# Serdica

## Bulgariacae mathematicae publicationes

# Сердика

## Българско математическо списание

# APPROXIMATION OF A CLASS OF BOUNDED CONVEX FUNCTIONS BY BERNSTEIN POLYNOMIALS IN $L_1$

VIOLETA AT. KOSTOVA

In the paper an estimate is obtained for the approximation of a class of bounded convex functions by Bernstein polynomials in $L_1$. It is proved that the estimate is exact with respect to the order.

We shall use the following notations:
$L[a,b]$ — the set of all bounded and measurable functions on $[a, b]$;

$$\|f(x)-g(x)\|_{L_1}=\int_a^b |f(x)-g(x)|\,dx$$

— the distance between $f, g\in L_{[a,b]}$; $K^M_{[a,b]}=\{f(x);\ f\in L_{[a,b]},\ f(\alpha x_1-(1-\alpha)x_2)\leq \alpha f(x_1)+(1-\alpha)f(x_2),\ x_1,\ x_2\in[a, b],\ 0\leq\alpha\leq 1,\ \max[\,|f(x)|,\ a\leq x\leq b]\leq M<\infty\}$ is the set of the bounded and convex functions in $L_{[a,b]}$;

$$B_n(f;\ x)=\sum_{\nu=0}^{n} f(\frac{\nu}{n})P_{n\nu}(x),\ \text{where } P_{n\nu}(x)=\binom{n}{\nu}x^\nu(1-x)^{n-\nu},$$

is the Bernstein polynomial for $f\in K^M_L$.

In [1; 3] is proved

Theorem 1. *Let $f$ be a function of bounded variation on* $[0, 1]$. *Then* $\|B_n(f;\ x)-f(x)\|_{L_1}\leq CV_0^1 fn^{-1/2}$.

We prove

Theorem 2. *Let* $f\in K^M_{[0,1]}$. *Then* $\|B_n(f;\ x)-f(x)\|_{L_1}=O(n^{-1})$, *where* $O(1)$ *depends only on* $M$.

For the proof of Theorem 2 we need two lemmas.

Lemma 1. *Let $g(\lambda)$ be the convex increasing function*

$$g(\lambda;\ x)=\max\{0,\ M(x-\lambda)/(1-\lambda)\},\quad \lambda\in(0,\ 1),\quad M>0.$$

*Then for* $n\geq 4$, $\|B_n(g(\lambda);\ x)-g(\lambda;\ x)\|_{L_1}\leq 3/2Mn^{-1}$ *holds.*

Proof. We integrate the Bernstein polynomial (see [1]) and obtain

$$(1)\qquad \int_0^1 B_n(g(\lambda);x)dx=\int_0^1\sum_{\nu=0}^{n}\frac{M}{1-\lambda}(\frac{\nu}{n}-\lambda)_+p_{n\nu}(x)dx$$

$$=\frac{M}{1-\lambda}\sum_{\nu=0}^{n}(\frac{\nu}{n}-\lambda)_+\int_0^1 p_{n\nu}(x)dx=\frac{M}{(1-\lambda)(n+1)}\sum_{\nu=0}^{n}(\frac{\nu}{n}-\lambda)_+$$

$$=\frac{M}{(1-\lambda)(n+1)}\sum_{\nu=[n\lambda]+1}^{n}\left(\frac{\nu}{n}-\lambda\right)=\frac{M}{1-\lambda}\Big\{\frac{1}{2}(1-\lambda)^2+\Big[\lambda-\frac{\lambda^2}{2}-\frac{[n\lambda]([n\lambda]+1)}{2n(n+1)}$$

$$-\frac{\lambda(n-[n\lambda])}{n+1}\Big]\Big\}=M(1-\lambda)^{-1}[1/2(1-\lambda)^2+D_1],$$

where $D_1=\lambda-\frac{\lambda^2}{2}-\frac{[n\lambda]([n\lambda]+1)}{2n(n+1)}-\frac{\lambda(n-[n\lambda])}{n+1}$. After some calculations we get

(2) $$D_1=\lambda(1-\lambda)(n+1)^{-1}+D_2,$$

where $D_2=\frac{\lambda^2}{2}-\frac{[n\lambda]([n\lambda]+1)}{2n(n+1)}$. Now we shall prove that for

(3) $$\lambda\in[3/2n^{-1},\quad 1-3/2n^{-1}],\quad n\geqq 4,$$

we have

(4) $$D_2\leqq 1/2\lambda(1-\lambda)(n+1)^{-1}.$$

Indeed under the restriction (3) it is true that $2\leqq 2\lambda(1-\lambda)n$. Applying elementary transformations we obtain a sequence of equivalent inequalities:

$$1+2n\lambda^2\leqq 2n\lambda-1\leqq n\lambda+[n\lambda];$$

$$(n\lambda)^2-[n\lambda]^2+2n\lambda^2\leqq n\lambda+[n\lambda];$$

$$(n\lambda)^2+n\lambda^2-[n\lambda]^2-[n\lambda]\leqq n\lambda-n\lambda^2;$$

$$n(n+1)\lambda^2-[n\lambda]([n\lambda]+1)\leqq n\lambda(1-\lambda);$$

$$\frac{\lambda^2}{2}-\frac{[n\lambda]([n\lambda]+1)}{2n(n+1)}\leqq\frac{\lambda(1-\lambda)}{2(n+1)},$$

which prove (4).

Further we conclude from (2) and (4) that

(5) $$D_1\leqq 3/2\lambda(1-\lambda)n^{-1}.$$

Due to (1) and (5) we obtain

(6) $$\int_0^1 B_n(g(\lambda);\ x)dx=\frac{M(1-\lambda)}{2}+\frac{3M\lambda}{2n}.$$

But in view of the definition of $g(\lambda)$ we have

(7) $$\int_0^1 g(\lambda;\ x)\,dx=\frac{M(1-\lambda)}{2}$$

and

(8) $$\|B_n(g(\lambda);\ x)-g(\lambda;\ x)\|_{L_1}=\int_0^1[B_n(g(\lambda);\ x)-g(\lambda;\ x)]dx.$$

Then (6), (7) and (8) yield

(9) $$\|B_n(g(\lambda);\ x)-g(\lambda;\ x)\|_{L_1}\leqq 3/2\,Mn^{-1}.$$

Lemma 1 is proved.

L e m m a 2. *Let $f \in K^M_{[0,1]}$ be a monotone increasing function for $x \in [0, 1]$ and $f(x)=0$ for $x=0$, $f(x)=M$ for $x=1$. Then $\| B_n(f; x)-f(x) \|_{L_1} \leqq 9/2Mn^{-1}$ holds.*

P r o o f. Without restriction of the generality we can consider $f$ to be continuous on $[0, 1]$. Then for every $\varepsilon>0$ there exists a linear combination of the functions $g(x)=\Sigma_{i=1}^{m(\varepsilon)}\mu_i g(\lambda_i; x)$ with the property

$$\max\{ |f(x)-g(x)|, \quad x \in [0, 1]\}<\varepsilon, \tag{10}$$

where $\mu_i \geqq 0, \quad i=1, 2, \ldots, m, \quad \sum_{i=1}^{m} \mu_i=1; \quad g(\lambda_i), \ i=1, 2, \ldots, m$

are the convex functions from Lemma 1.

According to Lemma 1 for every function $g(\lambda_i)$, $i=1, 2, \ldots, m$, (9) holds. Then for the polynomial $B_n(g; x)=\Sigma_{i=1}^m \mu_i B_n(g(\lambda_i); x)$ from (9) follows

$$\| B_n(g; x)-g(x) \|_{L_1}=\| \sum_{i=1}^{m} \mu_i [B_n(g(\lambda_i); x)-g(\lambda_i; x)] \|_{L_1} \tag{11}$$

$$\leqq \sum_{i=1}^{m} \mu_i \| B_n(g(\lambda_i); x)-g(\lambda_i: x) \|_{L_1} \leqq 3/2Mn^{-1}.$$

Due to (10) we have

$$\| B_n(g; x)-B_n(f; x) \|_{L_1} \leqq \int_0^1 \sum_{\nu=0}^{n} |g(\tfrac{\nu}{n})-f(\tfrac{\nu}{n})| p_{n\nu}(x)dx<\varepsilon. \tag{12}$$

Using (10), (11) and (12) we obtain

$$\| B_n(f; x)-f(x) \|_{L_1} \leqq \| B_n(g; x)-g(x) \|_{L_1}+\| B_n(g; x)-B_n(f; x) \|_{L_1}$$
$$+\| g(x)-f(x) \|_{L_1} \leqq 9/2Mn^{-1}.$$

Lemma 2 is proved. Now we shall prove Theorem 2.

Let $f \in K^M_{[0,1]}$. Without restriction of the generality we can consider $f$ to be continuous. We denote $a=f(\alpha)=\min\{f(x); x \in [0, 1]\}$, $\alpha \in [0, 1]$, and define the non-negative function $\tilde{f}(x)=f(x)-a$ for $x \in [0, 1]$. Further we express the function $\tilde{f}$ as follows: $\tilde{f}(x)=p(x)+h(x)$ for $x \in [0, 1]$, where

$$p(x)=\begin{cases} 0, & x \in [0, \alpha]; \\ \tilde{f}(x), & x \in (\alpha, 1]; \end{cases}$$

$$h(x)=\begin{cases} \tilde{f}(x), & x \in [0, \alpha]; \\ 0, & x \in (\alpha, 1]. \end{cases}$$

The functions $p$ and $q(x)=h(1-x)$, $x \in [0, 1]$, satisfy the conditions of Lemma 2. Hence it holds: $\| B_n(p; x)-p(x) \|_{L_1} \leqq 9/2Mn^{-1}$; $\| B_n(q; x)-q(x) \|_{L_1} \leqq 9/2Mn^{-1}$. Then for the Bernstein polynomial $B_n(\tilde{f}; x)=B_n(p; x)+B_n(h; x)=B_n(p; x)+B_n(q; x)$ one has

$$\| B_n(\tilde{f}; x)-\tilde{f}(x) \|_{L_1} \leqq \| B_n(p; x)-p(x) \|_{L_1}+\| B_n(q; x)-q(x) \|_{L_1} \leqq 9Mn^{-1}. \tag{13}$$

The obtained estimate holds for the function $\tilde{f}(x)=f(x)-a$, $x\in[0, 1]$. This is not a restriction of the generality since by definition

$$\| B_n(\tilde{f}; x)-\tilde{f}(x)\|_{L_1}=\| B_n(f; x)-a-f(x)+a\|_{L_1}=\| B_n(f; x)-f(x)\|_{L_1}.$$

Now we shall prove that the order of approximation $0(n^{-1})$ of the functions $f\in K^M_{[0,1]}$ by Bernstein polynomials can not be improved in $L_1$.

Let us consider the function $g(\frac{1}{2}; x)=\max\{0,2M(x-\frac{1}{2})\}$, $x\in[0, 1]$. For the order of approximation of the function $g(1/2)$ by Bernstein polynomials we obtain:

a) Let $n=2k$, $k=1, 2, \ldots$, $\lambda=1/2$. From (1) follows

$$\int_0^1 B_{2k}(g(\frac{1}{2}); x)dx=M[1-0,5(3k+1)(2k+1)^{-1}]=0,25M[1+(2k+1)^{-1}].$$

Then from (8) we get

$$(14)\qquad \| B_{2k}(g(1/2); x)-g(1/2; x)\|_{L_1}=0,25M(2k+1)^{-1}\geqq 0,125Mn^{-1}.$$

b) Let $n=2k+1$, $k=0, 1, 2, \ldots$ In this case we have

$$\| B_{2k+1}(g(1/2); x)-g(1/2; x)\|_{L_1}\geqq \| B_{2k+2}(g(1/2); x)-g(1/2; x)\|_{L_1}\geqq 0,125Mn^{-1}.$$

Therefore for every $n$ it holds

$$(15)\qquad \| B_n(g(1/2); x)-g(1/2; x)\|_{L_1}\geqq 0.125Mn^{-1}.$$

Due to (13) and (15) we obtain for $f\in K^M_{[0,1]}$

$$\| B_n(f; x)-f(x)\|_{L_1}=O(n^{-1}).$$

Theorem 2 is proved.

Corollary. *Let* $f\in K^M_{[0,1]}$, $f(x)\geqq 0$ *for* $x\in[0, 1]$. *Then* $\inf\{\| P^*_n(x)-f(x)\|_{L_1}, P^*_n\in H^*_n\}\leqq 6Mn^{-1}$, *where* $H^*_n=\{P^*_n; P^*_n(x)=\Sigma_{i+j\leqq n}a_{ij}x^i(1-x)^j, a_{ij}\geqq 0\}$, *is the set of polynomials with positive coefficients of degree* $\leqq n$.

## REFERENCES


1. A. S. Andreev, V. A. Popov. Approximation of functions by means of linear summation operators in $L_1$. *Coll. Math. Soc. Janos Bolyai*, **35**, 1980, 3-24.
2. V. A. Popov. Approximation of convex functions by algebraic polynomials in Hausdorff's metric. *Serdica*, **1**, 1975, 386-398.
3. W. Hoeffding. The $L_1$-norm of the approximation error for Bernsteintype polynomials. *J. of Aprrox. Theory*, **4**, 1971, 347-356.


*VTU "Angel Kanchev", 7004 Russe, Bulgaria*

*Received 30. 3. 1983*