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AN APPLICATION OF NEWTON POLYGONS

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We prove the non-existence of Distance-Regular Graphs with diameters $d=24$, 25 and intersection array

$$I=\begin{Bmatrix} * & 1 & \ldots & 1 & \ldots & 1 & \ldots & c \\ 0 & 0 & & 0 & \ldots & 0 & \ldots & 0 \\ k & k-1 & & k-1 & \ldots & k-1 & \ldots & * \end{Bmatrix}, \quad c\neq 1,\ k \text{ and } k>2,$$

by means of Newton polygons.

1. Distance-Regular Graphs. If Γ is a connected graph, and $\delta(x, y)$ denotes the distance between the vertices x and y of Γ, then number

$$S_{hi}(u, v)=|\{w\in V(\Gamma)\mid \delta(u, w)=h \text{ and } \delta(v, w)=i\}|,$$

where $V(\Gamma)$ the vertex set of Γ, is the number of vertices of Γ whose distance from u and v is h and i respectively.

Definition 1.1. *The connected graph Γ with diameter d is distance-h-regular (where h is a given natural number) if for all integers i and j $(0\leq i, j\leq d)$ and for all pairs of vertices u,v with $\delta(u, v)=j$ the number*

$$S_{hi}(u, v)=S_{hij} \quad \text{(say)}$$

depends only on i, j and not on the individual pair (u, v). Γ is distance-regular if it is distance-h-regular for all h, $0\leq h\leq d$.

Theorem 1.2. *If Γ is distance-1-regular then Γ is distance-regular.*

For a proof see [3].

If j is fixed, then the number S_{1ij} counts the vertices w of Γ such that w is adjacent to u and $\delta(v, w)=i$, where $\delta(u, v)=j$. Now if w is adjacent to u and $\delta(u, v)=j$, then $\delta(v, w)=i$ must be one of the numbers $j-1, j, j+1$; In other words $S_{1,i,j}=0$ if $i\neq j-1, j, j+1$. We introduce the notation $a_j=S_{1,j,j}$, $b_j=S_{1,j+1,j}$, $c_j=S_{1,j-1,j}$ where $0\leq j\leq d$, except that c_0 and b_d are undefined.

Definition 1.3. *The intersection array of a distance-regular graph Γ is the array*

$$I(\Gamma)=\begin{Bmatrix} * & c_1 & \ldots & c_j & \ldots & c_d \\ a_0 & a_1 & \ldots & a_j & \ldots & a_d \\ a_0 & b_1 & \ldots & b_j & \ldots & * \end{Bmatrix}.$$

For a detailed treatment of the theory of the distance-regular graphs see [2] and [3].

In this paper we deal with distance-regular graphs of diameter $d=24$ and 25 with intersection array

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$$(1) \qquad I=\begin{Bmatrix} * & 1 & 1 & \dots & 1 & \dots & c \\ 0 & 0 & 0 & \dots & 0 & \dots & k-c \\ k & k-1 & k-1 & \dots & k-1 & \dots & * \end{Bmatrix}$$

2. Newton polygons. Consider the following polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$, $a_n.a_0\neq 0$ with rational coefficients.

For the prime number q, each non-zero coefficient a_i can be written in the form $a_i=q^{k_i}m_i/l_i$, where $(m_i, l_i)=1$, $(m_i, q)=1$, and $(l_i, q)=1$, $i=0, 1, \dots, n$.

From these expressions of non-zero a_i we form the ordered pairs (i, k_i) and plot them on a rectangular coordinate system (no points correspond to zero coefficients).

Using these points we construct the Newton polygon of $P(x)$, which is the convex line enclosing all the points from below. For these polygons Dumas [4] has proved the following.

Theorem 2.1. *The polygon of a product is obtained from the polygons of the factors by joining their sides end to end according to nondecreasing slope.*

For example, for the prime 3, the factors x^2+3x+9, $x^3-3x+27$ have the polygons shown in Fig. 1 and 2, respectively. These polygons are then combined in order of non decreasing slopes to form the polygon shown in Fig. 3. This polygon corresponds to the product $(x^2+3x+9)(x^3-3x+27)=x^5+3x^4+6x^3+18x^2+54x+243$.

Corollary 2.2. *If the polynomial $P(x)$, with rational coefficients is a product of quadratic factors, over the rationals, then its Newton polygon is combined of sections of (horizontal) length 2. These sections are flat or have slope* 1/2 *an integer.*

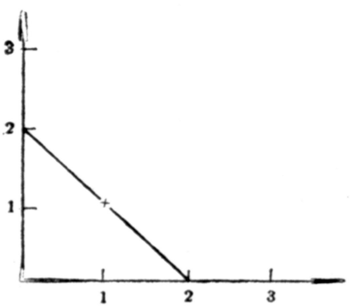

Fig. 1

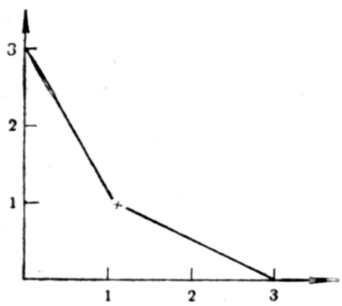

Fig. 2

3. The main theorem. Here we prove that the distance-regular graph with intersection array (1.4) and diameter $d=25$ or 25 does not exist. The proof is based on a theorem due to Bannai and Ito [1] and it makes use of the technique of Newton polygons [4]. We ommit graphs corresponding to $c=1$ or k and $k=2$. These graphs have been studied in [2; 3].

Definition 3.1. *Following Biggs* [2, ch. 23] *we define $h=k-1$ and $e=c-1$. We also define the polynomial $H_d(Y)=(Y-2)H_{d-1}(Y)-H_{d-2}(Y)+M(Y)$, $d\geq 2$, where $H_0(Y)=(k-c)^2$, $H_1=h^2Y-hc^2$, $M(Y)=2[h(c-1)Y+(h-c+1)^2-hc^2]$.*

Theorem 3.2. *Suppose that a distance-regular graph with intersection array* (1.4) *and diameter d exists, then the roots of the corresponding polynomial* $H_d(Y)$ *are all rational or quadratic.* (For a proof of this theorem see [1, th. 4]).

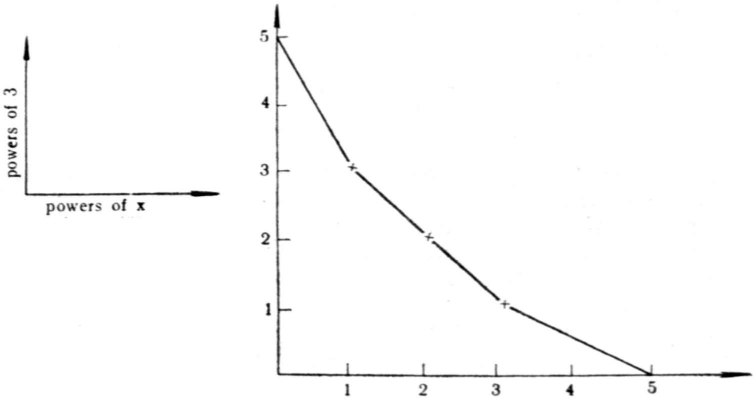

Fig. 3

Definition 3.3. *If n is an integer, then the order of n, written* ord (n), *is the largest number j such that* 2^j *divides n.*

Definition 3.4. *We define* $r = \text{ord}(h)$, $s = \text{ord}(e)$, $u = \text{ord}(h-e)$, $v = \text{ord}(h+e)$, $w = \text{ord}(e+1)$ *and* $L_d(Y) = 2^{-p}H_d(Y)$, *where* $p = \min(r, 2s)$.

Lemma 3.5. *If the distance regular graph with intersection array I and diameter* $d = 24$ *or* 25 *exists then for every odd* θ, $\text{ord}\, H_d(\theta) \geqq 8 + p$.

Lemma 3.6. $H_{24}(1) = (h-e)^2$ *and* $H_{25}(1) = h^2 - h(e+1)^2$. (For a proof of the above two lemmas see [5, Lemmas 4.4 and 4.5].)

It has been proved [5, ch. 4] that

$d = 25$ is impossible unless $r > 2s \geqq 0$,

$d = 24$ is impossible unless $2s > r > 0$ or $r = s = 0$.

Call the above graphs Γ_1, Γ_2, respectively.

Theorem 3.7. *The distance regular graphs* Γ_1 *and* Γ_2 *do not exist.*

Proof. Suppose that Γ_1 or Γ_2 exist. Then Lemmas 3.5 and 3.6 gne: that $2u \geqq 8 + r$ when $d = 24$, $\min(2r, r + 2w) \geqq 8 + 2s$ when $d = 25$. Also from [5, ch. 2] we get that for $d = 25$ or 24 and $Y = z + 1$

$$L_d(Y) \equiv \begin{cases} (z+1)^8 z^{16}, & r = s = 0 \\ (z+1)^7 z^{16}, & 2s > r > 0 \\ & r > 2s \geqq 0 \end{cases} \pmod 2.$$

Thus in the polynomial $L_d(Y)$, $d = 24$ or 25, the coefficient of z^m for any m such that $0 \leqq m \leqq 16$ is even while the coefficient of z^{16} is odd. Thus for the prime 2 we have a family of Newton Polygons with parameters r and s that have the point (16, 0) in common. Now by Theorem 3.2 the polynomial $L_d(Y)$, $d = 24$ or 25 is a product of quadratic factors over the rationals. Thus by corollary 2.2 the non flat sections of the relevant Newton Polygon will have slope greater than or equal 1/2 an integer.

Thus the coefficient of z^8 will have order greater than or equal $1/2 . 8 = 4$. Now by computing coefficients we get that the actual coefficient of z^8 is

$$3 \cdot [10505\, h^2 - 407\, h(e^2+1) - 5716\, e^2] \quad \text{if} \quad d = 25,$$

$$3 \cdot [5716 h(e^2+1) + 4372 e^2 + 407\, (h^2 - e^2)] \quad \text{if} \quad d = 24,$$

but for $d = 25$ the orders of the terms of the coefficient of z^8 in $L_{25}(Y)$ are

$$(1) \qquad \begin{array}{l} 2r - 2s,\ r - 2s,\ 2 \ \text{ when } \ s \neq 0, \\ 2r,\ r+1,\ 2 \ \text{ when } \ s = 0. \end{array}$$

Let $s \neq 0$. Then $w = \operatorname{ord}(e+1) = 0$ and $\min(2r, r+2w) \geqq 8 + 2s$ hence $r \geqq 8 + 2s$. Thus by (1) we get that order of the coefficient is 2. Let $s = 0$. Since $r \geqq 1$ we have that the coefficient is again of order 2. Finally when $d = 24$ the orders of the terms of the coefficient of z^8 in $L_{24}(Y)$ are

$2,\ 2s - r + 2,\ u + v$ when $2s > r > 0$,

$3,\ 2,\ u + v$ when $r = s = 0$.

Now since $2u \geqq 8 + r$ we have that $u \geqq 4$ for any $r \geqq 0$ hence $u + v > 2$. Thus the coefficient of z^8 in both $L_{24}(Y)$ and $L_{25}(Y)$ is not divisible by 2^4. This proves the theorem.

REFERENCES

1. E. Bannai, I. Ito. On the spectra of certain distance-regular graphs, I. *J. Comb. Theory*, **27**, 1979, 274—293.
2. N. L. Biggs. Algebraic graph theory. *Cambridge tracts in Math.*, **67**. Cambridge, 1974.
3. R. M. Damerell. On Moore graphs. *Proc. Camb-idge Philos. Soc.*, **74**, 1973, 227—236.
4. M. G. Dumas. Sur quelques cas d'irreductibilité de polynomes à coefficients rationnels. *J. Math.*, *Sér.*, *6*, **2**, 1906, 191—258.
5. M. A. Georgiacodis. On a class of Distance-Regular graphs. Ph. D. thesis, London, 1978.

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