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VECTOR OPTIMIZATION AND EXISTENCE THEOREMS IN GAME THEORY

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It is well known that the Pareto optimality plays an important role in the mathematical economy and game theory. The infinite dimensional generalization of this optimality concept has been investigated for two decades (see [1]). Joining to this research, in this paper the game-theoretical applications of existence theorems of vector optimization are investigated. In section 1 the basic notions are defined and a simple Weierstrass type theorem is proved. In section 2, on the basis of the formalization proposed in [2], the existence of a cooperative solution of a game with infinite number of players is considered. In section 3 the existence of a minimax type solution of a general antagonistic game with infinite number of pay-off functions is proved.

1. In our investigations the following definitions will be needed.

Definition. 1. *Let Z be a real Banach space, $K\subset Z$ a convex sharp cone. (A cone K is called sharp if $z\in K$, $-z\in K$ imply $z=0$.) The pair (Z, K) is said to be an ordered Banach space. In convention with this ordering the following notations will be used. For any $z_1, z_2\in Z$ we shall write $z_1\leqq z_2$, $z_1\leq z_2$ and $z_1<z_2$, if $z_2-z_1\in K$, $z_2-z_1\in K\setminus\{0\}$ and $z_2-z_1\in \operatorname{int} K$, respectively.*

Definition 2. *Let $\mathcal{U}\neq\emptyset$ be an arbitrary set, (Z, K) be an ordered Banach space and $f:\mathcal{U}\to Z$ be a given function. An element $u_0\in\mathcal{U}$ is said to be a K-minimum point of the function f, if there exists no $u_1\in\mathcal{U}$ with $f(u_1)\leq f(u_0)$.*

We notice that in case $Z:=\mathbf{R}^N$ the ordering is usually given by the cone $K:=\mathbf{R}^N_{+0}$ of non-negative vectors. Then, as a special case, the Pareto minimality is obtained. In the classical function spaces, in most cases, the order cone is that of (a. e.) non-negative functions.

The K-maximum point is defined analogously.

Definition 3. *Let $\mathcal{U}\neq\emptyset$ be a topological space, (Z, K) be an ordered Banach space and suppose that $\operatorname{int} K\neq\emptyset$. A function $f:\mathfrak{U}\to Z$ is said to be K-lower semicontinuous (briefly K—l. s. c.) at a point $u_0\in\mathfrak{U}$, if for every $\gamma\in\operatorname{int} K$ there exists a neighbourhood $k(u_0)$ of u_0 such that $f(u)\geqq f(u_0)-\gamma$ $(u\in k(u_0))$. f is called K—l. s. c, if it is K—l. s. c. at every $u_0\in\mathcal{U}$.*

The notation of K-upper semicontinuity (K-u. s. c.) is defined analogously.

Remark 1. It follows from $\operatorname{int} K\neq\emptyset$ that any continuous function $f:\mathcal{U}\to Z$ is also K-l. s. c. and K-u. s. c. It is also not hard to see that in case of $Z:=\mathbf{R}^N$, $K:=\mathbf{R}^N_{+0}$ $f:\mathcal{U}\to Z$ is K-l. s. c. if and only if every coordinate function of f is l. s. c. in the usual sense.

Definition 4. *Let (Z, K) be an ordered Banach space. Then a continuous linear functional $p\in Z^*$ is called strictly positive, if for each $z\in K\setminus\{0\}$ $\langle p, z\rangle>0$.*

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The following theorem is a generalization of the Weierstrass theorem to ordered Banach spaces.

Theorem 1. *Let $\mathcal{U}\neq\emptyset$ be a compact topological space, (Z, K) be an ordered Banach space,* int $K\neq\emptyset$ *and suppose that there exists a strictly positive functional $p\in Z^*$, and $f:\mathcal{U}\to Z$ is K-l. s. c. Then the function f has a K-minimum point.*

Proof. First, we show that $p\circ f:\mathcal{U}\to\mathbf{R}$ is l. s. c. in the usual sense. Let $u'\in\mathcal{U}$ and $\varepsilon\in\mathbf{R}_+$ be arbitrary. Since p is continuous, there exists a $\delta\in\mathbf{R}_+$ such that for any $z\in Z$ with $\|z\|<\delta$ we have $|\langle p, z\rangle|<\varepsilon$. By int $K\neq\emptyset$, there is a $\gamma\in$ int K with $\|\gamma\|<\delta$. By the K-l. s. c. of f there exists a neighbourhood $k(u')$ such that $f(u)-f(u')+\gamma>0$ $(u\in k(u'))$. Thus we get

$$0<\langle p, f(u)\rangle-\langle p, f(u')\rangle+\langle p, \gamma\rangle<(p\circ f)(u)-(p\circ f)(u')+\varepsilon,$$

that is, $p\circ f$ is l. s. c. By Weierstrass's theorem the function $p\circ f$ has a minimum point $u_0\in\mathcal{U}$. u_0 is a K-minimum point of f. Indeed, supposing the contrary, there exists a $u_1\in\mathcal{U}$ for which $f(u_1)\leqq f(u_0)$. Since p is strictly positive we have $(p\circ f)(u_1)<(p\circ f)(u_0)$, which is a contradiction.

Remark 2. If the Banach space Z is separable and K is closed then by Krein's theorem there exists a strictly positive functional $p\in Z^*$ [3, theorem 2. 1.]. A counterexample in [3] shows that in nonseparable spaces a strictly positive functional does not necessarily exist. However, there exists such a functional, for example, in the ordered Banach space

$$Z:=L_\infty([t_0, t_1], \mathbf{R}),$$

$$K:=\{z\in L_\infty([t_0, t_1], \mathbf{R}): z(t)\geqq 0 \text{ for a. e. } t\in[t_0, t_1]\}.$$

Indeed, the continuous linear functional $p:Z\to\mathbf{R}$, $\langle p, z\rangle:=\int_{t_0}^{t_1}z$ $(z\in Z)$ is clearly strictly positive.

2. In the following, we shall show how Theorem 1 guarantees the existence of a cooperative solution of games with infinite number of players.

Let $(\mathcal{E}, \rho)$ be a metric space, $(\mathcal{N}, \lambda)$ be a compact metric space, $\pi:\mathcal{E}\to\mathcal{N}$ be a continuous surjection. A mapping $\mathcal{U}:\mathcal{N}\to\mathcal{E}$ is called a section of π, if $\pi_\circ u=\mathrm{id}_{\mathcal{N}}$. Denote by $\mathcal{U}$ the set of all continuous sections of π. The elements of the space $\mathcal{N}$ are interpreted as players while the elements of $\mathcal{U}$ are called cooperative strategies. For any $\nu\in\mathcal{N}$ the metric space $\mathcal{U}_\nu:=\pi^{-1}(\nu)$ is considered as the strategy space of the player ν. Fot any $u', u''\in\mathcal{U}$

$$d(u', u''):=\sup\{\rho(u'(\nu), u''(\nu)): \nu\in\mathcal{N}\} \tag{1}$$

is a metric on $\mathcal{U}$. Suppose that each player $\nu\in\mathcal{N}$ a pay-off function $f_\nu:\mathcal{U}\to\mathbf{R}$ is given such that for every $u\in\mathcal{U}$ the mapping $\nu\to f_\nu(u)$ $(\nu\in\mathcal{N})$ is continuous. (i. e. given a cooperative strategy, "close" players have "close" pay-offs). The joint pay-off function f is defined as follows:

$$f:\mathcal{U}\to C(\mathcal{N}),\quad f(u)(\nu):=f_\nu(u)\quad(u\in\mathcal{U}, \nu\in\mathcal{N}).$$

The pair (π, f) if called a cooperative game.

Now, let $\emptyset\neq\mathcal{U}_0\subset\mathcal{U}$ be an arbitrary set of cooperative strategies. Then a cooperative strategy $u_0\in\mathcal{U}_0$ is said to be a solution of the game (π, f) with respect to $\mathcal{U}_0$, if there exists no $u_1\in\mathcal{U}_0$ such that for every $\nu\in\mathcal{N}$ the inequality $f_\nu(u_1)\geqq f_\nu(u_0)$ holds and besides $f_{\nu_0}(u_1)>f_{\nu_0}(u_0)$ for some $\nu_0\in\mathcal{N}$. It is clear that $u_0 \in \mathcal{U}_0$ is a solution of (π, f) with respect to $\mathcal{U}_0$ if and only if u_0 is a K-maximum point of the function $f|_{\mathcal{U}_0}: \mathcal{U}_0 \to Z: = C(\mathcal{N})$, where $K: = \{z \in C(\mathcal{N}): z(\nu) \geqq 0 \ (\nu \in \mathcal{N})\}$.

Suppose the set $\mathcal{U}_0$ is compact with respect to the metrics d, and $f_{\mathcal{U}_0}$ is K-u. s. c. Since $C(\mathcal{N})$ is separable, by Krein's theorem there exists a strictly positive functional in $C(\mathcal{N})^*$. Thus, from Theorem 1, it follows that the game (π, f) has a solution with respect to $\mathcal{U}_0$.

Now consider the connection between the classical cooperative games with a finite number of players and the general formalization given above. If the number of players is $N \in \mathbf{N}$, then $\mathcal{N}: = \{1, \ldots, N\}$ is compact with respect to the discrete metrics. For each $\nu \in \mathcal{N}$ let the compact metric space $(\mathcal{U}_\nu, \rho_\nu)$ be interpreted as the strategy space of player ν. Then, on the disjoint union $\mathcal{E}: = \cup_{\nu=1}^{N} \mathcal{U}_\nu$, define the following mapping: $\pi: \mathcal{E} \to \mathcal{N}$, $\pi(e): = \nu$, if $e \in \mathcal{U}_\nu$. Define on the set $\mathcal{E}$ a metric ρ such that for any $e', e'' \in \mathcal{E}$ put

$$\rho(e', e''): = \begin{cases} 1, & \text{if } \pi(e') \neq \pi(e''), \\ \dfrac{\rho_\nu(e', e'')}{1+\rho_\nu(e', e'')}, & \text{if } \pi(e') = \pi(e'') = : \nu. \end{cases}$$

Since the metrics ρ induces the topological sum of the spaces $\mathcal{U}_\nu (\nu \in \mathcal{N})$, the mapping is continuous. Clearly, any section of π is continuous, thus $\mathcal{U} = \mathrm{X}_{\nu=1}^{N} \mathcal{U}_\nu$. Now, the metrics d defined in (1) induces the product topology on the set $\mathcal{U}$, which is compact. For each $\nu \in \mathcal{N}$, let $f_\nu: \mathcal{U} \to \mathbf{R}$ be the pay-off function of player ν, and suppose that they are u. s. c. Then, by Remark 1, the joint pay-off function $f: \mathcal{U} \to C(\mathcal{N}) = \mathbf{R}^N$ is K-u. s. c. on the compact space $\mathcal{U}_0 = \mathcal{U}$ with respect to the order cone $K: \mathbf{R}^N_{+0}$. Thus, from the above, we obtain the well-known existence theorem for $\mathcal{N}$-person cooperative games.

3. In this section the existence of a minimax type solution of antagonistic games with infinite number of pay-off functions is investigated.

Let $\mathcal{U} \neq \emptyset$ be a compact topological space, and let $(\mathcal{V}, \rho)$ and (S, λ) be non-empty compact metric spaces, suppose the function $F: \mathcal{U} \times \mathcal{V} \times S \to \mathbf{R}$ is l. s. c., and for each $u \in \mathcal{U}$ define the continuous function $F_u: \mathcal{V} \times S \to \mathbf{R}$, $F_u(v, s): = F(u, v, s)$. Put

$$G: \mathcal{U} \times \mathcal{V} \to C(S), \quad G(u, v)(s): = F(u, v, s) \quad (s \in S).$$

The pair $(\mathcal{U} \times \mathcal{V}, G)$ is called an antagonistic game with vector-valued pay-off function. $\mathcal{U}$ and $\mathcal{V}$ are interpreted as the strategy sets of player I and II, respectively, and G is called the pay-off function of player II.

Definition 5. *A strategy $u_0 \in \mathcal{U}$ is called a K-minimax solution for player I, if there is no $u_1 \in \mathcal{U}$ such that for each $s \in S$*

$$\max \{G(u_1, v)(s): v \in \mathcal{V}\} \leqq \max (G(u_0, v(s): v \in \mathcal{V}\},$$

and

$$\max \{G(u_1, v)(s_1): v \in \mathcal{V}\} < \max \{G(u_0, v)(s_1): v \in \mathcal{V}\}$$

for some $s_1 \in S$.

Under the above condition the following existence theorem is valid:

Theorem 2. *Let $\mathcal{U}\neq\emptyset$ be a compact topological space, let $(\mathcal{V}, \rho)$ and (S, λ) be non-empty compact metric spaces, suppose the function $F\colon \mathcal{U}\times\mathcal{V} \times S \to \mathbf{R}$ is l. s. c. and let be given the vector-valued pay-off function*

$$G\colon \mathcal{U}\times\mathcal{V} \to C(S), \quad G(u, v)(s) := F(u, v, s)\ (s \in S),$$

where the function $F_u\colon \mathcal{V}\times S \to \mathbf{R}$, $F_u(v, s) := F(u, v, s)$ is continuous for each $u \in \mathcal{U}$. Then in the antagonistic game with vector-valued pay-off function $(\mathcal{U}\times\mathcal{V}, G)$ there exists K-minimax solution for player I.

Proof. a) For every $(v, s) \in \mathcal{V}\times S$ define the following function: $F_{vs}\colon \mathcal{U} \to \mathbf{R}$, $F_{vs}(u) := F(u, v, s)$. We show that at each point $u_0 \in \mathcal{U}$ the elements of the set $\{F_{vs}\colon (v, s) \in \mathcal{V}\times S\}$ are equally l. s. c., that is for any $\varepsilon \in \mathbf{R}_+$ there exists a neighbourhood $k(u_0)\subset\mathcal{U}$ such that for each $u \in k(u_0)$ and $(v, s) \in \mathcal{V}\times S$

$$F_{vs}(u) > F_{vs}(u_0) - \varepsilon. \tag{2}$$

Now, let $\varepsilon \in \mathbf{R}_+$ be given arbitrarily. Since F is l. s. c. and F_{u_0} is uniformly continuous, for every pair $(v_0, s_0) \in \mathcal{V}\times S$ there exists an open neighbourhood $k(u_0, v_0, s_0)\subset\mathcal{U}\times\mathcal{V}\times S$ such that for any $(u, v, s) \in k(u_0, v_0, s_0)$

$$F_{vs}(u) - F_{v_0 s_0}(u_0) > -\varepsilon/2, \tag{3}$$

$$F_{vs}(u_0) - F_{v_0 s_0}(u_0) < \varepsilon/2. \tag{3'}$$

From the open covering $\{k(u_0, v_0, s_0)\colon (v_0, s_0) \in \mathcal{V}\times S\}$ of the compact set $\{u_0\}\times\mathcal{V}\times S$ choose a finite covering $\{k(u_0, v_1, s_1), \ldots, k(u_0, v_n, s_n)\}$. Then the set

$$k(u_0) := \bigcap_{i=1}^{n} pr_1\, k(u_0, v_i, s_i) \subset \mathcal{U}$$

is obviously a neighbourhood of u_0. Let $u \in k(u_0)$ and $(v, s) \in \mathcal{V}\times S$ be arbitrary. Then there exists an index $j \in \overline{1, n}$ for which $(u, v, s) \in k(u_0, v_j, s_j)$. Thus, from (3) and (3′) choosing $(v_0, s_0) := (v_j, s_j)$, it follows that

$$F_{vs}(u) - F_{vs}(u_0) = F_{vs}(u) - F_{v_j s_j}(u_0) + F_{v_j s_j}(u_0) - F_{vs}(u_0) > -\varepsilon$$

b) For each $s \in S$ define the function $F_s\colon \mathcal{U} \to \mathbf{R}$, $F_s(u) := \max\{F_{vs}(u) : v \in \mathcal{V}\}$. Then, for every $u_0 \in \mathcal{U}$ the elements of the set $\{F_s\colon s \in S\}$ are equally l. s. c. Indeed, let $\varepsilon \in \mathbf{R}_+$ be arbitrary. Then, by the arguments of a), there exists a neighbourhood $k(u_0)$ such that for every $u \in k(u_0)$ and $(v, s) \in \mathcal{V}\times S$ we have (2). Hence, it follows that for any $u \in k(u_0)$ and $s \in S$

$$F_s(u) = \max\{F_{vs}(u) : v \in \mathcal{V}\} > \max\{F_{vs}(u_0) : v \in \mathcal{V}\} - \varepsilon = F_s(u_0) - \varepsilon.$$

c) Define the function

$$f\colon \mathcal{U} \to \mathbf{R}^S,\ f(u)(s) := F_s(u) = \max\{F(u, v, s)\colon v \in \mathcal{V}\}\ (s \in S).$$

We show that $R_f \subset C(S)$. Fix a point $u_0 \in \mathcal{U}$ arbitrarily. Since $f(u_0)$, as a pointwise supremum of continuous functions is l. s. c., it is enough to prove that $f(u_0)$ is u. s. c. Let $\varepsilon \in \mathbf{R}_+$ and $s_0 \in S$ be arbitrary and consider the metrics

$$d\colon (\mathcal{V}\times S)\times(\mathcal{V}\times S) \to \mathbf{R},$$

$$d((v', s'), (v'', s'')) := \rho(v', v'') + \lambda(s', s''),$$

which defines the product topology on $\mathscr{V}\times S$. Since $F_{u_0}: \mathscr{V}\times S \to \mathbf{R}$ is uniformly continuous, there exists a $\delta \in \mathbf{R}_+$ such that

$$F_{u_0}(v, s) < F_{u_0}(v, s_0) + \varepsilon, \tag{4}$$

whenever $(v, s) \in \mathscr{V}\times S$ and $d((v, s), (v, s_0)) < \delta$. The last condition is equivalent to $\lambda(s, s_0) < \delta$. Thus, for every $s \in k_\delta(s_0)$ relation (4) holds for each $v \in \mathscr{V}$. Therefore,

$$f(u_0)(s) = \max\{F_{u_0}(v, s):\ v \in \mathscr{V}\} < \max\{F_{u_0}(v, s_0):\ v \in \mathscr{V}\} + \varepsilon = f(u_0)(s_0) + \varepsilon$$
$$(s \in k_\delta(s_0)).$$

d) The function $f: \mathscr{U} \to C(S)$ is K-l. s. c. with respect to the order cone

$$K := \{z \in C(S):\ z(s) \geqq 0 (s \in S)\} \tag{5}$$

of the space $C(S)$. Indeed, choose $\gamma \in \operatorname{int} K$ arbitrarily. Since S is compact and the continuous function $\gamma: S \to \mathbf{R}$ is everywhere positive, $\varepsilon := \min \gamma > 0$. From b) it follows the existence of a neighbourhood $k(u_0)$ such that for every $u \in k(u_0)$ and $s \in S$ we have $F_s(u) > F_s(u_0) - \varepsilon \geqq F_s(u_0) - \gamma(s)$, consequently $f(u) > f(u_0) - \gamma\ (u \in k(u_0))$.

e) Finally, (S, λ) is a compact metric space. Thus the Banach space $C(S)$ is separable. Since the cone (5) is closed, by Krein's theorem there exists a strictly positive functional $p \in C(S)^*$. Thus, using Theorem 1, statement d) implies that the function f has a K-minimum point $u_0 \in \mathscr{U}$. From Definition 5, it immediately follows that u_0 is a K-minimax solution for player I.

Theorem 2 is proved.

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