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SOLUBILITY OF FINITE GROUPS WITH A TWO-VARIABLE COMMUTATOR IDENTITY

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The subject of our recent research has been groups which satisfy a commutator identity of the type: $[x,_m y]=[x,_n y]$, $m, n \in \mathbb{N}$, $m<n$. If G is such a group, then there exists a minimal law $[x,_{m_0} y]=[x,_{n_0} y]$, $m_0<n_0$, for it. The invariants m_0, n_0 depend essentially on the structure of the group. On the other hand, the structure of the group can be deduced in terms of m_0, n_0. With the purpose of obtaining a characterization in terms of the first invariant m_0 for small values of m_0, in this paper we are interested in the class of finite groups with a minimal law $[x,_2 y]=[x,_n y]$, $2<n$. Such groups turn out to be soluble.

1. An Engel word in variables x, y is a left-normed commutator $e_m(x, y)=[x, \underbrace{y, y, \ldots, y}_{m \text{ entries}}]$. We are interested in the laws of the type

$$e_m(x, y)=e_n(x, y), \quad m<n \tag{1}$$

which hold in a group G. Such a law is said to be minimal, if and only if G has no similar law with a lexicographically smaller pair (m, n).

Every finite group has such a law for some m, n. We want to deduce the structure of finite groups in terms of the first invariant m. In this paper we are interested in finite groups with a minimal law

$$e_2(x, y)=e_n(x, y), \quad 2<n, \tag{2}$$

because, on one hand, finite groups with a minimal law $e_1(x, y)=e_n(x, y)$, $1<n$, are abelian [3] and, on the other hand, there exist finite simple groups with a minimal law $e_3(x, y)=e_n(x, y)$, $3<n$, as $PSL(2,4) \simeq PSL(2,5) \simeq A_5$ and $PSL(2,8)$ for which we obtain $e_3(x, y)=e_{63}(x, y)$ [4] and $e_3(x, y)=e_{129}(x, y)$, respectively. The main result here is the following

Theorem. *Every finite group in which the minimal law of type* (1) *is* $e_2(x, y)=e_n(x, y)$, $2<n$, *is soluble.*

1.1. Notations and definitions. We write $x^y=yxy^{-1}$, $[x, y]=xyx^{-1}y^{-1}$, $[x_1, x_2, \ldots, x_n]=[[x_1, x_2, \ldots, x_{n-1}], x_n]$. We set $e_n(x, y)=[x,_n y]=[x, \underbrace{y, y, \ldots, y}_{n \text{ entries}}]$.

Let $1=Z_0 \leq Z_1 \leq Z_2 \leq \ldots$ and $G=\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq \ldots$ be, respectively, the upper and the lower central series of a group G.

We denote by $\mathrm{Syl}_p(G)$ the family of Sylov p-subgroups of G.

Following Gruenberg, we say that a group G is *an Engel group*, if for every pair of elements $x, y \in G$ there is an integer $k=k(x, y)$ such that $e_k(x, y)=1$. If $e_n(x, y)$ is a law in G, then G satisfies the n-th Engel condition. If G is an Engel group which satisfies the n-th but not the $(n-1)$-st Engel condition, then G has Engel class n.

A minimal simple group is a finite simple group of a composite order all of whose proper subgroups are soluble. Those groups have been described by Thompson [7].

Consider the matrices $E_{ij} \in GL(n, K)$, where K is a field, which contain on the (i, j)-th position 1 and 0 elsewhere. We remind that the matrix $t_{ij}(\alpha)=E+\alpha E_{ij}$, $\alpha \in K$, $\alpha \neq 0$, is called *a transvection.*

All unexplained notations are standard and can be found in [1] or [2].

1.2. Some preliminary results. Lemma 1. *If G is a nilpotent group whose minimal law of type* (1) *is* (2), *then*

(i) *G is an Engel group of class* 2,

(ii) *G is nilpotent of class* ≤ 3,

(iii) *G is metabelian,*

(iv) *the centralizer of each element $x \in G$ comprises the contjugacy class generated by x.*

Proof. (i) G is an Engel group of class 2 by Lemma 3 of [3].

(ii) Then we use a well-known result of Levi [5], that is: each group all of whose 2-generated subgroups are nilpotent of class 2, is nilpotent of class 3 (and the exponent of the third term of its lower central series is a divisor of 3).

(iii) Since each commutator of weight n is a product of left-normed commutators and their inverses of weight n (i. e. is their consequence [2, 33.35]), the group G is metabelian. This fact follows as well by the well-known group-theoretical inclusion: $G^{(k)} \leq \Gamma_{2k}$.

(iv) $\forall x \in G$, $\forall y \in G$, $[y\ x, x]=1$, which give $[y, x]x=x[y, x]$, $yxy^{-1}=xyxy^{-1}x^{-1}$, $(yxy^{-1})x=x(yxy^{-1})$.
Thus we have $[x^y, x]=1$, $\forall x \in G$, $\forall y \in G$.

Lemma 2. *Every finite nilpotent group G, such that 3 does not divide its order $|G|$, and he minimal law of type* (1) *in G is* (2), *has nilpotency class* 2.

Proof. Consider the lower central series. By Lemma 1 we get $G=\Gamma_1 > \Gamma_2=G' > \Gamma_3 > \Gamma_4=1$ and $G=\Gamma_1 \neq \Gamma_2=G' \neq \Gamma_3 > \Gamma_4=1$.

The same result of Levi [5] gives $\exp(\Gamma_3)|3$. Since there exist no elements of order 3 in G, we have $\Gamma_3=1$ and hence $[x_1, x_2, x_3]=1$ is a law in G. In particular, if G is a finite group with a law of type (2), every subgroup of $\mathrm{Syl}_p(G)$, $p \neq 3$, is either abelian, or nilpotent of class 2.

2. Proof of the Theorem. The theorem is proved by examining a counter-example G_0 of least possible order. So, every proper subgroup of G_0 is soluble. Assume that G_0 is not simple, i. e. there exist $H_0 \lhd G_0$, $1 \neq H_0 \neq G_0$. Both groups H_0 and G_0/H_0 are soluble, as they satisfy laws of type (2) (because G_0 does), and have orders smaller than G_0. Then G_0 is soluble too, contrary to our assumption. Hence, G_0 is a minimal simple group, belonging to the list given by Thompson [7]:

(i) $PSL(2, 2^p)$, for any prime p;

(ii) $PSL(2, 3^p)$, where p is an odd prime;

(iii) $PSL(2, p)$, where p is a prime, $p\ 3$, $p^2+1=0 \pmod 5$);

(iv) $PSL(3, 3)$;

(v) $Sz(2^p)$, where p is an odd prime number.

2.1. $G_0 \neq PSL(3,3)$. Assume that in $G_0 = PSL(3, 3) = SL(3,3)$ the minimal law is (2). Then, the minimal identity in $H_0 = SL(2,3) < G_0$ is either of type $e_1(x, y) = e_n(x, y)$, $1 < n$, or of type (2). Since the former case is not possible, because H_0 is not abelian (see [3]), the minimal law in H_0 is of type (2) as well. The only normal subgroups in H_0 are $S = \mathrm{Syl}_2(H_0)$ and $Z(H_0) = gp(z)$ of order 2. Consider an element $s \in S$, $s \neq 1, z$. Since $[h, s] \in S$, $\forall h \in H_0$, we have $[h,_3 s] = 1$. But then $e_n(h, s) = 1 = e_2(h, s)$ and $[h,_2 s] = 1$ yields $[s^h, s] = 1$ as in Lemma 1 (iv). Hence, the normal closure of s in H_0: (s^{H_0}) is abelian, which is impossible since $S = (s^{H_0})$.

2.2. G_0 is not a Suzuki group. Assume $G_0 = Sz(q)$, where $q = 2^p$ and p is an odd prime number. Then [6] the (ZT)-group G_0 is of order $q^2(q-1)(q^2+1)$. Let a, b be two of the symbols on which G_0 acts. Consider the following three subgroups:

H: consisting of all elements in G_0, leaving a invariant,

Q: consisting of all elements in $H \setminus \{1\}$, leaving only a invariant,

K: consisting of all elements in H, leaving b invariant.

Then H is the split extention of Q by $K: H = QK$. We know further that:

(a) $Q \in \mathrm{Syl}_2(G_0)$, $Q \lhd H$, $|Q| = q^2$, $\exp(Q) = 4$;

(b) $|Z(Q)| = q$ and $Z(Q)$ consists of all involutions in Q together with the unit;

(c) $\forall \sigma \in Q$, $\sigma \neq 1 \Rightarrow C_{G_0}(\sigma) \subseteq Q$;

(*d*) K is a cyclic group of automorphisms; its order coincides with the number of the involutions of $Q: q^{-1}$; an element $\neq 1$ of K leaves only the identity invariant; K permutes the set of involutions transitively and cyclically.

Consider an element $s \neq 1$ of Q whlch is not an involution, i. e. $|s| = 4$. The number of those elements is $q^2 - q$ and they are divided into two conjugacy classes in G_0 (since the number of irreducible characters of G_0 is equal to the number of classes of conjugate elements (see [6, Proposition 18]). Moreover, two elements of the group Q are conjugate in G_0 if and only if they are conjugate in H. Find the order of $C_H(s)$. The centralizer $C_H(s)$ contains $gp(s, Z)$ of order $2q$, where $Z = Z(Q)$. Thus,

$$|s^H| = \frac{|H|}{|C_H(s)|} \underset{(c)}{=} \frac{|H|}{|C_Q(s)|} \leq \frac{q^2(q-1)}{2q} = \frac{q(q-1)}{2},$$

i. e. the two classes together contain at most $2q(q-1)/2 = q^2 - q$ elements. This means that each class of elements of order 4 contains exactly $q(q-1)/2$ elements. Hence, $|C_H(s)| = 2q^2(q-1)/q(q-1) = 2q$.

On the order hand, $\forall h \in H$, $[h, s] \in Q$. As by our assumption the minimal law in G_0, and so in H is of type (2), then we have $[h,_3 s] = 1$. Hence, $[h,_2 s] = 1$ and $[s, s^h] = 1$. But then $s^H \subseteq C_H(s)$, which is impossible since in $C_H(s)$ there exist only q elements of order 4, while $|s^H| = q(q-1)/2$. Thus G_0 could not be a Suzuki group.

2.3. G_0 doesn't belong to the series (i) and (ii). It remains to consider the groups of the type $PSL(2, q)$. The idea here consists in the following: find two matrices A, $B \in PSL(2, q)$, such that $[A,_k B]$ are all transvections, $\forall k > 2$, while $[A, B]$ and $[A,_2 B]$ are not. Then the minimal equality for the elements A and B would be of the type $e_m(A, B) = e_n(A, B)$, $3 \leq m < n$, and $PSL(2, q)$ could not have a minimal law of type (2). Hence, $G_0 \neq PSL(2,q)$.

Since in the cases of consideration $q>4$, $q\neq 5$, there exists an element $\varepsilon\in GF(q)$, such that $\varepsilon^2\neq\pm 1$. Consider the matrix

$$B=\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}.$$

It has the following interesting property: $\forall\lambda\in GF(q)$, $[t_{12}(\lambda),_k B]=t_{12}(\lambda(1-\varepsilon^2)^k)$, i. e. if $p=\operatorname{char} GF(q)+\lambda$ we obtain nontrivial transvections for any k. If we find a matrix A, such that $C=[A,_2 B]$ is not a transvection, while $[C, B]=t_{12}(\lambda)$, this would complete the proof of the theorem.

Let

$$A=\begin{pmatrix}x & y\\ z & t\end{pmatrix},\quad xt-yz=1,\quad C=[A,_2 B]=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad ad-bc=1.$$

By the straightforward computation we look for a C of the form

$$C=\begin{pmatrix}\alpha & *\\ 0 & \alpha^{-1}\end{pmatrix},\qquad \text{where}\quad \alpha\neq 1,$$

since we don't want C to be a transvection. Thus we get

$$C=[A,_2 B]=\begin{pmatrix}\varepsilon^{-2} & u\\ 0 & \varepsilon^2\end{pmatrix}.$$

Now, we impose on C the condition $[C, B]=t_{12}(1)$ which gives

$$C=\begin{pmatrix}\varepsilon^{-2} & \varepsilon^2(1-\varepsilon^2)^{-1}\\ 0 & \varepsilon^2\end{pmatrix}.$$

The variables x, y, z, t we determine by the following system of equations $\det A=1$, $c=0$, $b=\varepsilon^2(1-\varepsilon^2)^{-1}$.

For $y=1$ we get

$$A=\begin{pmatrix}-\varepsilon^4(\varepsilon^2+1)^{-1}(\varepsilon^2-1)^{-3} & 1\\ (\varepsilon^2-1)^{-1} & -\varepsilon^{-2}(\varepsilon^2+1)(\varepsilon^2-1)^2\end{pmatrix}.$$

Thus $[A,_2 B]\neq[A,_k B]$, $\forall k>2$ and $[A, B]\neq[A,_l B]$, $\forall l>1$, since $[A, B]$ and $[A,_2 B]$ are neither transvections, nor can coincide with the unit matrix. This completes the proof of the theorem.

3. Further on, it seems interesting to characterize the class of soluble groups with a minimal law (2). Some examples explored by the author as the symmetric group S_3, the two nonabelian groups of order 8: D_4, Q_8, the groups D_6, A_4, the varieties $\mathfrak{A}_k\mathfrak{A}_l$ for $(k, 1)=1$, yielded the assumption that the solubility length of such groups is 2. However, this is not true as there exist groups in the class which are not metabelian.

P r o p o s i t i o n. *There exist soluble groups with a law* (2), *which are not metabelian.*

P r o o f. Consider the nonabelian group N of order p^3 and exponent p, where p is an odd prime number $N=\langle a, b\mid a^p=b^p=[a, b]^p=1\rangle$. If we denote by c the commutator $[a, b]$, each element x of the subgroup N has the form $x=a^k b^l c^t$, $0\leq k, l, t\leq p-1$. Two elements of N are multiplied in the following way: $(a^k b^l c^t)(a^{k_1} b^{l_1} c^{t_1})=a^{k+k_1} b^{l+l_1} c^{t+t_1-k_1 l}$.

Since $|N/Z(N)|=p^2$, $N'\subseteq Z(N)$ then $[x, y, z]=1$ is a law in N. We get $N'=Z(N)=(C)$.

Consider the mapping $\alpha: x=a^k b^l c^t \overset{\alpha}{\longrightarrow} a^{-k}b^{-l}c^t$. It is easy to see that α is an automorphism (of order 2) of the group N.

Consider the split extension G of N by the cyclic group $H=(\alpha)$, i. e.: $G=N\lambda H$, $N\lhd G$, $N\cap H=1$, $G=NH$.

Since $G'=N$, we get $G''\neq 1$ and G is not metabelian. Actually, G is soluble of class 3: $G\rhd N\rhd Z(N)=N'\rhd 1$.

Let us find the law in G

(a) $y\in N$, i. e. $y=a^k b^l c^t$, $0\leq k, l, t\leq p-1$.

An element $z=a^\varkappa b^\lambda c^\theta$ belongs to the centralizer $C_N(y)$ if and only if $\varkappa l=\lambda k \pmod p$. Thus $y^\alpha\in C_N(y)$. Consider $x\in G$, i. e. $x=\alpha^\varepsilon n$, $\varepsilon=0, 1$, $n\in N$:

$$[x, y]=[\alpha^\varepsilon n, y]=[n, y]^{\alpha^\varepsilon}[\alpha^\varepsilon, y]=n' y^{\alpha^\varepsilon} y^{-1}\in C_N(y).$$

So that $[x,_2 y]=1$ and the minimal equality for those two elements is $[x,_2 y]=[x,_3 y]$.

(b) $y\notin N$, i. e. $y=\alpha a^{k_2} b^{l_2} c^{t_2}$, $0\leq k_2, l_2, t_2\leq p-1$.

Let $x\in G$, $x=\alpha^{\varepsilon_1} a^{k_1} b^{l_1} c^{t_1}$, where $0\leq k_1, l_1, t_1\leq p-1$, $\varepsilon_1=0, 1$. Consider two cases

b.1. $\varepsilon_1=0$. We can prove by induction on m that $[x,_m y] = a^{2^m k_1} b^{2^m l_1} c^{2^{m-1}[k_1(l_2-2^m l_1)-k_2 l_1]}$.

If m is the smallest positive integer with the property $p\,|\,(2^{m-1}-1)$, then the minimal equality for this pair of elements is $[x, y]=[x,_m y]$.

b.2. $\varepsilon_1=1$. Here again by induction on m we get

$$[x,_m y]=a^{2^m(k_2-k_1)} b^{2^m(l_2-l_1)} c^{2^{m-1}\{k_1[(2^m-1)l_2-2^m l_1]+k_2[(2^m+1)l_1-2^m l_2]\}}.$$

If m is the same as in b. 1., we obtain $[x, y]=[x,_m y]$.

Hence the minimal law in the group G is $e_2(x, y)=e_{m+1}(x, y)$, where m is the smallest positive integer with the property $p\,|\,(2^{m-1}-1)$ ($m\leq p$ by the theorem of Ferma-Oiler).

By an exhaustive search computer program we have shown as well that the minimal identity in the symmetric group S_4 is $e_2(x, y)=e_{14}(x, y)$.

A characterization of soluble groups with a minimal law (2) is still not known to the author.

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Received 8. 2. 1983
Revised 30. 7. 1983