and

$$|\tilde{\psi}(x)-\tilde{f}(x)|\leq[\sup_{|x|<\infty}\tilde{f}(x)]\,(2r-1)^{-1}\,[\pi/m\delta_x]^{2r-1}+\omega_1(\tilde{f},\,x,\,\delta_x).$$

Theorem 2. *For the function*

$$f(x)=\begin{cases}M>0, & x\in[-1,\,1],\\ 0, & x\notin[-1,\,1]\end{cases}$$

we have $\widetilde{\text{Exp}}\,(\sigma)\,(f)_{L_1}\leq cM\sigma^{-1}$, *where c is an absolute constant.*

Proof. We consider the subsidiary function

$$\tilde{f}(x)=\begin{cases}M(1+\alpha/\sigma), & x\in[-1-(\pi e\ln\sigma)/2\sigma,\ 1+(\pi e\ln\sigma)/2\sigma],\\ 0\quad, & x\notin[-1-(\pi e\ln\sigma)/2\sigma,\ 1+(\pi e\ln\sigma)/2\sigma],\end{cases}$$

where we shall determine α later and now we want only $\alpha>0$. Obviously $\tilde{f}(x)\geq f(x)$. We substitute $m=2\sigma/\ln\sigma$ and $r=2^{-1}\ln\sigma$. Let now $-1\leq x\leq 1$. Then, if $\delta_x=(\pi e\ln\sigma)/2\sigma$ for every $x\in[-1,\,1]$ we obtain from Lemma 2, considering the function $\tilde{\psi}(x)$, which we have examined in the same Lemma:

$$|\tilde{f}(x)-\tilde{\psi}(x)|\leq M(1+\alpha/\sigma)\,(\ln\sigma-1)^{-1}\,(2\pi\sigma\ln\sigma)^{\ln\sigma-1}\,(2\pi\,e\ln\sigma)^{-\ln\sigma+1}$$
$$=M\,(1+\alpha/\sigma)\,e\,\sigma^{-1}.$$

If $\sigma>e^2$, then $|\tilde{f}(x)-\tilde{\psi}(x)|\leq M\,(1+\alpha/\sigma)e\,\sigma^{-1}$. We want $\tilde{f}(x)-\tilde{\psi}(x)\leq\tilde{f}(x)-f(x)$ for $-1\leq x\leq 1$. Evidently $\underset{[-1,1]}{\tilde{f}(x)}\geq\underset{[-1,1]}{\tilde{\psi}(x)}$ because $\int_{-\infty}^{+\infty}D_{m,r}(t)dt=1$. One sufficient condition for $f(x)\leq\tilde{\psi}(x)$ is $M(1+\alpha/\sigma)\,e\sigma^{-1}\leq\alpha M\sigma^{-1}$ or $\alpha\leq e\,(1-e/\sigma)$. Since $\sigma>e^2$ we can choose $\alpha=e^2/(e-1)$, for example.

Let's consider the domain

$$\mu=(-\infty,\,-1-(\pi(e+1)\ln\sigma)/(2\sigma)]\cup[1+(\pi(e+1)\ln\sigma)/(2\sigma),\,+\infty)$$

and choose $\delta_x=|x|-1-(\pi\,e\ln\sigma)/2\sigma$. Applying Lemma 2 we obtain:

$$\|\tilde{\psi}(x)-f(x)\|_{L_1(\mu)}=\|\tilde{\psi}(x)-\tilde{f}(x)\|_{L_1(\mu)}$$

$$\leq 2M\int_{1+(\pi(e+1)\ln\sigma)/2\sigma}^{\infty}(2r-1)^{-1}\,[\pi/(m(x-1-2^{-1}\sigma^{-1}\pi e\ln\sigma))]^{2r-1}\,dx=I\,(m,\,r,\,\sigma).$$

We change the argument $y=x-1-(\pi e\ln\sigma)/2\sigma$. Then we have

$$I\,(m,\,r,\,\sigma)=2M\int_{(\pi\ln\sigma)/2\sigma}^{+\infty}(\pi/m)^{2r-1}\,y^{-2r+1}dy$$

$$=2M\,(2r-1)^{-1}\,(2r-2)^{-1}\,(\pi/m)^{2r-1}\,[(\pi\ln\sigma)/2\sigma]^{-2r+2},$$

for $r>1$. After that we have

$$I\,(m(\sigma),\,r(\sigma),\,\sigma)=2M[(\ln\sigma-1)\,(\ln\sigma-2)]^{-1}\,(\pi\ln\sigma/2\sigma)^{\ln\sigma-1}\,(\pi\ln\sigma/2\sigma)^{-\ln\sigma+2}$$
$$=M\ln\sigma\,[(\ln\sigma-1)\,(\ln\sigma-2)]^{-1}\,\pi\,.\,\sigma^{-1}.$$

Now for $\sigma > l^2 A$, where $A > 1$, we obtain

$$\ln\sigma - 2 > \ln A\,; \quad \ln\sigma/(\ln\sigma - 1) = 1/(1 - 1/\ln\sigma) < (2 + \ln A)/(1 + \ln A)$$

$$\Rightarrow \ln\sigma\,[(\ln\sigma - 1)(\ln\sigma - 2)]^{-1} < (2 + \ln A)\,[(1 + \ln A)\ln A]^{-1}.$$

Let's consider the following domain

$$\gamma = [-1 - (\pi(e+1)\ln\sigma)/2\sigma,\ -1] \cup [1,\ 1 + (\pi(e+1)\ln\sigma)/2\sigma]$$

and let's estimate in $L_1(\gamma)$ the function $\int_{-\infty}^{+\infty} \widetilde{f}(x+t)\, c_{m,r}\, [t^{-1}\sin mt/2]^{2r} dt$ or find another method for approximation in $L_1(\gamma)$ of $f(x)$ with E_σ. Now we can write

$$(\pi(e+1)\ln\sigma)/2\sigma \leqq 2^{-1}\pi(e+1)(\ln A + 2)(e^2 A)^{-1}$$

$$= \pi(e+1)(2e^2)^{-1}(A^{-1}\ln A + A^{-1}2) < \pi(e+1)(2e^2)^{-1}(e^{-1} + 2) < 3,$$

because $\sigma > e^2 A$, where $A > 1$ is a fixed number. Here we use that the function $x^{-1}\ln x$ is decreasing for $x > e$.

So we obtain the following:

1. $\widetilde{\psi}(x) \leqq 2M$;
2. $\int_4^{\infty} [\widetilde{\psi}(x) - f(x)]dx \leqq C_1 M\sigma^{-1}$;
3. $\int_{-\infty}^{-4} [\widetilde{\psi}(x) - f(x)]d_x \leqq C_1 M\sigma^{-1}$;
4. $\int_{-1}^{1} [\widetilde{\psi}(x) - f(x)]dx \leqq C_4 M\sigma^{-1}$.

We define the following function:

$$f_1(x) = \begin{cases} 1, & x \in [2B_k - 1,\ 2B_k + 1], \quad k = 0, \pm 1, \pm 2, \dots \\ 0, & x \notin [2B_k - 1,\ 2B_k + 1], \quad k = 0, \pm 1, \pm 2, \dots, \end{cases}$$

where $B \geqq 4$ is a fixed number.
From [10] we can approximate $f_1(x)$ onesidedly in $[-B, B]$ with an element belonging to $E[\sigma]\pi B^{-1}$ and from the solution of this problem we know, that it is periodic with a period $2B$. We denote this solution by $\widetilde{\psi}_1(x)$ and for it we know the following:

$$\widetilde{\psi}_1(x) \leqq C_5\,; \quad \int_1^B [\widetilde{\psi}_1(x) - f_1(x)]dx \leqq C_2\sigma^{-1}\,; \quad \int_{-1}^{1} [\widetilde{\psi}_1(x) - f(x)]dx \leqq C_3\sigma^{-1}.$$

Let's consider the entire function $\widetilde{\psi}(x)\widetilde{\psi}_1(x) \in E_\sigma(1 + \frac{[\sigma]}{\sigma}\pi B^{-1}) \subset E_\sigma(1 + \pi B^{-1})$. We know, that $\widetilde{\psi}(x)\psi_1(x) \geqq f(x)$ and what is left is to estimate how close is $\widetilde{\psi}(x)\widetilde{\psi}_1(x)$ to $f(x)$ in $L_1(-\infty, +\infty)$:

$$\int_{-\infty}^{+\infty} [\widetilde{\psi}\widetilde{\psi}_1(x) - f(x)]dx = \int_{-1}^{1} [\widetilde{\psi}\widetilde{\psi}_1(x) - f(x)]dx + 2\int_1^B \widetilde{\psi}(x)\widetilde{\psi}_1(x)dx + 2\int_B^{+\infty} \widetilde{\psi}(x).\widetilde{\psi}_1(x)dx,$$

and from here we obtain

a) $$2\int_1^B \widetilde{\psi}\widetilde{\psi}_1(x)dx \leqq 4M\int_1^B (\widetilde{\psi}_1(x) - f_1(x))dx \leqq 4MC_1\,\sigma^{-1};$$

b) $$\int_{-1}^{1}(\tilde{\psi}\tilde{\psi}_1(x)-f(x))dx=\int_{-1}^{1}(\tilde{\psi}\tilde{\psi}_1(x)-M\,.\,1)dx=\int_{-1}^{1}\tilde{\psi}(x)[\tilde{\psi}_1(x)-1]dx$$

$$+\int_{-1}^{1}(\tilde{\psi}(x)-M)dx\leqq 2M\int_{-1}^{1}(\tilde{\psi}_1-1)dx+\int_{-1}^{1}(\tilde{\psi}(x)-M)dx\leqq M(2C_3+C_4)\sigma^{-1};$$

c) $$2\int_{B}^{+\infty}\tilde{\psi}(x)\tilde{\psi}_1(x)dx\leqq 2C_5\int_{B}^{+\infty}\tilde{\psi}(x)dx=2C_5\int_{B}^{+\infty}(\tilde{\psi}(x)-f(x)dx\leqq 2C_5C_1M\sigma^{-1}.$$

Finally we obtained the following integral estimate in $L_1(-\infty,+\infty)$:

$$\int_{-\infty}^{+\infty}[\tilde{\psi}\tilde{\psi}_1(x)-f(x)]dx\leqq M[(1+\pi B^{-1})(4C_2+2C_5C_1+2C_3+C_4)]\,[(1+\pi B^{-1})\sigma]^{-1},$$

which proves the theorem after the following note. If we want to approximate with an entire function of order σ we choose $\tilde{\psi}(x)$ with order $\sigma(1-\beta)$, $\tilde{\psi}_1(x)$ with order $[\sigma\beta]$, $\beta<1$. Then $\tilde{\psi}(x)\tilde{\psi}_1(x)\in E_\sigma$. In this way we obtain the approximation given above. By analogy we may approximate the step-function from below.

N o t e. Let's consider the functions

$$f(x)=\begin{cases}M, & x\in[-1,1]\\ 0, & x\notin[-1,1]\end{cases},\qquad g(x)=\begin{cases}M, & x\in[a,b]\\ 0, & x\notin[a,b]\end{cases}.$$

Evidently $f(x)=g[2^{-1}(b-a)x+2^{-1}(b+a)]$ and $g(t)=f[2t(b-a)^{-1}-(b+a)(b-a)^{-1}]$. We take a function $\tilde{\psi}(x)\in E_\sigma(b-a)2^{-1}$, which approximates onesidedly $f(x)$ with order $O(2M/\sigma(b-a))$. The function $\chi(t)=\tilde{\psi}\,[2t(b-a)^{-1}-(b+a)(b-a)^{-1}]$ belongs to $E_\sigma 2^{-1}(b-a)2(b-a)^{-1}=E_\sigma$. On the other hand $\chi(t)\geqq g(t)$ and

$$\int_{-\infty}^{+\infty}(\chi(x)-g(x))dx=\int_{-\infty}^{+\infty}(\tilde{\psi}(t)-f(t))dt\cdot\frac{b-a}{2}\leqq CM\sigma^{-1}.$$

Now we know that the step-function is approximated onesidedly by the set E_σ in $L_1(-\infty,+\infty)$ with order $O(M\sigma^{-1})$ independently from the length and the place of the support.

D e f i n i t i o n. *Let $f(x)$ be a real valued function defined in* $(-\infty,+\infty)$. *If there exists the limit*

$$\bigvee_{-\infty}^{+\infty}f=\lim_{n\to\infty}\bigvee_{-n}^{n}f<\infty,$$

we call it a variation of the function $f(x)$ in $(-\infty,+\infty)$.

E x a m p l e s. 1) If $f(x)=0$ for $x<x_0$, then it is evident that

$$\bigvee_{-\infty}^{+\infty}f=\bigvee_{x_0}^{+\infty}f+\bigvee_{-\infty}^{x_0}f=f(x_0)+\bigvee_{x_0}^{+\infty}f.$$

2) Let $f(x)$ has a first derivative, which is an absolute integrable. Applying the well known theorem for the final growths and the definition of Rieman's integral we obtain

$$\bigvee_{-+}^{+\infty}f=\int_{-\infty}^{+\infty}|f'(t)|\,dt=\|f'\|_{L_1}(-\infty,+\infty).$$

For example:

$$\bigvee_{-\infty}^{+\infty} e^{-|x|} = \bigvee_{-\infty}^{0} e^{x} + \bigvee_{0}^{+\infty} e^{-x} = \int_{-\infty}^{0} e^{x}dx + \int_{0}^{+\infty} e^{-x}dx = 2.$$

Lemma 3. *Let $f(x)$ be a function with a bounded variation and let's consider the functions $f^{+}(x) = \max(f(x), 0)$; $f^{-}(x) = \min(f(x), 0)$. Then*

i) $f(x) = f^{+}(x) + f^{-}(x)$;

ii) $f^{+}(x)$ *and* $f^{-}(x)$ *are also with a bounded variation and*

$$\bigvee_{-\infty}^{+\infty} f^{+} \leqq \bigvee_{-\infty}^{+\infty} f; \quad \bigvee_{-\infty}^{+\infty} f^{-} \leqq \bigvee_{-\infty}^{+\infty} f.$$

The proof of this lemma is trivial.

Let $f(x)$ be a function, for which

1) $\tau_1(f, \sigma^{-1})_{L_1} = \int_{-\infty}^{+\infty} \omega_1(f, x, \sigma^{-1})dx < \infty$,

(2) $\forall \varepsilon > 0$, $\forall R > 0$ $\exists\, x_i^{+}, x_i^{-}$; $(i = 1, 2)$: $x_1^{+} > 0$, $x_1^{-} > 0$, $x_2^{+} < 0$, $x_2^{-} < 0$ $|x_i^{+}| > R$, $|x_i^{-}| > R$ and $|f^{+}(x_i^{+})| < \varepsilon$; $|f^{-}(x_i^{-})| < \varepsilon$, $(i = 1, 2)$. If σ is a positive number, $\sigma \gg 4Ae^{2}$, we construct the functions $S_\sigma(x)$ and $J_\sigma(x)$ in the following way: we divide the real line to intervals with length σ^{-1} each. In every interval $S_\sigma(x)$ is equal to $\sup f(x)$ in this interval and $J_\sigma(x)$ is equal to $\inf f(x)$ in this interval. Then it's evident that

$$J_\sigma(x) \leqq f(x) \leqq S_\sigma(x) \;\forall\, x \in (-\infty, +\infty).$$

Let's consider the function $S_\sigma(x)$. According to the definition it's variation is the sum of its jumps. Then it is obvious from the definition of $\tau_1(f, \delta)_{L_1}$ and from its property on page 1 that

$$\sigma^{-1} \bigvee_{-\infty}^{+\infty} S_\sigma \leqq \tau_1(S_\sigma, 2^{-1}\sigma^{-1})_{L_1} \text{ and } \tau_1(S_\sigma, \sigma^{-1})_{L_1} \leqq \tau_1(f, 2\sigma^{-1})_{L_1} \leqq 9^{4}\tau_1(f, \sigma^{-1})_{L_1}.$$

It follows, that

$$\bigvee_{-\infty}^{+\infty} S_\sigma \leqq \sigma(2^{-1}4+1)^{4}\, \tau_1(S_\sigma, \sigma^{-1})_{L_1} \leqq 27^{4}\, \sigma\tau_1(f, \sigma^{-1})_{L_1} < \infty.$$

Here we conclude that $S_\sigma(x)$ has a bounded variation and $|S_\sigma(x)|$ is bounded from a constant depending on σ. The functions $S_\sigma^{+}(x)$ and $S_\sigma^{-}(x)$ also have a bounded variation according to Lemma 3.

Now we shall prove Theorem 1. There are three cases in the proof and every following contains the previous.

a) $\operatorname{supp} f(x)$ is bounded. Here 2) (page 9) is automatically fulfilled. Then $S_\sigma(x)$ and hence $S_\sigma^{+}(x)$ and $S_\sigma^{-}(x)$ have a finite number different from zero jumps. Considering $S_\sigma^{+}(x)$ we divide the ordinate on the non-zero values of $S_\sigma^{+}(x)$, which are finite numbers. We keep the following law: if the dividing line $y = y_0$ cuts a stem, where $S_\sigma^{+}(x)$ have a value greater or equal to y_0, we approximate the function, which is zero out of this stem and has a value equal to the distance from this cutting line to the next lower cutting line in it.

In this way we obtain the following construction:
For example from the functions

$$d_1(x)=\begin{cases} d, & x\in[A,B) \\ 0, & x\notin[A,B) \end{cases} \quad \text{and} \quad d_2(x)=\begin{cases} d, & x\in[B,C] \\ 0, & x\notin[B,C) \end{cases}$$

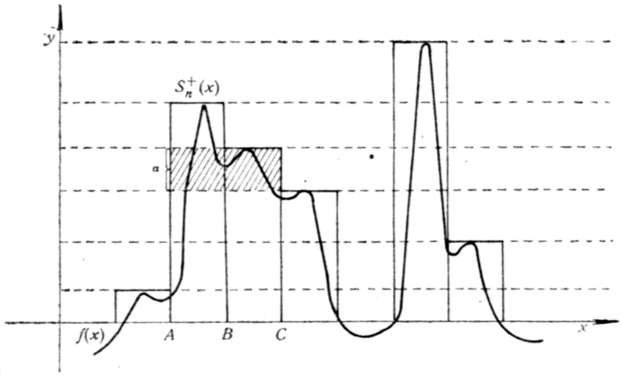

Fig. 1

we construct for approximation the function $d_1(x)+d_2(x)$ and we make this not only with two neighbour ones, but with all similar to them neighbour functions.

Applying Theorem 2 we approximate each one of the step-functions, obtained in this way with order $O(M'_i\sigma^{-1})$, where M'_i is the height of the corresponding step and σ^{-1} is the order of the exponential class of entire functions. Let the steps, which we have to approximate be k, and $T_1(x)$, $T_2(x)$, $T_3(x)$, $\dots$, $T_k(x)$ be the corresponding entire functions, which approximate the corresponding step-function. Then $T_\sigma^+(x)=\Sigma_{i=1}^k T_i(x)$ is an entire function, which belongs to E_σ, such that

$$\| S_\sigma^+(x)-T_\sigma^+(x)\|_{L_1}\leq c\,(\sum_{i=1}^{k} M'_i)\,\sigma^{-1}\leq c\,(\bigvee_{-\infty}^{+\infty} S_\sigma^+)\,\sigma^{-1}$$

and $T_\sigma^+(x)\geq S_\sigma^+(x)$. $T_\sigma^+(x)$ is an entire function, but it isn't clear whether $T_\sigma^+\in E_\sigma$. That will be proved in case b).

b) supp $f(x)$ is bounded from the left and unbounded from the right (for example). Then we make the division from the left end of the support to the right. We obtain $S_\sigma(x)$ and from there $S_\sigma^+(x)$. Here the values nonequal to zero can be infinite number. $S_\sigma^+(x)$ is a function with bounded variation and $\bigvee_{-\infty}^{+\infty} S_\sigma^+=\Sigma_{k=1}^\infty M_k$, from where it follows, that $M_k\to 0\ (k\to\infty)$, where M_k are differences of the heights of all neighbour stems, which belong to $S_\sigma^+(x)$.

Since $\Sigma M_k<\infty$, it follows, that the series, whose m-th sum is equal to the height of the m-th stem, is summable. Let Γ is the sum of this series. Then from

$$0\leq S_\sigma(x)-J_\sigma(x)\leq\omega_1(f,x,\sigma^{-1})\Rightarrow\| S_\sigma-J_\sigma\|_{L_1}\leq\tau_1(f,\sigma^{-1})_{L_1}<\infty,$$

and from 1) and 2) (page 9) it follows that $\Gamma=0$.

We consider $B_m=$[the set of the steps (which are finite numbers) with corresponding heights $M'_{m1}, M'_{m2}, \ldots, M'_{mi_m}$ for which the upper cutting line, participating in the forming of these steps has a level less or equal to $m^{-1}\sup f(x)$ and greater then $(m+1)^{-1}\sup f(x)$, where $m=1, 2, 3, \ldots$]. We consider B_1 and

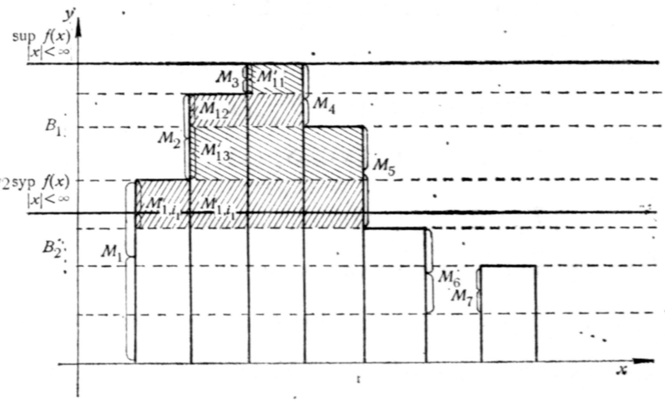

Fig. 2

make the same construction like in a). The functions, which we are approximating, are obtained to the last cut line, which is the first in B_2 and s. on. In this way we obtain the series $\Sigma_{k=1}^{\infty}\Sigma_{j=1}^{i_k} M'_{kj}$, and

$$\sum_{k=1}^{\infty}\sum_{j=1}^{i_k} M'_{k_j} \leqq \sum_{l=1}^{\infty} M_l = \bigvee_{-\infty}^{+\infty} S_\sigma^+$$

which is evidently from Fig. 2.

The series of entire functions, belonging to E_σ, which approximate the steps with corresponding heights $M'_{11}, M'_{12}, \ldots, M'_{1i_1}, M'_{21}, M'_{22}, \ldots, M'_{2i_2}, \ldots, M'_{k1}, M'_{k2}, \ldots, M'_{ki_k}, \ldots$ are denoted by $T_1, T_2, \ldots, T_{i_1}, T_{i_1+1}, T_{i_1+2}, \ldots, T_{i_1+i_2+\cdots+i_{k-1}+1}, \ldots, T_{i_1+i_2+\cdots+i_{2-1}+i_k}, \ldots$. We require that $T_\sigma^+(x)=\Sigma_{q=1}^{\infty} T_q(x)$ belongs to E_σ. Now we put a new condition, which follows from this requirement:

3) $$f(x)\in L_1(-\infty, +\infty).$$

Since

$$0\leqq S_\sigma(x)-f(x)\leqq \omega(f, x, \sigma^{-1}) \quad \text{so} \quad \| S_\sigma - f\|_{L_1} \leqq \tau(f, \sigma^{-1})_{L_1} < \infty,$$

i. e. $S_\sigma(x)-f(x)\in L_1$ it follows, that $S_\sigma(x)\in L_1$. In this way from 1) (page 9) and 3) it follows 2) (more precisely from the good asymptotic behaviour of the function $f(x)$), i. e. the condition 2) (page 9) drops off.

From [11, p. 126, Theorem 336] it follows, that $T_\sigma^+\in E_\sigma$.

Now we can write down, that

$$\| S_\sigma^+ - T_\sigma^+ \|_{L_1(-\infty, +\infty)} \leqq \sigma^{-1} c \Big(\sum_{k=1}^{\infty}\sum_{j=1}^{i_k} M'_{kj}\Big) \leqq c\sigma^{-1} \bigvee_{-\infty}^{+\infty} S_\sigma.$$

c) supp$f(x)$ is unbounded. We fix one stem like in b) and make the same like in b) to the right and to the left of it. We obtain $T^+_{\sigma,0}(x)$ and $T^+_{\sigma,1}(x)$ such, that

$$\| S^+_\sigma(x) - T^+_{\sigma,0}(x) - T^+_{\sigma,1}(x) \| L_1 \leqq \sigma^{-1} c \Big(\sum_{k=1}^{\infty} \sum_{i=1}^{q_k} M^{1,0}_{k,i} + \sum_{l=1}^{\infty} \sum_{p=1}^{j_l} M'_{l,p} \Big) \leqq \sigma^{-1} c \bigvee_{-\infty}^{+\infty} S^+_\sigma .$$

In this way for $S^+_\sigma(x)$ we find $T^+_\sigma(x)$ such, that

i) $$T^+_\sigma(x) \in E_\sigma,\ T^+_\sigma(x) = T^+_{\sigma,0}(x) + T^+_{\sigma,1}(x),$$

ii) $$T^+_\sigma(x) \geqq S^+_\tau(x),$$

iii) $$\| T^+_\sigma(x) - S^+_\sigma(x) \|_{L_1(-\infty,+\infty)} \leqq \sigma^{-1} c \bigvee_{-\infty}^{+\infty} S^+_\sigma .$$

For $S^-_\sigma(x)$ we find $T^-_\sigma(x)$, which satisfies:

i) $$T^-_\sigma(x) \in E_\sigma,$$

ii) $$T^-_\sigma(x) \geqq S^-_\sigma(x),$$

iii) $$\| T^-_\sigma(x) - S^-_\sigma(x) \|_{L_1(-\infty,+\infty)} \leqq \sigma^{-1} c \bigvee_{-\infty}^{+\infty} S^-_\sigma .$$

Then applying Lemma 3 for $T^+_\sigma + T^{-1}_\sigma$ and $S_\sigma = S^+_\sigma + S^-_\sigma$ we have

$$\| S_\sigma(x) - T^-_\sigma(x) - T^+_\sigma(x) \|_{L_1} \leqq \| S^+_\sigma(x) - T^+_\sigma(x) \|_{L_1} + \| S^-_\sigma(x) - T^-_\sigma(x) \|$$

$$\leqq \sigma^{-1} c \Big(\bigvee_{-\infty}^{+\infty} S^+_\sigma + \bigvee_{-\infty}^{+\infty} S^-_\sigma \Big) \leqq 2c\, \sigma^{-1} \bigvee_{-\infty}^{+\infty} S_\sigma .$$

It is obvious also that $S_\sigma(x) \leqq T^+_\sigma(x) + T^-_\sigma(x)$ and $\bigvee_{-\infty}^{+\infty} S_\sigma \leqq \sigma \tau_1 (S_\sigma, \sigma^{-1})_{L_1}$.

We constructed $S_\sigma(x)$, which satisfies

i) $$S_\sigma(x) \geqq f(x),$$

ii) $$\| S_\sigma(x) - f(x) \|_{L_1} \leqq \tau_1 (f, \sigma^{-1})_{L_1} .$$

It is evident also that

$$\omega_1 (S_\sigma, x, \sigma^{-1}) \leqq w_1 (f, x, 2\sigma^{-1}) \Rightarrow \tau_1 (S_\sigma, \sigma^{-1})_{L_1} \leqq \tau_1 (f, 2\sigma^{-1})_{L_1} .$$

On the other hand $(T^+_\sigma(x) + T^-_\sigma(x)) \in E_\sigma$ and we have

$$\| f(x) - T^-_\sigma(x) - T^+_\sigma(x) \|_{L_1} \leqq \| f(x) - S_\sigma(x) \|_{L_1} + \| S_\sigma(x) - T^-_\sigma(x) - T^+_\sigma(x) \|_{L_1} .$$

$$\leqq \tau_1 (f, \sigma^{-1})_{L_1} + \sigma^{-1} c_1 \bigvee_{-\infty}^{+\infty} S_\sigma \leqq \tau_1 (f, \sigma^{-1})_{L_1} + c_1 \tau_1 (S_\sigma, \sigma^{-1})_{L_1}$$

$$\leqq \tau_1 (f, \sigma^{-1})_{L_1} + c_1 \tau_1 (f, 2\sigma^{-1})_{L_1} \leqq (1 + 9^4 c_1)\, \tau_1 (f, \sigma^{-1})_{L_1} = c \,.\, \tau_1 (f, \sigma^{-1})_{L_1},$$

where c is an absolute constant.

It follows that $\widetilde{\operatorname{Exp}}(\sigma)(f)_{L_1} \leqq c \tau_1 (f, \sigma^{-1})_{L_1}$, which establishes Theorem 1.

Now we give an example for a function, for which $\tau_1(f, \sigma^{-1})_{L_1}$ is infinite and the approximation with the class E_σ is infinite in L_1.

Let $f(x)$ be defined as follows

$$f(x)=\begin{cases}(x-k+k^{-2})k^2, & x\in[k-k^{-2}, k],\\ (k+k^{-2}-x)k^2, & x\in[k, k+k^{-2}],\\ 0, & x\notin[k-k^{-2}, k+k^{-2}],\ k=1, 2, 3, \dots\end{cases}$$

One may see, that

i) $$\|f\|_{L_1(-\infty, +\infty)}=\sum_{k=1}^{\infty}\int_{k-k^{-2}}^{k}(x-k+k^{-2})k^2dx$$

$$+\sum_{k=1}^{\infty}\int_{k}^{k+k^{-2}}(k+k^{-2}-x)k^2dx=2\int_0^1 v\,dv\sum_{q=1}^{\infty}\frac{1}{q^2}=\sum_{q=1}^{\infty}\frac{1}{q^2}<\infty,$$

ii) $$\forall\varepsilon>0,\ \forall R>0\ \exists\ x_i^+,\ x_i^-\ (i=1, 2): x_1^+>0,\ x_2^+<0,\ x_1^->0,$$

$$x_2^-<0,\ |x_i^+|>R,\ |x_i^-|>R \ \text{ and } \ |f^+(x_i^+)|<\varepsilon,\ |f^-(x_i^-)|<\varepsilon,$$

iii) $$\tau_1(f, r^{-1})_{L_1}\geq\sum_{q^2>\sigma}\left(\frac{1}{\sigma}-\frac{1}{q^2}\right)=\infty.$$

Let's assume, that $\omega(x)\in E_{\sigma_0}$ and $\omega(x)\geq f(x)$, $\forall x\in(-\infty, +\infty)$. Then from Bernstein's inequality we have

$$\sup\{|\omega'(x)|,\ |x|<\infty\}\leq\sigma_0\sup\{|\omega(x)|,\ |x|<\infty\}<k_0,\ k_0=\text{const}.$$

Let p_0 be such positive integer, that $p_0^{-2}<k_0^{-1}$ and $(p_0-1)^{-2}\geq k_0^{-1}$. Considering the following function

$$\Delta_{k_0}(x)=\begin{cases}(k+k_0^{-1}-x)k_0, & x\in[k, k+k_0^{-1}], & k=1, 2, \dots\\ (x-k+k_0^{-1})k_0, & x\in[k-k_0^{-1}, k], & k=1, 2, \dots\\ 0, & x\notin[k-k_0^{-1}, k+k_0^{-1}], & k=1, 2, \dots\end{cases}$$

and assume, that there exists x_0 such that $\omega(x_0)<\Delta_{k_0}(x_0)$, where $x_0\in[k_{x_0}, k_{x_0}+k_0^{-1}]$. Then we assert, that

a) there is only finite number values of x, belonging to $[k_{x_0}, k_{x_0}+k_0^{-1}]$ such that $\omega(x)=\Delta_{k_0}(x)$.

If we suppose, that exists a sequence $x_1, x_2, x_3, \dots, x_n, \dots$ such that $\Delta_{k_0}(x_i)=\omega(x_i)$, $i=1, 2, \dots$ we consider the function $\Omega(x)=\omega(x)-(k_{x_0}+k_0^{-1}-x)k_0$ which is an entire function and $\Omega(x_i)=0$, $i=1, 2 \dots$. But from the Cantor's theorem we know that there exists $\{x_{i_l}\}_{l=1}^{\infty}<\{x_p\}_{p=1}^{\infty}$ and x^* such that $\lim x_{i_l}=x^*$. It follows, that $\Omega(x)\equiv 0$. Hense $\omega(x)=(k_{x_0}+k_0^{-1}-x)k_0$ which is impossible.

b) from a) there exist x' and x'', which belong to $[k_{x_0}, k_{x_0}+k_0^{-1}]$ such that $\Delta_{k_0}(x')=\omega(x')$, $\Delta_{k_0}(x'')=\omega(x'')$ and $\omega(x)<\Delta_{k_0}(x)$, $\forall x\in[x', x'']$.

Then from Rolle's theorem there exists a point $\zeta\in(x', x'')$ such that $\omega'(\zeta)=\Delta'_{k_0}(\zeta)=k_0$ but $\sup\{|\omega'(x)|,\ |x|<\infty\}<k_0$.

It follows that $\omega(x) \geqq \Delta_{k_0}(x)$, $\forall x$, and then we have the following estimate

$$\| f-\omega \|_{L_1(-\infty, +\infty)} \geqq \| f-\Delta_{k_0} \|_{L_1(p_0-k_0^{-1}, +\infty)} = \infty .$$

Note. We obtain covergence from the following new property of $\tau_1(f, \delta)_{L_1(-\infty+\infty)}$: $\tau_1(f, \delta)_{L_1} \underset{\delta\to 0}{\to} 0 \Leftrightarrow \tau_1(f, \delta)_{L_1} < \infty$ and $f(x)$ is continuous almost everywhere.

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