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ASYMPTOTIC ESTIMATES FOR THE BEST APPROXIMATION OF STRICTLY n-CONVEX FUNCTIONS WITH CHEBYSHEVIAN SPLINES

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A function $f(t)$ is said to be strictly n-convex on $[a, b]$ if $f(t) \in C^n[a, b]$ and $f^{(n)}(t) > 0$ on $[a, b]$. (The concept of n-convex functions is considered by other authors, e. g. Bl. Sendov [6], who introduces it without the restrictive assumptions for $f(t)$ being in C^n.)

The set $T_k = \{t_i\}_{i=0}^k$ is said to be a k-partition of $[a, b]$ if $a = t_0 < t_1 < \cdots < t_k = b$.

The set $T_k = \{t_i\}_{i=0}^k$, where $a \geqq t_0 < t_1 < \cdots < t_k \leqq b$, is said to realize a Chebyshev alternance for f if

$$f(t_i) = \gamma \, . (-1)^i \| f \|, \quad i = 0, 1, \ldots, k; \ \gamma = \pm 1.$$

Here $\| f \|$ denotes the norm of f in $C[a, b]$.

Denote by M_k^n the class of the continuous on $[a, b]$ functions $p(t)$ for any of which there exists a k-partition $T_k = \{t_i\}_{i=0}^k$ of $[a, b]$, such that the restriction of p on each subinterval $[t_{i-1}, t_i]$ $(i = 1, 2, \ldots, k)$ coincides with a polynomial of degree at most n. The functions of M_k^n are deficient Chebyshevian splines and here they are called simply splines.

The purpose of this paper is to obtain an asymptotic estimate for the best approximation

$$H_{n,k}(f) = \| f - p_k \| = \min_{p \in M_k^{n-1}} \sup_{t \in [a, b]} | f(t) - p(t)$$

as $k \to \infty$. The function $p_k \in M_k^{n-1}$ is called a function of the best approximation for f and its existence is established in the following theorem:

Theorem 1. *Let $f(t)$ be a strictly n-convex function of even order n on $[a, b]$. A necessary and sufficient condition $p_k \in M_k^{n-1}$ to be of the best approximation for $f(t)$ is the existence of a kn-partition $T_{kn} = \{t_i\}_{i=1}^{kn}$ of $[a, b]$ realizing a Chebyshev alternance for $f(t) - p_k(t)$, i. e. $f(t_i) - p_k(t_i) = (-1)^i \| f(\cdot) - p_k(\cdot) \|$ $(i = 0, 1, \ldots, kn)$, where $\| \cdot \|$ stands for the uniform norm on $[a, b]$. The function of the best approximation $p_k \in M_k^{n-1}$ exists, it is unique and for each $i = 1, 2, \ldots, k$, p_k is the polynomial of the best approximation for $f(t)$ on the interval $[t_{(i-1)n}, t_{in}]$.*

We give here only the idea of the proof without any details.

The necessity is obtained by induction in k. For $k = 1$ it follows immediately from the well known Chebyshev theorem for approximation of continuous functions by polynomials.

Suppose $T_k=\{t_i\}_{i=0}^k$ is a k-partition of $[a, b]$ such that the function of the best approximation $p_k \in M_k^{n-1}$ is a polynomial p_k^i of degree at most $n-1$ on each interval $[t_{i-1}, t_i]$. To prove the necessity for an arbitrary k the function $q(t)$ is defined by

$$q(t)=\begin{cases} q_1(t), & t_0 \leqq t < t_{k-1}, \\ q_2(t), & t_{k-1} \leqq t \leqq t_k, \end{cases}$$

where $q_1 \in M_{k-1}^{n-1}$ is of the best approximation for f on $[t_0, t_{k-1}]$ and $q_2 \in M_1^{n-1}$ is of the best approximation for f on $[t_{k-1}, t_k]$.

Using the fact that n is even and the Rolle's theorem one can show that $q \in M_k^{n-1}$. From the inductive uniqueness argument it follows $p_k=q$ and satisfies the conditions of the theorem.

The uniqueness of p_k is obtained from the uniqueness of the determination of the knot t_{k-1}, which must satisfy the equation $d_1(t)=d_2(t)$, where

$$d_1(t)=\inf_{p \in M_{k-1}^{n-1}} \sup_{\tau \in [a, t]} |f(\tau)-p(\tau)|, \quad d_2(t)=\inf_{p \in M_1^{n-1}} \sup_{\tau \in [t, b]} |f(\tau)-p(\tau)|.$$

Here the continuity and monotonity of these functions are used.

The proof of the sufficiency is based on the pigeon — hole principle.

Remark. If n fails to be even the theorem is not true.

Let us mention that this result was inspired by the work of P. Kenderov [1], who studies the approximation of convex figures by polygons. In this case the Hausdorff distance appears to be more suitable than the uniform one.

We give some facts which we use later.

Lemma 1 (Polya's mean value theorem [4]). *Determine an integral $h(t)$ of the homogeneous equation*

$$Lx \equiv \frac{d^n x}{dt^n}+\Phi_1 \frac{d^{n-1}x}{dt^{n-1}}+\cdots+\Phi_n x=0 \tag{1}$$

with continuous coefficients $\Phi_1(i=1, 2, \ldots, n)$, assuming the same values as $f(t) \in C^n(a, b)$ at n given points of (a, b); determine further an integral $g(t)$ of the non — homogeneous equation

$$Lx \equiv \frac{d^n x}{dt^n}+\Phi_1 \frac{d^{n-1}x}{dt^{n-1}}+\cdots+\Phi_n . x=1$$

that vanishes at the n points in question. Suppose that the homogeneous equation (1) admits $n-1$ integrals $h_1(t), \ldots h_{n-1}(t)$ satisfying the $n-1$ inequalities $h_1(t)>0$, $W(h_1(t), h_2(t))>0, \ldots, W(h_1(t), h_2(t), \ldots, h_{n-1}(t))>0$ throughout the open interval (a, b) (here $W(h_1(t), \ldots, W_k(t))$ stands for the Wronskian of the functions $h_1(t), \ldots, h_k(t)$). Then for an arbitrary t of (a, b) there exists a point ξ intermediate between the given n points and t such that $f(t)=h(t)+g(t)Lf(\xi)$.

We use the asymptotic analysis developed by D. McClure [2] and D. McClure and Vitale [3]. In fact we need the following

Lemma 2 (Lemma 5 on p. 350 in [3]). *Let f be a real-valued function on an interval $[a, b]$ and let the function $e(f; \alpha, \beta)$ be defined for arbitrary $a \leqq \alpha < \beta \leqq b$. Let the following assumptions be satisfied:*

i) *For any* (α, β) *satisfying* $a \leqq \alpha < \beta \leqq b$, $e(f;\ \alpha, \beta) \geqq 0$. *Further, if* $a \leqq \alpha < \beta < \gamma \leqq b$, *then* $\max\{e(f;\ \alpha, \beta), e(f;\ \beta, \gamma)\} \leqq e(f;\ \alpha, \gamma)$.

ii) *There is a function* J_f *on* $[a, b]$ *associated with* f *and a constant* $m > 0$ *such that* J_f *is non-negative and piecewise continuous on* $[a, b]$, *admitting at worst a finite number of jump discontinuities, and* $\lim_{h\to+0} e(f;\ \alpha, \alpha+h)/h^m = J_f(\alpha+)$. *This limit is uniform in that the difference* $|J_f(\alpha+) - e(f;\ \alpha, \alpha+h)/h^m|$ *can be made uniformly small when* $(\alpha, \alpha+h)$ *is contained in an interval where* J_f *is continuous.*

iii) $e(f;\ \alpha, \beta)$ *depends continuously on* (α, β).

Define $E_k(f) = \min_{T_k} \max_{1\leqq i\leqq k} e(f;\ t_{i-1}, t_i)$, *where the minimum is taken under all* **k**-*partitions* $T_k = \{t_i\}_{i=0}^k$ *of* $[a, b]$. *Then*

$$\lim_{k\to\infty} k^m E_k(f) = \left(\int_a^b (J_f(s))^{\frac{1}{m}} ds\right)^m.$$

Lemma 3. *Let* $f(t) \in C^n[a, b]$ *and for each* $\alpha, \beta \in [a, b]$ *define* $e_n(f;\ \alpha, \beta)$ *to be the best approximation of* f *with polynomials of degree at most* $n-1$ *on* $[\alpha, \beta]$. *Then for each* $\alpha < \beta$

$$\lim_{h\to+0} h^{-n} e_n(f;\ \alpha, \alpha+h) = 2^{-2n+1}\,|f^{(n)}(\alpha)|/n! \tag{2}$$

The limit in (2) *is uniform in* α, *in the sense that the difference* $|h^{-n}\cdot e_n(f;\ \alpha, \alpha+h) - 2^{-2n+1}|f^{(n)}(\alpha)|/n!|$ *can be made uniformly small in* α *when the interval* $[\alpha, \alpha+h]$ *is contained in* $[a, b]$ *and* $h \to +0$.

Proof. Let p be a polynomial of degree at most $n-1$ interpolating f at n distinct points $\alpha < t_1 < \cdots < t_n < \alpha+h$. Obviously $(d^n/dt^n)p(t) = 0$. Then accordingly with the Polya's mean value theorem (Lemma 1) we see that $f(t) - p(t) = f^{(n)}(\xi)\cdot\widehat{p}(t)$ $(\alpha < t < \alpha+h)$, where $\min(t, t_1) < \xi < \max(t, t_n)$ and $\widehat{p}(t)$ is the unique solution $[\alpha, \alpha+h]$ of the equation $(d^n/dt^n)\widehat{p}(t) = 1$ satisfying $\widehat{p}(t_i) = 0$ $(i = 1, 2\ldots, n)$. Simple computations give $\widehat{p}(t) = \frac{1}{n!}(t-t_1)(t-t_2)\cdots(t-t_n)$, whence

$$f(t) - p(t) = \frac{1}{n!} f^{(n)}(\xi)(t-t_1)(t-t_2)\cdots(t-t_n)$$

$$= \frac{1}{n!} f^{(n)}(\alpha)(t-t_1)(t-t_2)\cdots(t-t_n) + o(h^n).$$

Consequently (see p. 29 in [7])

$$\sup_{\alpha\leqq t\leqq\alpha+h} |f(t) - p(t)| \geqq \frac{1}{n!}|f^{(n)}(\alpha)| \sup_{\alpha\leqq t\leqq\alpha+h} |T_n(t;\ \alpha, \alpha+h)| + 0(h^n) \tag{3}$$

$$= 2^{-2n+1}h^n\cdot|f^{(n)}(\alpha)|/n! + o(h^n),$$

where $T_n(t;\ \alpha, \alpha+h) = 2^{-2n+1}\cdot h^n \cos(n \arccos((2t-2\alpha-h)/)h)$ is the Chebyshev polynomial of degree n on $[\alpha, \alpha+h]$.

Since the polynomial $p(t)$ interpolating $f(t)$ in the zeros of $T_n(t;\ \alpha, \alpha+h)$ gives equality in (3) and accordingly with the Chebyshev theorem the polynomial of best approximation $p_{n-1}(t)$ for $f(t)$ on $[\alpha, \alpha+h]$ interpolates $f(t)$ at n distinct points, hence

$$(4) \qquad e_n(f;\ \alpha, \alpha+h) = 2^{-2n+1} \cdot h^n \,|\, f^{(n)}(\alpha) \,|/n! \ + o(h^n).$$

As a straightforward consequence we obtain inequality (2). Since the term $o(h^n)$ in the used inequality can be taken independent on α, therefore the limit in (2) is uniform in α.

Theorem 2. *For the best approximation of the strictly n-convex function $f(t)$ by functions of M_k^{n-1} on $[a, b]$ the following asymptotic estimate is satisfied:*

$$(5) \qquad \lim_{k\to\infty} k^n H_{n,k}(f) = \frac{2^{-2n+1}}{n!} \Big[\int_a^b \sqrt[n]{f^{(n)}(t)}\,dt\Big]^n.$$

Proof. For each k-partition $T_k = \{t_i\}_{i=0}^k$ of $[a, b]$ we introduce $E_n(f, T_k) = \max_{1\leqq i\leqq k} e_n(f;\ t_{i-1}, t_i)$, where $e_n(f;\ t_{i-1}, t_i)$ are defined as in Lemma 3. Let us denote

$$E_{n,k}(f) = \min_{T_k} \max_{1\leqq i\leqq k} e_n(f;\ t_{i-1}, t_i) = \min_{T_k} E_n(f, T_k).$$

Obviously $e_n(f;\ \alpha, \beta)$ depends continuously on the interval ends of $[\alpha, \beta]$, and for each $a \leqq \alpha < \beta < \gamma \leqq b$ the inequality $\max\{e_n(f;\ \alpha, \beta),\ e_n(f;\ \beta, \gamma)\} \leqq e_n(f;\ \alpha, \gamma)$ holds. The proof follows as a direct consequence of Lemma 2 and 3 and from the fact that $E_{n,k}(f) = H_{n,}(f)$, which one easily shows applying the construction in the proof of Theorem 1.

Analogous asymptotic estimates concerning the best approximations of convex sets by polygons related to the Hausdorff metric are obtained by Toth [8], McClure and Vitale [3] and V. Popov [5].

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