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PERIODIC AUTOMORPHISMS ON A SMOOTH MANIFOLD, PRESERVING A CLOSED DIFFERENTIAL 2-FORM

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The problem of the equivalence of the preservation of a closed 2-form and certain 1-forms is considered. The periodic automorphisms are proved to realize this equivalence. As in general these 1-forms depend on the mappings, the question for simultaneous preservation is considered. Some sets of periodic automorphisms are proved to preserve the 2-form iff all of them preserve a fixed 1-form, the same for all the elements of the sets.

Let M be a (finite-dimensional) smooth manifold and $\Phi \in \Lambda^2 T^*M$, $d\Phi=0$. Let us assume first $\pi_1(M)=0$ and $\Phi=d\psi$. ψ is free-chosen and fixed.

Definition 1. *The smooth mapping* $f: M\to M$ *satisfies the condition* $(*)$ *if there exist* $g, \alpha \in \mathcal{F}(M)$, $g=g(f)$, $\alpha=\alpha(f)$, *such that*

$$(*)\quad \begin{aligned}&1.\ f^*\psi-\psi=dg;\\ &2.\ \alpha-\alpha f=g.\end{aligned}$$

Clearly $f^*\Phi=\Phi$ if f satisfies $(*)$ and 1. of $(*)$ follows from $f^*\Phi=\Phi$, because of $\pi_1(M)=0$, universal coefficient formula ([3], Chap. 6, § 4) and the equality $H^1(M, \mathbf{R})=\mathrm{Hom}_{\mathbf{R}}(H_1(M, \mathbf{R}); \mathbf{R})$ (see [1], Theorem A1). Thus $f^*\Phi=\Phi$ is equivalent to 1. of $(*)$. An example where a smooth mapping f, $f^*\Phi=\Phi$ and 2. of $(*)$ is not satisfied will be given later.

The importance of the condition $(*)$ is determinated by the following

Proposition 1. *The smooth mapping* $f: M\to M$ *satisfies the condition* $(*)$ *iff there exists 1-form* $\psi_1=\psi_1(f)\in T^*M$ *such that* $d\psi_1=\Phi$ *and* $f^*\psi_1=\psi_1$.

Proof. Let f satisfy $(*)$ and g, α be the corresponding functions. Let $\psi_1=\psi+d\alpha$. Then $f^*\psi_1=f^*\psi+d(\alpha f)=\psi+dg+d(\alpha f)=\psi+d(g+\alpha f)=\psi+d\alpha=\psi_1$. Now let there exist $\psi_1=\psi_1(f)\in d^{-1}\Phi$, $f^*\psi_1=\psi_1$. As $d\psi_1=\Phi$, $[\psi-\psi_1]\in H^1(M, \mathbf{R})=0$, i. e. there exists $\alpha\in\mathcal{F}(M)$, $d\alpha=\psi_1-\psi$. Then $dg=f^*\psi-\psi=f^*\psi_1-\psi_1+d(\alpha-\alpha f)$ and as $f^*\psi_1=\psi_1$, the equality $d(\alpha-\alpha f-g)=0$ holds. Thus $\alpha-\alpha f=g+c$, $c=\text{const}$. Then f satisfies $(*)$ with $g'=g+c$ and α.

Let us note that Proposition 1 proves the independence of Definition 1 on the specific choice of ψ.

Definition 2. $Sp_2(M, \Phi)=\{f: M\to M, f^*\Phi=\Phi\}$; $Sp_1(M, d^{-1}\Phi)=\{f$, there exists $\psi_f\in d^{-1}\Phi$, $f^*\psi_f=\psi_f\}$; $*(M)=\{f: M\to M$, satisfies the condition $(*)\}$.

Theorem 1. Let $f: M\to M$, $f^k=id (k=k_0\in\mathbf{Z}_+)$. $f\in Sp_2(M, \Phi)$ iff $f\in Sp_1(M, d^{-1}\Phi)$.

Proof. It is enough to find a solution of the 2. of $(*)$. Let $f^*\psi-\psi=dg$. Let $\beta=r_0 g+\sum_{\nu=1}^{k-1} r_\nu g.f^\nu$; $\beta\in\mathcal{F}(M)$. $\beta-\beta f=(r_0-r_{k-1})g+\sum_{\nu=1}^{k-1}(r_\nu-r_{\nu-1})gf^\nu$. Let

$rj=-j,\ 0\leq j\leq k-1$. Then $\beta-\beta f=(k-1)g-\sum_{\nu=1}^{k-1} gf^{\nu}=kg-(g+\sum_{\nu=1}^{k-1} gf^{\nu})$.

On the other hand, $f^*\psi-\psi=dg$, $f^{\nu *}\psi-f^{\nu-1*}\psi=dgf^{\nu-1}$, so $d(g+\sum_{\nu=1}^{k-1} gf^{\nu})=0$.

Thus $\beta-\beta f=kg+c$, $c=\text{const}$. Let $g'=g+c/k$, $\alpha=\beta/k$. Then $f^*\psi-\psi=dg'$ and $\alpha-\alpha f=g'$, i. e. f satisfies $(*)$.

Example 1. Let $M=\mathrm{R}^2(x, y)$, $\Phi=dx\wedge dy$. Let $f(x, y)=(x+\sin(x+y)+\cos(x+y),\ y-\sin(x+y)-\cos(x+y))$. $f\in Sp_2(\mathrm{R}^2 . \Phi)\setminus Sp_1(\mathrm{R}^2,\ d^{-1}\Phi)$. This fact is proved in [7]. Thus there is no function $\alpha\in\mathscr{F}M$, $\alpha-\alpha f=g$, where $dg=f^*\psi-\psi$ $(d\psi=\Phi)$, i. e. 2. of(*) is not satisfied.

Corollary 1. *For any* $\varepsilon>0$ *there exists* $f\in Sp_2(\mathrm{R}^2, dx\wedge dy)$, $\|f-id\|<\varepsilon$ *and* $f^*\psi\neq\psi$ *for any* $\psi\in d^{-1}(dx\wedge dy)$.

Proof. Let $f_1(x, y)=(x+\varepsilon_1\sin(x+y)+\varepsilon_2\cos(x+y),\ y-\varepsilon_1\sin(x+y)-\varepsilon_1\cos(x+y))$. As Jacoby matrix $Df_1(x, y)\in SL_2(\mathrm{R})$ for any $(x, y)\in\mathrm{R}^2$, $f_1^*\Phi=\Phi$. On the other hand, $f_1^*\psi\neq\psi$ for any $\psi\in d^{-1}\ \Phi$ (this fact is easy to obtain using Example 1). $\|f_1-id\|\leq 6\varepsilon_1$. Thus $\|f_1-id\|\leq\varepsilon$ for $\varepsilon_1=\varepsilon/6$.

Corollary 1 characterizes the set of the symplectic mappings close to the identity. It is possible to generalize it using Darboux theorem [4] on a class of manifolds. Let us note that the set of the immovable points of the symplectic mappings on a smooth simply connected, exact symplectic manifold are considered in [5]. The mapping f in Corollary 1 possesses (more than one) immovable points and this fact is the basic one, used in the proof of Example 1. ([7])

Now let M be a paracompact manifold (no requirement for the fundamental group of M) and $\Phi\in\Lambda^2T^*M$, $d\Phi=0$ (Φ is not required exact).

Theorem 2. *Let* $f: M\to M$ *be a smooth mapping and* $f^k=id$ $(k=k_0\in\mathrm{Z}_+)$. $f^*\Phi=\Phi$ *iff there exists 1-form* $\psi=\psi(f)$ *such that for a convenient partition* $\{\eta_\gamma\}_\Gamma$ *of unity*

1. $f^*\psi=\psi$;
2. $d\psi-\Phi=\sum_{\gamma\in\Gamma} d\eta_\gamma\wedge\psi_\Gamma$, *where* $n_\gamma=\eta_\gamma\circ f$, $f^*\psi_\gamma=\psi_\gamma$ *on* $\operatorname{supp}\eta_\gamma=F_\gamma$, $d\psi_\gamma/_{F_\gamma}=\Phi/_{F_\gamma}$ *and* Γ *is a convenient index set*, $\sum_\gamma\eta_\gamma=1$.

Proof. Let $x\in M$ and U_x be a neighbourhood of x such that $f^i(U_x)\cap f^j(U_x)=\emptyset$ or $f^i(U_x)=f^j(U_x)$ for any pair $(i, j)\in\mathrm{Z}\oplus\mathrm{Z}$. Let U_x be so small that $j_x^*\Phi=d\psi_x$ in U_x. Let $V_x=\bigcup_{j\in\mathrm{Z}} f^j(U_x)$ and Γ be a subset of M such that $\{V_\gamma\}_{\gamma\in\Gamma}=\mathscr{V}$ is a covering of M. Let ψ_γ be extended to a smooth 1-form on M. Let $\{\sigma_\gamma\}_{\gamma\in\Gamma}$ be a partition of unity, connected with $\mathscr{V}$. Then $\eta_\gamma=k^{-1}(\sigma_\gamma+\sum_{\nu=1}^{k-1}\sigma_\gamma f^\nu)$ defines a partition of unity, connected with $\mathscr{V}$ and $\eta_\gamma=\eta_\gamma\circ f$. Let $\psi=\sum_\gamma\eta_\gamma\psi_\gamma$ and $f^*j_\gamma^*\psi_\gamma=\psi_\gamma$, $\gamma\in\Gamma$, $j_\gamma: v_\gamma\subset M$. $f^*\psi=\sum\eta_\gamma f$. $f^*\psi_\gamma=\sum_\gamma\eta_\gamma f^*j_\gamma^*\psi_\gamma=\sum_\gamma\eta_\gamma\psi_\gamma=\psi$, $d\psi=\sum_\gamma d\eta_\gamma\wedge\psi_\gamma+\sum_\gamma\eta_\gamma d\psi_\gamma=\sum_\gamma d\eta_\gamma\wedge\psi_\gamma+\Phi$, i. e. $d\psi-\Phi=\sum_\gamma d\eta_\gamma\wedge\psi_\gamma$.

$$f^*\psi_\gamma=\psi_\gamma \text{ on } \operatorname{supp}\eta_\gamma,\ j_\gamma^*d\psi_\gamma=j_\gamma^*\Phi \text{ implies } f^*\Phi=\Phi.$$

Let us note that the definitions and statements proved above are possible to be formulated or proved for $\Phi\in\Lambda^pT^*M$ $2\leq p\leq\dim M$. One ought to modify slightly some details according to de Rham complex and Hurevich theorem.

So the periodic authomorphisms on a smooth manifold M preserve a (fixed) closed 2-form on M iff they preserve convenient 1-forms on M. These 1-forms in general depend on the mappings. So the question arises when a set of authomorphisms on M preserves a closed 2-form if and only if it preserves a convenient 1-form — the same for all the mappings of the set?

Let us consider the simplest case again: $\pi_1(M)=0$ and $d\psi=\Phi$.

Definition 3. *Let $F=\{f_\gamma\}_{\gamma\in\Gamma}$ be a set of smooth authomorphisms on M. F satisfies the condition* $(*)$ *if there exist smooth functions g_γ, α_γ, β_γ on M such that*

$$(*_\Gamma)\qquad \begin{aligned} &1.\ f_\gamma^*\psi-\psi=dg_\gamma, \quad \gamma\in\Gamma;\\ &2.\ \alpha_\gamma-\alpha_\gamma f_\gamma=g_\gamma, \quad \gamma\in\Gamma;\\ &3.\ \beta_\gamma f_\gamma=\beta_\gamma+c_\gamma, \quad c_\gamma=\text{const},\ \gamma\in\Gamma;\\ &4.\ \beta_\gamma-\beta_\delta=\alpha_\delta-\alpha_\gamma, \quad \gamma,\ \delta\in\Gamma. \end{aligned}$$

Proposition 2. *Let $F^*\Phi=\Phi$. There exists $\psi_0\in T^*M$, $d\psi_0=\Phi$ and $F^*\psi_0=\psi_0$ iff F satisfies* $(*_\Gamma)$.

Here $F^*\Phi=\Phi$ means $f_\gamma^*\Phi=\Phi$ for any $\gamma\in\Gamma$; the same is for ψ_0.

Proof. Let F satisfy $(*_\Gamma)$ and let $\psi_0=\psi+d\alpha_\gamma+d\beta_\gamma$. ψ_0 is correctly defined because of $d(\alpha_\gamma+\beta_\gamma)=d(\alpha_\delta+\beta_\delta)$.

$$f_\gamma^*\psi_0=f_\gamma^*\psi+d(\alpha_\gamma f_\gamma+\beta_\gamma f_\gamma)=\psi+d(g_\gamma+\alpha_\gamma f_\gamma+\beta_\gamma)=\psi+d(\alpha_\gamma+\beta_\gamma)=\psi_0.$$

Let there exist ψ_0, $d\psi_0=\Phi$ and $F^*\psi_0=\psi_0$. As f_γ satisfies $(*)$ there exist $\alpha_\gamma\in\mathcal{F}(M)$, $\alpha_\gamma-\alpha_\gamma f_\gamma=g_\gamma$ and $f_\gamma^*(\psi+d\alpha_\gamma)=\psi+d\alpha_\gamma$. As $d(\psi+d\alpha_\gamma)=d\psi_0=\Phi$, there exist $\beta_\gamma\in\mathcal{F}(M)$, $d\beta_\gamma=\psi_0-\psi-d\alpha$, i. e. $\psi_0=\psi+d\alpha_\gamma+d\beta_\gamma$. Clearly $\beta_\gamma f_\gamma=\beta_\gamma+c_\gamma$, $c_\gamma=\text{const}$ and $d(\alpha_\gamma+\beta_\gamma)=d(\alpha_\delta+\beta_\delta)$, i. e. $\beta_\gamma-\beta_\delta=\alpha_\delta-\alpha_\gamma+c_{\gamma\delta}$, $c_{\gamma\delta}=\text{const}$. It is possible to make $c_{\gamma\delta}$ disappear by using $\beta_\gamma'=\beta_\gamma+c_{\gamma_0\gamma}$, where γ_0 is a fixed index in Γ.

The condition $(*)$ is not trivial. There are examples of mappings each of them satisfying $(*)$ but the set of them doesn't satisfy $(*)$.

Example 2. Let $M=\mathbf{R}^2$, $\Phi=dx\wedge dy$, $\psi=-ydx+xdy$ (as f^* is a linear mapping, the consideration is correct).

Let $f_1(x, y)=(-y, x)$ — rotation of $-\pi/2$; $f^4=id$. $f_2(x, y)=(x+1, y)$. It is easy to prove that $f_1^*\Phi=f_2^*\Phi=\Phi$ and $\alpha_2-\alpha_1=-xy$. Let us assume that $F=\{f_1, f_2\}$ satisfies $(*_{\{1,2\}})$, i. e. there exist β_1, β_2 such that

$$\begin{aligned} &\beta_1(x, y)=\beta_1(-y, x);\\ &\beta_2(x+1, y)=\beta_2(x, y)+c;\\ &\beta_1(x, y)-\beta_2(x, y)=-xy. \end{aligned}$$

Thus $\beta_2(-y, x)=\beta_2(x, y)-2xy$. Let $(m, n)\in\mathbf{Z}\oplus\mathbf{Z}$. $\beta_2(m, n)=\beta_2(m-1, n)+c=\cdots=\beta_2(0, n)+mc=\beta_2(-n, 0)+mc=\beta_2(0, 0)+(m-n)c$. $\beta_2(m, n)=\beta_2(-n, m)+2mn=\beta_2(0, 0)+2mn-(m+n)c$. So $c=2n$ which is impossible because of the choice of (m, n). Now some properties of the periodic mapping on will be proved.

Theorem 3. *Let* $F=\{f_0, f_1, \ldots, f_p\}$, $p \in \mathbf{Z}$, $f_i f_j = f_j f_i$ *for any pair* (i, j) *and* $f_i^2 = id$, $i=0, 1, 2, \ldots, p$. $F^*\Phi=\Phi$ *iff there exists* $\psi_0 \in d^{-1}\Phi$, $F^*\psi_0=\psi_0$.

Proof. Let $F^*\Phi=\Phi$. So $f_i^*\psi-\psi=dg_i$. As $f_i^2=id$, $\alpha_i=g_i/2$ (it is clear that $\alpha_i-\alpha_i f_i=g_i$) and $g_i-g_j=g_i f_j-g_j f_i$ as $f_i f_j=f_j f_i$.

Let $p=2$.

$$\beta_0=g_1+g_1 f_0; \quad \beta_0=\beta_0 f_0;$$

$$\beta_1=g_0+g_0 f_1; \quad \beta_1=\beta_1 f_1;$$

$$\beta_0-\beta_1=g_1-g_0+g_1 f_0-g_0 f_1=2(g_1-g_0)=4(\alpha_1-\alpha_0).$$

Thus $\{f_0, f_1\}$ satisfies $(*_{\{1,2\}})$.

Let $p=3$. As $f_i f_j=f_j f_i$, the following equalities hold: $g_1 f_2 f_3+g_2 f_3+g_3 = g_2 f_1 f_3+g_3 f_1+g_1 = g_3 f_1 f_2+g_1 f_2+g_2$. Let

$$\beta_i=g_{i+1}+g_{i+2}+g_{i+1}f_i+g_{i+2}f_i+g_{i+1}f_{i+2}+g_{i+2}f_{i+1}$$

$$+g_{i+1}f_i f_{i+2}+g_{i+2}f_i f_{i+1}, \quad i=0, 1, 2, \quad i+k=(i+k) \bmod 3.$$

It is easy to verify $\beta_i-\beta_j=4(\alpha_j-\alpha_i)$, $i=0, 1, 2$, i. e. $\{f_0, f_1, f_2\}$ satisfies $(*_{\{1,2,3\}})$.

Using the same kind of expression of β_i it is possible to prove the proposition for any integer p.

Proposition 3. *Let* $f_1^4=id$, $f_2^4=id$ *and* $f_1 f_2=f_2 f_1$. $f^*\Phi=\Phi$ $(i=1, 2)$ *iff there exists* $\psi_0 \in d^{-1}\Phi$, $f_i^*\psi_0=\psi_0$, $i=1, 2$.

Proof. Let $h_1=f_1^2$ and $h_2=f_2^2$. As $h_1^2=h_2^2=id$ and $h_1 h_2=h_2 h_1$, there exists $\psi_1 \in d^{-1}\Phi$, $h_1^*\psi_1=h_2^*\psi_1=\psi_1$.

Let $g_i^0=f_i^*\psi_1-\psi_1$. $0=f_i^{*2}\psi_1-\psi_1=d(g_i^0 f_i+g_i)=f_i^*(f_i^*\psi_1-\psi_1)+(f_i^*\psi_1-\psi_1)$, i. e. $g_i^0 f_i+g_i^0=c_i=\text{const}$. It is easy to prove that $g_1^0 f_2^2=g_1^0$ and $g_2^0 f_1^2=g_2^0$. Let $\beta_1=g_2^0+g_2^0 f_1$ and $\beta_2=g_1^0+g_1^0 f_2$. Then $\beta_1-\beta_2=2(g_2^0-g_1^0)=2(\alpha_2-\alpha_1)$, i. e. $\{f_1, f_2\}$ satisfies $(*_{\{1,2\}})$. Proposition 3 is possible to be generalized for the mappings of the type $f^k=id$, $k=2^n$, using the same way.

So two statements for simultaneous preservation on a potential of Φ of periodic authomorphisms are proved. Nevertheless the question of simultaneous preservation of a general class of authomorphisms is open. However, three theorems hold:

Theorem 4. *Let M be a smooth closed, not boardant to zero manifold and Φ be an exact 2-form on M. Let F be a set of mappings* $\{f_\gamma\}_{\gamma \in \Gamma}$ *satisfying* $(*_\Gamma)$. *Let f_0 be a smooth involution on M. Then there exists submanifold M_1 of M,* $\dim M_1 \geqq (2/5)\dim M$ *such that* $F \cup \{f_0\}$ *satisfies* $(*_{\Gamma \cup 0})$ *on* M_1.

Proof. Let M_2 be the max. dimension component of the set of immovable points of f_0. According to [2], [6] $\dim M_2 \geqq (2/5)\dim M$ and let $M_1 \subset M_2$ be chosen in a proper way. According to Proposition 2, there exists ψ_0, $d\psi_0=\Phi$ and $F^*\psi_0=\psi_0$. As $f^* \mid M_1\ \psi_0=\psi_0$, then the restriction of $F \cup \{f_0\}$ to M_1 preserves ψ_0, restricted to M_1. Thus $F \cup \{f_0\}$ satisfies $(*)$.

Theorem 5. *Let M^{2k} be a closed manifold with odd Euler characteristic. Let Φ be an exact 2-form on M and f_0 be a smooth involution on M.*

Let $F=\{f_\gamma\}_{\gamma\in\Gamma}$ *satisfy* $(*_\Gamma)$. *Then there exists submanifold* M_1^m *of* M *such that the restriction of* $F\cup\{f_0\}$ *to* M_1 *satisfies* $(*_{\Gamma\cup 0})$, $m\geqq k$.

Proof. M_1 is the set of the immovable points of f_0 according to [2].

Theorem 6. *Let* (M^{2n}, J) *be an almost complex manifold and* f_0 *be a smooth involution on* M, *possessing at least* 1 *immovable point. Let* $J\cdot f_{0*}+f_{0*}J=0$, *and* $F=\{f_\gamma\}_{\gamma\in\Gamma}$ *satisfy* $(*_\Gamma)$. *Then there exists* $M_1\subset M$, $\dim M_1=n$ *and the restriction of* $F\cup\{f_0\}$ *to* M_1 *satisfies* $(*_{\Gamma\cup 0})$.

Proof. As the proof of Theorem 5.

Finally, let us note that Theorem 2 and Theorem 3 immediately imply

Theorem 7. *Let* M *be a paracompact manifold,* Φ *be a closed 2-form on* M *and* $F=\{f_0, f_1, \dots, f_p\}$, $f_j^2=id$, $f_i f_j=f_j f_i$, $i, j=0, 1, \dots, p$. $F^*\Phi=\Phi$ *iff there exists* $\psi_0\in T^*M$, *such that*:

1. $F^*\psi_0=\psi_0$;
2. $d\psi_0-\Phi=\sum_\gamma d\eta_\gamma\wedge\psi_\gamma$, *where* $\eta_\gamma=\eta_\gamma f_j$, $f_j^*\psi_\gamma=\psi_\gamma$ *on* $\operatorname{supp}\eta_\gamma$, $d\psi_\gamma=\Phi$ *on* $\operatorname{supp}\eta_\gamma$ $(\gamma\in\Gamma, j=0, 1, \dots, p)$, *where* Γ *is a convenient index set, and* $\{\eta_\gamma\}_{\gamma\in\Gamma}$ *is a partition of unity.*

REFERENCES

1. С. Милнор. Характеристические классы. М., 1979.
2. Ф. Коннер. Гладкие периодические отображения. М., 1969.
3. A. Dold. Lectures on algebraic topology. Berlin — Heidelberg — New York, 1972.
4. S. Sternberg. Lectures on differential geometry. Englewood Hills, N. Y., 1964.
5. J. Moser. A fixed point theorem in symplectic geometry. *Acta Math.*, **141**, 1978, No 1-2.
6. Boordman. On manifolds with involution. *Bull. Ammer. Math. Soc.*, **73**, 1967, No 1.
7. Ts. Pelov. A class of mappings preserving a closed differential form on a smooth manifold. *C. R. Acad. bulg. Sci.*, **35**, 1982, No 10.

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