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CHARACTERIZATION OF CONVEX SUBSETS OF THE PLANE THROUGH THEIR LOCAL APPROXIMATION PROPERTIES

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The smooth planar convex compacta are characterized (up to translation) by means of their local approximation properties. The same is done for polygons.

1. Let R^2 be the two-dimensional plane with the Euclidean norm $|\cdot|$. Denote by CONV the set of all convex compact subsets of R^2:

CONV$=\{A\subseteq R^2:\ A$ is compact and convex$\}$.

The Hausdorff distance h between A_1 and A_2 in CONV is defined by

$$h(A_1, A_2)=\inf\{t\geqq 0:\ A_1\subset A_2+tB,\ A_2\subset A_1+tB\},$$

where $B=\{P\in R^2:\ |P|\leq 1\}$ is the unit circle.

We denote the usual inner product of two points $P_1, P_2\in R^2$ by $\langle P_1, P_2\rangle$ The support function S_A corresponding to $A\in$CONV is given by $S_A(P)=\max\{\langle P, X\rangle:\ X\in A\}$.

This function, defined for each point P in R^2, is continuous, convex and positively homogeneous. Therefore it is completely determined by its values on the set $S=\{\theta\in R^2:\ |\theta|=1\}$.

It is well-known that the mapping $(A\mapsto S_A(\cdot))$ from CONV into the space $C(S)$ of all continuous functions in S, is one to one and that

$$h(A_1, A_2)=\max\{|S_{A_1}(\theta)-S_{A_2}(\theta)|:\ \theta\in S\}=\|S_{A_1}-S_{A_2}\|.$$

Let us accept the counterclockwise direction on S as positive. For $\theta_1, \theta_2\in S$ we denote by $[\theta_1, \theta_2]$ the arc on S with end points θ_1 and θ_2 which connects θ_1 and θ_2 "in the counterclockwise direction". The meaning of (θ_1, θ_2), $(\theta_1, \theta_2]$, $[\theta_1, \theta_2)$ is clear now.

The following operation, defined in [3], plays an important role in our considerations. To each pair $\theta', \theta''\in S$, $0<(\theta', \theta'')<\pi$ and $A\in$CONV we assign a point $M=M(A;\ \theta', \theta'')$, a number $d=d(A;\ \theta', \theta'')\geq 0$ and a vector $\theta^*=\theta^*(A;\ \theta', \theta'')\in(\theta', \theta'')$ so that

$$S_A(\theta')-\langle\theta', M\rangle=S_A(\theta'')-\langle\theta'', M\rangle=\langle\theta^*, M\rangle-S_A(\theta^*)=d,$$

$$\max\{|S_A(\theta)-\langle\theta, M\rangle|:\ \theta\in[\theta', \theta'']\}=d.$$

The geometric interpretation of the function $d(A;\ \theta', \theta'')$ is shown in Fig. 1. One important extremal property of this construction is revealed in [3], where P. Kenderov had proved that in the arc $[\theta', \theta'']$ the function $\langle\theta, M\rangle$ approximates $S_A(\theta)$ better than any other function of the type $\langle\theta, P\rangle$.

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In this paper we prove that the function $d(A;\ \theta', \theta'')$ determines the set A up to translation (in the case when A is "smooth" set or a polygon). We need the local asymptotic analysis developed in [5], which is connected

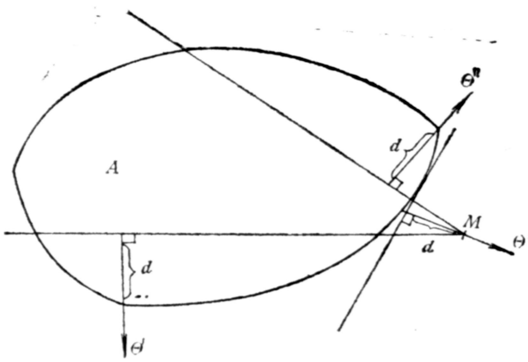

Fig. 1

with the approximation of the set A by a sequence $\{\Delta_n\}_{n=3}^{\infty}$ of "best approximating n-gons", P. Kenderov [4].

Asymptotic estimates of this approximation are obtained by Toth [9], V. Popov [7], McClure and Vitale [5]. It is an open problem to find necessary and sufficient conditions for a given n-gon to be a best approximating for some $A \in$ CONV. N. Zhivkov in [10] has announced a stronger necessary condition for the best approximating n-gons. Unknown is also the answer of the following question of Bl. Sendov and V. Popov: *Is it true that among all the elements of* CONV *with perimeter* 1, *the equilateral* $(n+1)$*-gon (with the same perimeter) is the worst one to be approximated by n-gons?* In support of the positive answer is the result of R. Ivanov [2].

2. From now on we will identify S with the set $[0, 2\pi]$.

Theorem. *Let* A_1 *and* A_2 *be sets of* CONV *with interior points. Let the support functions* S_{A_i} *be twice continuously differentiable and* $r_{A_i}>0\ (i=1, 2)$, *where* $r_{A_i}=S_{A_i}+\ddot{S}_{A_i}$ *is the familiar radius of curvature function. Let* $d(A_1;\ \theta', \theta'')=d(A_2;\ \theta', \theta'')$ *for any pair* $\theta', \theta'' \in S$, $0<(\theta', \theta'')<\pi$. *Then* A_2 *is obtained from* A_1 *by a translation.*

The proof of this Theorem is based on the following Lemmas:

Lemma 1. *Let* $A \in$ CONV *have interior points. Let* S_A *be twice continuously differentiable and* $r_A(\theta) \geq \delta_0$ *for* $\theta \in [0, 2\pi]$. *For any* $\delta \leq \delta_0$ *a convex subset* A_δ *exists satisfying* $A_\delta + B_\delta = A$ (B_δ *is a circle with a centre in the origin and radius* δ).

Proof. Consider the set $A_\delta=\{x \in A;\ \rho(x, R^2 \setminus A) \geq \delta\}$, where ρ is the distance with respect to the Euclidean norm, $\rho(x, R^2 \setminus A)=\inf_{y \in R^2 \setminus A} \rho(x, y)$. At first we show that A_δ is convex.

Let $x \in A_\delta$, $y \in A_\delta$ and $\rho(x, R^2 \setminus A)=\delta_1 \geq \delta$; $\rho(y, R^2 \setminus A)=\delta_2 \geq \delta$. That implies $x+B_{\delta_1} \subset A$ and $y+B_{\delta_2} \subset A$. From convexity of A we obtain $\text{conv}\{x+B_\delta, y+B_\delta\} \subset A$.

Let $z=(1-t)x+ty$ for $0<t<1$ and $r=\delta_1(1-t)+\delta_2 t\geqq\delta(1-t)+\delta t=\delta$ (see Fig. 2). Explicitly $z+B_r\subset\operatorname{conv}\{x+B_{\delta_1},\ y+B_{\delta_2}\}\subset A$. Then $\rho(z, R^2\setminus A)\geqq\delta$ or $z\in A_\delta$, from which it follows that A_δ is convex.

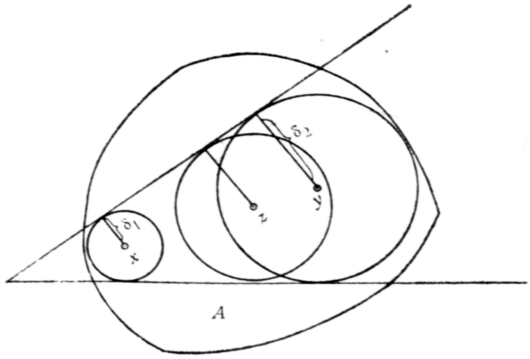

Fig. 2

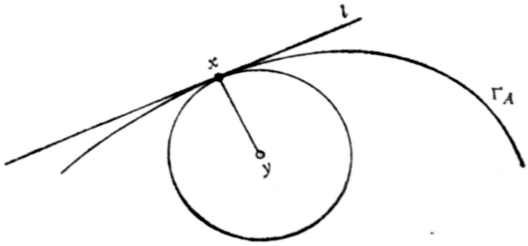

Fig. 3

Now we show that $A_\delta+B_\delta=A$.

Let $x\in A_\delta+B_\delta=\bigcup_{y\in A_\delta}(y+B_\delta)$. This implies $x\in y_0+B_\delta$ for some $y_0\in A_\delta$. But $\rho(y_0, R^2\setminus A)\geqq\delta$ yields $x\in y_0+B_\delta\subset A$ or

$$A_\delta+B_\delta\subset A. \tag{21}$$

We have to prove the opposite relation.

Let $x\in\Gamma_A$ (Γ_A be the boundary of A). Let l be the support line of A that passes through the point x and $y+B_\delta$ osculates l in the point x by the side of Γ_A (Fig. 3). From the corollary of Blaschke's theorem [1, p. 140] we obtain $y+B_\delta\subset A$. This implies $\rho(y, R^2\setminus A)\geqq\delta$ or $y\in A_\delta$. Then $x\subset y+B_\delta\subset A_\delta+B_\delta$, from which $\Gamma_A\subset A_\delta+B_\delta$.

From the theorem of Krein-Milman [8]

$$A\subset A_\delta+B_\delta. \tag{22}$$

Comparing (21) and (22) we obtain $A=A_\delta+B_\delta$.

Lemma 2. *Let* $A \in \mathrm{CONV}$ *has interior points. Let* S_A *be twice continuously differentiable and* $r_A > 0$ *for* $\theta \in [0, 2\pi]$. *Then*

$$d(A\,;\,\theta', \theta'+k) = \frac{r_A(\theta')\,.\,k^2}{16+k^2} + o(k^2)$$

for a fixed θ' *and sufficiently small* k.

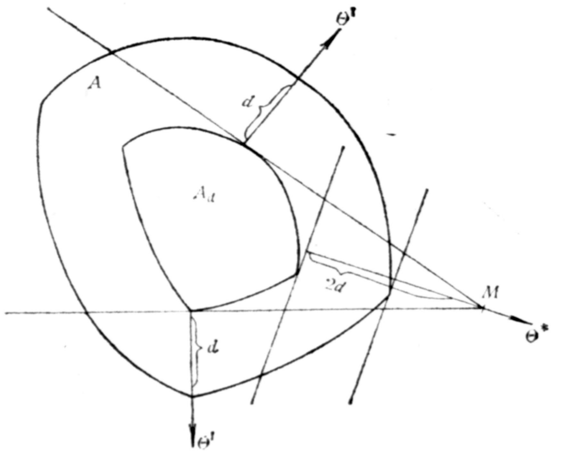

Fig. 4

Proof. From the assumption $r_A(\theta) > 0$ on $[0, 2\pi]$ and the continuity of r_A it follows that $\delta_0 > 0$ exists that $r_A(\theta) \geqq \delta_0$. Let $0 < d \leqq \delta_0$. This assumption and the Lemma above yield $A = A_d + B_d$, which implies

$$S_{A_d}(\theta) + d = S_A(\theta) \quad \text{and} \quad r_{A_d}(\theta) = r_A(\theta) - d \tag{23}$$

for any $\theta \in [0, 2\pi)$.

Let us fix now $\theta' \in [0, 2\pi)$. Unique vectors θ'', θ^* and a point M exist, for which

$$\begin{gathered} S_A(\theta') - \langle M, \theta' \rangle = S_A(\theta'') - \langle M, \theta'' \rangle = \langle M, \theta^* \rangle - S_A(\theta^*) = d, \\ \max\{\,|S_A(\theta) - \langle \theta, M \rangle|:\ \theta \in [\theta'. \theta'']\} = d \end{gathered} \tag{24}$$

(see Fig. 4). This follows from the results of P. Kenderov [3].

It is easy to show that $0 < \theta'' - \theta' < \pi$.

From (23) and (24) it follows $h(M, A_d) = \langle M, \theta^* \rangle - S_{A_d}(\theta^*) = \langle M, \theta^* \rangle - S_A(\theta^*) + d = 2d$. That is

$$h(M, A_d) = 2d(A\,;\,\theta', \theta''). \tag{25}$$

In [5] it is obtained that

$$h(M, A_d) = \frac{1}{8}\, r_{A_d}(\theta')k^2 + o(k^2), \tag{26}$$

where $k = \theta'' - \theta' < \pi$. A powerful tool in the obtaining of this result is Polya's mean value theorem [6].

From (25) and (26) it follows $2d(A;\ \theta',\theta'')=\frac{1}{8}r_{A_d}(\theta')k^2+O(k^2)$, where $k=\theta''-\theta'<\pi$. Having in mind (23) this becomes

$$2d(A;\ \theta',\theta'')=\frac{1}{8}r_A(\theta').k^2-\frac{d(A;\ \theta',\theta'')}{8}k^2+o(k^2).$$

Then

$$d(A;\ \theta',\theta'')=\frac{r_A(\theta').k^2}{16+k^2}+o(k^2).$$

Proof of the theorem. From the hypothesis of the theorem and the result of Lemma 2, for any $\theta\in[0,2\pi)$ and sufficiently small k we have

$$d(A_1;\ \theta,\theta+k)=\frac{r_{A_1}(\theta).k^2}{16+k^2}+o_1(k^2),\quad d(A_2;\ \theta,\theta+k)=\frac{r_{A_2}(\theta).k^2}{16+k^2}+o_2(k^2).$$

Thus

$$\frac{r_{A_1}(\theta).k^2}{16+k^2}+o_1(k^2)=\frac{r_{A_2}(\theta).k^2}{16+k^2}+o_2(k^2).$$

This is equivalent to

$$r_{A_1}(\theta)+\frac{o_1(k^2)}{k^2}=r_{A_2}(\theta)+\frac{o_2(k^2)}{k^2}.$$

We obtain the limit by $k\downarrow 0$:

(27) $$r_{A_1}(\theta)=r_{A_2}(\theta)\ \text{ for any }\ \theta\in[0,2\pi].$$

Equality (27) implies that the function $\varphi=S_{A_1}-S_{A_2}$ satisfies the differential equation $\ddot{\varphi}+\varphi=0$. Then $S_{A_1}(\theta)=C_1\cos\theta+C_2\sin\theta+S_{A_2}(\theta)$.

This means A_2 may be obtained from A_1 by a translation, determined by a vector with coordinates C_1 and C_2.

3. Let A be a polygon with vertexes $P_1, P_2, \dots, P_n$, and side-directions (directions of the outward normals to the sides) $\theta_1<\theta_2<\dots<\theta_n<\theta_1+2\pi$.

Having in mind only the definition of the function $d(A;\ \theta',\theta'')$, it is easy to show that

i) $$d(A;\ \theta,\theta_{\nu+1})=0\ \text{ for }\ \theta\in[\theta_\nu,\theta_{\nu+1}],$$

ii) $$d(A;\ \theta_{\nu-1},\theta)=\ \text{ for }\ \theta\in[\theta_{\nu-1},\theta_\nu],$$

iii) $$d(A;\ \theta',\theta'')>0\ \text{ for } \theta'\in[\theta_{\nu-1},\theta_\nu) \text{ and }\ \theta''\in(\theta_\nu,\theta_{\nu+1}].$$

Theorem. *Let A_1 and $A_2\in$ CONV be polygons. Let $d(A_1;\ \theta',\theta'')=d(A_2;\ \theta',\theta'')$ for any pair $\theta',\theta''\in S$, $0<(\theta',\theta'')<\pi$. Then A_2 is obtained from A_1 by a translation.*

Proof. From the properties stated above i), ii), iii) and the equation $d(A_1;\ \theta',\theta'')=d(A_2,\theta',\theta'')$, it follows that the corresponding side-directions of the polygons A_1 and A_2 are equal. In order to obtain A_2 from A_1 by a translation, it is sufficient to show that the lengths of the corresponding sides are equal too. That follows immediately from the congruence of the triangles (see Fig. 5)

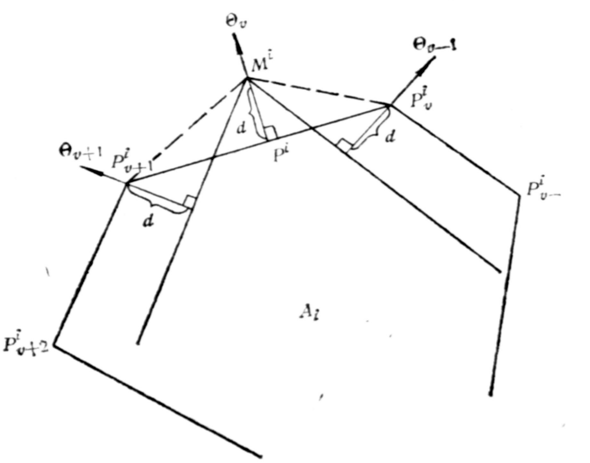

Fig. 5

$$\Delta P^1P^1_vM^1 \cong \Delta P^2P^2_vM^2, \ \Delta P^1P^1_{v+1}M^1 \cong \Delta P^2P^2_{v+1}M^2.$$

Then $P^1_vP^1_{v+1} = P^2_vP^2_{v+1}$. The proof is completed.

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