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ON THE EXTREME AND L^2 DISCREPANCIES OF SYMMETRIC FINITE SEQUENCES

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In the present paper we obtain estimates for the extreme and L^2 discrepancies of any symmetric finite sequence of points in s-dimensional unit cube $E^s=[0, 1]^s$. Our estimate for the L^2 discrepancy of symmetric sequences is an analogue of Erdös-Turán-Koksma's inequality. The estimate for the extreme discrepancy of symmetric sequences in the case $s=1$ coincides with LeVeque's inequality.

1. Introduction. Let $s\geq 1$ and let E^s denotes the unit cube consisting of points $x=(x_1,\ldots, x_s)$ with $0\leq x_j\leq 1$ $(j=1,\ldots, s)$. Let $X=\{a_k\}_{k=1}^N$ be a finite sequence of points in E^s. For every $\gamma=(\gamma_1,\ldots,\gamma_s)$ in E^s we write $A(X;\gamma)$ for the number of terms of X lying in the box $0\leq x_j<\gamma_j$ $(j=1,\ldots, s)$ and put $D(X;\gamma)=N^{-1}A(X;\gamma)-\gamma_1\ldots\gamma_s$. The numbers

$$D(X)=\sup_{\gamma\in E^s}|D(X;\gamma)| \quad \text{and} \quad T(X)=\Big(\int_{E^s}|D(X;\gamma)|^2 d\gamma\Big)^{1/2}$$

are called the extreme and L^2 discrepancies of X, respectively.

We shall make use of the following notations:

For every integer m we write $\bar{m}=\max(1, |m|)$.

For every lattice point $m=(m_1,\ldots, m_s)$ in $\mathbf{Z}^s$ we define

$$\|m\|=\max_{1\leq j\leq s}|m_j| \quad \text{and} \quad R(m)=\bar{m}_1\ldots\bar{m}_s.$$

For $\alpha=(\alpha_1,\ldots,\alpha_s)$ and $\beta=(\beta_1,\ldots,\beta_s)$ in $\mathbf{R}^s$, let $\langle\alpha, \beta\rangle$ denote the standard inner product, that is $\langle\alpha,\beta\rangle=\alpha_1\beta_1+\ldots+\alpha_s\beta_s$.

It is well known that for any finite sequence $X=\{a_k\}_{k=1}^N$ in E^s and any natural number n we have

$$(1)\qquad D(X)\leq c(s)\Big(\frac{1}{n}+\sum_{0<\|m\|\leq n}(R(m))^{-2}\Big|\frac{1}{N}\sum_{k=1}^N e^{2\pi i\langle m,\, a_k\rangle}\Big|\Big),$$

where $c(s)>0$ is an absolute constant depending only on the dimension s.

It is also well known that in the case $s=1$ the discrepancy $D(X)$ of any finite sequence $X=\{a_k\}_{k=1}^N$ in $E=[0, 1]$ satisfies

$$(2)\qquad D(X)\leq\Big(\frac{6}{\pi^2}\sum_{m=1}^\infty\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^N e^{2\pi i m a_k}\Big|^2\Big)^{1/3}.$$

Inequalities (1) and (2) are called Erdös-Turán-Koksma's inequality (see [1], p. 116) and LeVeque's inequality (see [1], p. 111), respectively.

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In the present paper we obtain an inequality (see Theorem 2) for the L^2 discrepancy of the so-called symmetric finite sequences in E^s which is an analogue of Erdös-Turán-Koksma's inequality. For the extreme discrepancy of the symmetric finite sequences in E^s we obtain an inequality (see Theorem 3) which in the case $s=1$ coincides with LeVeque's inequality.

2. Symmetric finite sequences. Let $X=\{a_k\}_{k=1}^N$ be a finite sequences in E^s. We shall say that a given point $x=(x_1, \ldots, x_s)$ in E^s has a multiplicity $p(0\leq p\leq s)$ with respect to the sequence X if exactly p terms of X coincide with the point x. We call the sequence X a symmetric one if for any point $x=(x_1, \ldots, x_s)$ in E^s all points of the type

$$(\tau_1+(-1)^{\tau_1}x_1, \ldots, \tau_s+(-1)^{\tau_s}x_s) \tag{3}$$

have one and the same multiplicity with respect to X, when $\tau_1, \ldots, \tau_s$ take independently the values 0 and 1.

It is obvious that if a sequence X in E^s is symmetric then any other sequence Y in E^s originating from X by means of a transposition of its terms, is also symmetric.

Let X be a finite sequence consisting of N points in E^s and let $\tilde{X}$ be a symmetric sequence, consisting of $M=2^sN$ points in E^s. We say that the symmetric sequence $\tilde{X}$ is produced by the sequence X if for every point $x=(x_1, \ldots, x_s)$ in E^s the following is valid: if a point x is a term of the sequence X, then each point of type (3) is a term of the sequence $\tilde{X}$, where $\tau_1, \ldots, \tau_s$ take the values 0 and 1.

Apparently, each sequence X in E^s produces at least one symmetric sequence $\tilde{X}$ in E^s. The inverse, of course, is true, too: if $\tilde{X}$ is a symmetric sequence, consisting of M points in E^s and if $M\equiv 0 \pmod{2^s}$ then there exists at least one sequence X in E^s which produces the sequence $\tilde{X}$.

3. Estimates for the L^2 discrepancy of symmetric finite sequences. Theorem 1. *Let $\tilde{X}$ be a symmetric finite sequence, consisting of $M=2^sN$ terms in E^s and let $X=\{a_k\}_{k=1}^N$ be any finite sequence in E^s producing $\tilde{X}$. Then*

$$T(\tilde{X})\leq\Big(c(s)\sum_{\|m\|>0}(R(m))^{-2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi i\langle m,a_k\rangle}\Big|^2\Big)^{1/2}, \tag{4}$$

where

$$c(s)=(3/4\pi^2)(1-2^{-s+1}+3^{-s}). \tag{5}$$

Proof. Let $X=\{a_k\}_{k=1}^N$ be a given sequence in E^s, and let $\tilde{X}=\{b_k\}_{k=1}^N$, where $M=2^sN$, be a symmetric sequence in E^s which is produced by X. We put

$$a_k=(\xi_1(k), \ldots, \xi_s(k)), \quad k=1, \ldots, N. \tag{6}$$

For $\gamma=(\gamma_1, \ldots, \gamma_s)$ in E^s, let $\varphi_\gamma(x)=\varphi_\gamma(x_1, \ldots, x_s)$ be the characteristic function of the box $0\leq x_j<\gamma_j$ $(j=1, \ldots, s)$. Then

$$A(\tilde{X};\gamma)=\sum_{k=1}^{N}\varphi_\gamma(b_k)=\sum_{k=1}^{N}\sum_{\tau_1,\ldots,\tau_s=0}^{1}\varphi_\gamma(\tau_1+(-1)^{\tau_1}\xi_1(k), \ldots, \tau_s+(-1)^{\tau_s}\xi_s(k)).$$

Therefore

(7) $$D(\widetilde{X};\ \gamma)=\frac{1}{2^sN}\sum_{k=1}^{N}\ \sum_{\tau_1,\ldots,\tau_s=0}^{1}\ \varphi_\gamma(\alpha_1(k),\ldots,\ \alpha_s(k))-\gamma_1\ldots\gamma,$$

where, for brevity, we have put $\alpha_j(k)=\tau_j+(-1)^{\tau_j}\xi_j(k)$, $j=1,\ldots,s$.

The function $D(\gamma_1,\ldots,\gamma_s)=D(\widetilde{X};\ \gamma)$ is a piecewise continuous function in E^s. Let

$$\sum_{m_1,\ldots,m_s=-\infty}^{\infty} c(m_1,\ldots,\ m_s)e^{2\pi i(m_1\gamma_1+\ldots+m_s\gamma_s)}$$

be the Fourier series of $D(\gamma_1,\ldots,\gamma_s)$. It is well known that the Fourier coefficients are given by

(8) $$c(m_1,\ldots,m_s)=\int_{E^s} D(\gamma_1,\ldots,\ \gamma_s)\ e^{-2\pi i(m_1\gamma_1+\ldots+m_s\gamma_s)}d\gamma_1\ldots d\gamma_s.$$

From (7) and (8) it follows that

(9) $$c(m_1,\ldots,m_s)=\frac{1}{N2^s}\sum_{k=1}^{N}\ \sum_{\tau_1,\ldots,\tau_s=0}^{1}\ \prod_{j=1}^{s}\int_{\alpha_j(k)}^{1} e^{-2\pi i m_j\gamma_j}\,d\gamma_j - \prod_{j=1}^{s}\int_0^1 \gamma_j e^{-2\pi i m_j\gamma_j}d\gamma_j$$

$$=\frac{1}{N2^s}\sum_{k=1}^{N}\prod_{j=1}^{s}\sum_{\tau_j=0}^{1}A(m_j,\ \alpha_j(k))-\prod_{j=1}^{s}B(m_j),$$

where we use the following notations

$$A(m,\ a)=\int_a^1 e^{-2\pi i m\gamma}d\gamma \qquad B(m)=\int_0^1 \gamma e^{-2\pi i m\gamma}d\gamma\ .$$

For every real number a and every integer m we have

(10) $$A(m,\ a)=\begin{cases}(1/2\pi i m)(e^{-2\pi i m a}-1), & \text{if}\quad m\neq 0;\\ 1-a, & \text{if}\quad m=0;\end{cases}$$

and

(11) $$B(m)=\begin{cases}-1/2\pi i m, & \text{if}\quad m\neq 0;\\ 1/2, & \text{if}\quad m=0.\end{cases}$$

We verify immediately that

(12) $$\sum_{\tau_j=0}^{1}A(0,\ \alpha_j(k))=1.$$

By (9), (11) and (12) we obtain

$$c(0,\ldots,\ 0)=\frac{1}{N2^s}\sum_{k=1}^{N}1-\frac{1}{2^s}=0.$$

Then by Parseval's identity we have

$$(13)\qquad (T(\tilde{X}))^2=\int_{E^s}|D(\gamma_1,\ldots,\gamma_s)|^2 d\gamma_1\ldots d\gamma_s$$

$$=\sum_{m_1,\ldots,m_s=-\infty}^{\infty}|c(m_1,\ldots,m_s)|^2=\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}'|c(m_1,\ldots,m_s)|^2,$$

where Σ' means that $(m_1,\ldots,m_s)\neq(0,\ldots,0)$.

Let p be a natural number with $1\leq p\leq s$ and let $\{j_1,\ldots,j_p\}$ be an arbitrary subset of the set $\{1,2,\ldots,s\}$. We shall introduce the notation

$$\sigma(j_1,\ldots,j_p)=\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}^{*}|c(m_1,\ldots,m_s)|^2,$$

where the sum Σ^* is over $(m_1,\ldots,m_s)$ in $\mathbf{Z}^s$ with $m_j\neq 0$ $(1\leq j\leq s)$ if and only if j coincides with any of the numbers $j_1,\ldots,j_p$.

It is easy to see that formula (13) can be written in the form

$$(14)\qquad (T(\tilde{X}))^2=\sum_{p=1}^{s}\sum_{j_1,\ldots,j_p}\sigma(j_1,\ldots,j_p),$$

where the inner sum is over $\{j_1,\ldots,j_p\}\subset\{1,2,\ldots s\}$.

Now let p be again a given natural number with $1\leq p\leq s$ and let $\{j_1,\ldots,j_p\}$ be a given subset of $\{1,2,\ldots,s\}$. We shall obtain an estimate for the sum $\sigma(j_1,\ldots,j_p)$ From lower exposition, it will become clear that, without any loss of generality, we can assume that $\{j_1,\ldots,j_p\}$ coincides with $\{1,2,\ldots,p\}$.

Let us suppose that $\{j_1,\ldots,j_p\}=\{1,2,\ldots,p\}$. Then

$$(15)\qquad \sigma(j_1,\ldots,j_p)=\sum_{\substack{m_1,\ldots,m_p=-\infty\\ m_1\neq 0,\ldots,m_p\neq 0}}^{\infty}|c(m_1,\ldots,m_p,0,\ldots,0)|^2.$$

Let $m_1\neq 0,\ldots,m_p\neq 0$ $(1\leq p\leq s)$. Then from (9), (11) and (12) we obtain

$$(16)\qquad c(m_1,\ldots,m_p,0,\ldots,0)$$

$$=\frac{1}{N2^s}\cdot\frac{1}{(2\pi i)^p}\cdot\frac{1}{m_1\ldots m_p}\sum_{k=1}^{N}\prod_{j=1}^{p}\sum_{\tau_j=0}^{1}(e^{-2\pi i m_j\alpha_j(k)}-1)-\frac{(-1)^p}{(2\pi i)^p}\frac{1}{2^{s-p}}\frac{1}{m_1\ldots m_p}.$$

We make the following transformations

$$(17)\qquad \prod_{j=1}^{p}\sum_{\tau_j=0}^{1}(e^{-2\pi i m_j\alpha_j(k)}-1)=\prod_{j=1}^{p}(e^{2\pi i m_j\xi_j(k)}+e^{-2\pi i m_j\xi_j(k)}-2)$$

$$=\sum_{\varepsilon_1,\ldots,\varepsilon_p}\beta(\varepsilon_1,\ldots,\varepsilon_p)e^{2\pi i(\varepsilon_1 m_1\xi_1(k)+\ldots+\varepsilon_p m_p\xi_p(k))},$$

where the sum is over $\varepsilon_1,\ldots,\varepsilon_p$ which take the values 0, -1 and 1 (their number is 3^p).

It is easily seen, that

$$(18)\qquad \beta(0,\ldots,0)=(-1)^p2^p$$

and

(19) $$|\beta(\varepsilon_1, \dots, \varepsilon_p)| \leqq 2^{p-1} \quad \text{for} \quad (\varepsilon_1, \dots, \varepsilon_p) \neq (0, \dots, 0).$$

Now, using (18), from (16) and (17) we conclude that

(20) $$c(m_1, \dots, m_p, 0, \dots, 0)$$
$$= \frac{1}{2^s} \cdot \frac{1}{(2\pi i)^p} \cdot \frac{1}{m_1 \dots m_p} \sum_{\varepsilon_1, \dots, \varepsilon_p}' \beta(\varepsilon_1, \dots, \varepsilon_p) S(\varepsilon_1 m_1, \dots, \varepsilon_p m_p, 0, \dots, 0),$$

where Σ' means that $(\varepsilon_1, \dots, \varepsilon_p) \neq (0, \dots, 0)$ and where for a lattice point $m=(m_1, \dots, m_s)$ in $\mathbf{Z}^s$ we use the notation

(21) $$S(m_1, \dots, m_s) = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \langle m, a_k \rangle} = \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i (m_1 \xi_1(k) + \dots + m_s \xi_s(k))}.$$

From (20) and (19) we obtain the estimate

$$|c(m_1, \dots, m_p, 0, \dots, 0)| \leqq \frac{1}{2^{s+1}\pi^p} \sum_{\varepsilon_1, \dots, \varepsilon_p}' \frac{|S(\varepsilon_1 m_1, \dots, \varepsilon_p m_p, 0, \dots, 0)|}{m_1 \dots m_p}.$$

From the latter estimate and the well known inequality $(\Sigma_{j=1}^n u_j)^2 \leqq n \Sigma_{j=1}^n u_j^2$ we find

$$|c(m_1, \dots, m_p, 0, \dots, 0)|^2 \leqq \frac{3^p - 1}{4^{s+1}\pi^{2p}} \sum_{\varepsilon_1, \dots, \varepsilon_p}' \frac{|S(\varepsilon_1 m_1, \dots, \varepsilon_p m_p, 0, \dots, 0)|^2}{(m_1 \dots m_p)^2}.$$

From that inequality and from (15) we get

(22) $$\sigma(j_1, \dots, j_p) \leqq \frac{3^p - 1}{4^{s+1}\pi^{2p}} \sum_{\varepsilon_1, \dots, \varepsilon_p}' \sum_{\substack{m_1, \dots, m_p = -\infty \\ m_1 \neq 0, \dots, m_p \neq 0}}^{\infty} \frac{|S(\varepsilon_1 m_1, \dots, \varepsilon_p m_p, 0, \dots, 0)|^2}{(m_1 \dots m_p)^2}.$$

Let us consider the inner sum of that inequality with fixed $\varepsilon_1, \dots, \varepsilon_p$ and denote it by $\Omega(\varepsilon_1, \dots, \varepsilon_p)$. Let $q(1 \leqq q \leqq p)$ be the number of those ε_j $(j=1, \dots, s)$ which are not equal to zero. With no loss of generality we can assume that $\varepsilon_1 \neq 0, \dots, \varepsilon_q \neq 0$. Then

$$\Omega(\varepsilon_1, \dots, \varepsilon_p) = \sum_{\substack{m_1, \dots, m_p = -\infty \\ m_1 \neq 0, \dots, m_p = 0}}^{\infty} \frac{|S(\varepsilon_1 m_1, \dots \varepsilon_p m_p, 0, \dots, 0)|^2}{(m_1 \dots m_p)^2}$$

$$= \left(\frac{\pi^2}{3}\right)^{p-q} \sum_{\substack{m_1, \dots, m_q = -\infty \\ m_1 \neq 0, \dots, m_q \neq 0}}^{\infty} \frac{|S(\varepsilon_1 m_1, \dots, \varepsilon_q m_q, 0, \dots, 0)|^2}{(m_1 \dots m_q)^2}$$

$$\leqq \left(\frac{\pi^2}{3}\right)^{p-1} \sum_{\substack{m_1, \dots, m_q = -\infty \\ m_1 \neq 0, \dots, m_q \neq 0}}^{\infty} \frac{|S(m_1, \dots, m_q, 0, \dots, 0)|^2}{(m_1 \dots m_q)^2}$$

$$\leqq \left(\frac{\pi^2}{3}\right)^{p-1} \sum_{m_1, \dots, m_s = -\infty}^{\infty}{}' \frac{|S(m_1, \dots, m_s)|^2}{(\bar{m}_1 \dots \bar{m}_s)^2}.$$

From here and from (22) we obtain

$$(23)\qquad \sigma(j_1,\dots,j_p)\leq\frac{3(3^p-1)}{4^{s+1}\pi^2 3^p}\sum_{m_1,\dots,m_s=-\infty}^{\infty}\frac{|S(m_1,\dots,m_s)|^2}{(\overline{m}_1\dots\overline{m}_s)^2}\sum_{\varepsilon_1,\dots,\varepsilon_p}1$$
$$=\frac{3(3^p-1)}{4^{s+1}\pi^2 3^p}\sum_{m_1,\dots,m_s=-\infty}^{\infty}{}'\frac{|S(m_1,\dots,m_s)|^2}{(\overline{m}_1\dots\overline{m}_s)^2}.$$

We obtained the estimate (23) under the condition that $\{j_1,\dots,j_p\}$ coincides with $\{1, 2,\dots, p\}$. However, it is evident from the proof that it holds for an arbitrary subset of the set $\{1, 2,\dots, s\}$.

From (14) and (23), using

$$\sum_{p=1}^{s}\frac{(3^p-1)^2}{3^p}\sum_{j_1,\dots,j_p}1=\sum_{p=1}^{s}\frac{(3^p-1)^2}{3^p}\binom{s}{p}=4^s-2^{s+1}+(4/3)^s$$

we obtain the estimate

$$(24)\quad (T(\widetilde{X}))^2\leq\frac{3}{4^{s+1}\pi^2}\sum_{m_1,\dots,m_s=-\infty}^{\infty}{}'\frac{|S(m_1,\dots,m_s)|^2}{(\overline{m}_1\dots\overline{m}_s)^2}\sum_{p=1}^{s}\frac{(3^p-1)^2}{3^p}\sum_{j_1,\dots,j_p}1$$
$$=C(s)\sum_{m_1,\dots,m_s=-\infty}^{\infty}{}'\frac{|S(m_1,\dots,m_s)|^2}{(\overline{m}_1\dots\overline{m}_s)^2},$$

where the constant $C(s)$ is defined by Eq. (5). Obviously the estimate (24) coincides with (4). Thus Theorem 1 is proved.

It should be mentioned that Theorem 1 (with a different constant) has been proved first in [2], but with a certain inaccuracy in our proof. We express our gratitude to Professor V. Popov (Sofia) for drawing our attention to it.

From Theorem 1 for $s=1$ we get the following assertion.

Corollary 1. *Let $\widetilde{X}$ be a symmetric finite sequence consisting of $M=2^sN$ terms in $E=[0, 1]$ and let $X=\{a_k\}_{k=1}^N$ be any finite sequence in E producing $\widetilde{X}$. Then*

$$(25)\qquad T(\widetilde{X})\leq\Big(\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi i m a_k}\Big|^2\Big)^{1/2}.$$

We shall note that this estimate is precise. Indeed, if $a_1=a_2=\dots=a_N=0$ then $\widetilde{X}$ will consist of N zeros and N ones. Now from (25) we obtain

$$T(\widetilde{X})\leq\Big(\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big)^{1/2}=1/\sqrt{12},$$

and from the definition of L^2 discrepancy we find that

$$T(\widetilde{X})=\Big(\int_0^1\Big(\frac{1}{2}-\gamma\Big)^2d\gamma\Big)^{1/2}=1/\sqrt{12}.$$

Therefore in this case (25) is an equality.

The following theorem is an analogue of Erdös-Turán-Koksma's inequality (1).

Theorem 2. *Let $X=\{a_k\}_{k=1}^N$ be any symmetric finite sequence in E^s. Then*

$$T(X)\leq\Big(C(s)\sum_{\|m\|>0}(R(m))^{-2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi i\langle m,a_k\rangle}\Big|^2\Big)^{1/2}, \tag{26}$$

where the constant $C(s)$ is defined by (5).

Proof. Let $X=\{a_k\}_{k=1}^N$ be a given symmetric sequence in E^s and let $\widetilde{X}=\{b_k\}_{k=1}^N$, where $M=2^sN$, be another symmetric sequence in E^s which is produced by X. Then for every point γ in E^s we have $A(\widetilde{X};\gamma)=2^sA(X;\gamma)$. Therefore $D(\widetilde{X};\gamma)=M^{-1}A(\widetilde{X};\gamma)-\gamma_1\dots\gamma_s=D(X;\gamma)$, and so

$$T(X)=T(\widetilde{X}). \tag{27}$$

It follows from Theorem 1 that for the symmetric sequence $\widetilde{X}$ will be valid the estimate (4). Now from (4) and (27) we get (26). Theorem 2 is proved.

We proved that Theorem 2 follows from Theorem 1. It is not difficult to prove the inverse. Indeed, let $X=\{a_k\}=_{k=1}^N$ be an arbitrary sequence in E^s and let $\widetilde{X}=\{b_k\}_{k=1}^M$ be a symmetric sequence which is produced by X.

From (26) we shall have

$$(T(\widetilde{X}))^2\leq C(s)\sum_{\|m\|>0}(R(m))^{-2}\Big|\frac{1}{M}\sum_{k=1}^{M}e^{2\pi i\langle m,b_k\rangle}\Big|^2. \tag{28}$$

Using the notation (6) and (21) we can write

$$\frac{1}{M}\sum_{k=1}^{M}e^{2\pi i\langle m,b_k\rangle}=\frac{1}{2^sN}\sum_{k=1}^{N}\sum_{\tau_1,\dots,\tau_s=0}^{1}e^{2\pi i((-1)^{\tau_1}m_1\xi_1(k)+\dots+(-1)^{\tau_s}m_s\xi_s(k))} \tag{29}$$

$$=\frac{1}{2^s}\sum_{\tau_1,\dots,\tau_s=0}^{1}S((-1)^{\tau_1}m^1,\dots,(-1)^{\tau_s}m_s).$$

From (28) and (29) it follows that

$$(T(\tilde{X}))^2\leq\frac{C(s)}{2^s}\sum_{\tau_1,\dots,\tau_s=0}^{1}\ \sideset{}{'}\sum_{m_1,\dots,m_s=-\infty}^{\infty}\frac{|S((-1)^{\tau_1}m_1,\dots,(-1)^{\tau}sm_s)|^2}{(\overline{m}_1\dots\overline{m}s)^2}$$

$$=\frac{C(s)}{2^s}\sum_{\tau_1,\dots,\tau_s=0}^{1}\ \sideset{}{'}\sum_{m_1,\dots,m_s=-\infty}^{\infty}\frac{|S(m_1,\dots,m_s)|^2}{(\overline{m}_1\dots\overline{m}_s)^2}$$

$$=C(s)\sum_{m_1,\dots,m_s=-\infty}^{\infty}\frac{|S(m_1,\dots,m_s)|^2}{\overline{m}_1\dots\overline{m}_s)^2}$$

$$=C(s)\sum_{\|m\|>0}(R(m))^{-2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi i\langle m,a_k\rangle}\Big|^2.$$

We see that Theorem 1 is equivalent to Theorem 2.

In the case $s=1$, it follows from Theorem 2 that for any symmetric sequence $X=\{a_k\}_{k=1}^N$ in E we have

$$T(X)\leqq\Big(\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi ima_k}\Big|^2\Big)^{1/2}, \tag{30}$$

but from Koksma's equality (see [1], p. 110):

$$(T(X))^2=\Big(\frac{1}{N}\sum_{k=1}^{N}\Big(a_k-\frac{1}{2}\Big)\Big)^2+\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi ima_k}\Big|^2,$$

which holds for any sequence $X=\{a_k\}_{k=1}^N$ in E it follows that

$$T(X)\geqq\Big(\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi ima_k}\Big|^2\Big)^{1/2}.$$

That inequality shows that in fact we have in (30) an equality.

Corollary 2. *Let $X=\{a_k\}_{k=1}^N$ be any symmetric finite sequence in E. Then*

$$T(X)=\Big(\frac{1}{2\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi ima_k}\Big|^2\Big)^{1/2}.$$

Of course, corollary 2 can also be derived only from Koksma's equality.

4. An estimate for the extreme discrepancy of symmetric finite sequences. In the following theorem we shall show that for symmetric sequences LeVeque's inequality (2) can be generalized for any dimension $s\geqq 1$.

Theorem 3. *Let $X=\{a_k\}_{k=1}^N$ be any symmetric finite sequence in E^s. Then*

$$D(X)\leqq\Big(C_1(s)\sum_{\|m\|>0}(R(m))^{-2}\Big|\frac{1}{N}\sum_{k=1}^{N}e^{2\pi i(m,a_k)}\Big|^2\Big)^{1/(s+2)}, \tag{31}$$

where $C_1(s)>0$ is an absolute constant depending only on s and $C_1(1)=3/\pi^2$.

Proof. It is known (see [3], Theorem 4.2 and Corollary 1.2) that for any sequence $X=\{a_k\}_{k=1}^N$ in E^s the following inequality holds $(D(X))^{(s+2)/2}\leqq C_2(s)T(X)$, where $C_2(s)$ is an absolute constant depending only on s and $C_2(1)=\sqrt{12}$. Therefore

$$D(X)\leqq(C_2(s)T(X))^{2/(s+2)}. \tag{32}$$

Now, if X is a symmetric sequence from (32) and from Theorem 2 we get the estimate (31) with the constant $C_1(s)=((C_2(s))^2C(s))^{1/(s+2)}$, where the constant $C(s)$ is defined by (5). It is easy to verify that $C_1(1)=3/\pi^2$. Theorem 3 is proved.

Finally, we shall remark that for $s=1$ the estimate (31) coincides with LeVeque's inequality (2).

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