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COMPACTLY DETERMINED EXTENSIONS OF TOPOLOGICAL SPACES

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The notion of supertopological space, introduced earlier by the author, is used for construction of some kind of extensions of topological spaces.

The notion of supertopological space comprising as special cases the notions of topological, proximity and uniform spaces was introduced in [4] under the name generalized topological space. In this paper it is used for description of some classes of extensions of topological spaces.

Let us recall that an *extension* of a topological space X is a pair (Y, φ) consisting of a topological space Y and a dense embedding $\varphi: X \to Y$. The extension (X, i), where $i: X \to X$ is the identical mapping, is called *trivial extension* of the space X. Two extensions (Y_1, φ_1) and (Y_2, φ_2) of the same topological space X are called equivalent to each other if there exists a homeomorphism $\lambda: Y_1 \to Y_2$ with $\lambda\varphi_1 = \varphi_2$. When the space Y has a given property (such as regularity, compactness, local compactness, etc.), then one usually says that the extension (Y, φ) itself possesses this property. Compact extensions are usually called compactifications.

Here in Section 1 the definition of supertopological space and its main properties are briefly reminded. In Sections 2 and 3 all the statements are given, as a rule, with their detailed proofs. Section 4 contains some results whose not difficult proofs are omitted.

1. Supertopological spaces. Supertopological space is a generalization of the notion of topological space considered with its classical Hausdorff axiomatic, wherein the concept of neighbourhood is taken as fundamental.

Let X be a set, $\mathscr{P}(X)$ be the power set of X, and $\mathscr{I}(X)$ be the collection of all single-point subsets of X.

A *supertopology* on X is a pair $(\mathscr{M}, \mathscr{V})$ consisting of a collection $\mathscr{M}$ of subsets of X with $\mathscr{I}(X) \subset \mathscr{M} \subset \mathscr{P}(X)$ and an operator $\mathscr{V}: \mathscr{M} \to \mathscr{P}(\mathscr{P}(X))$ assigning to each $A \in \mathscr{M}$ a filter $\mathscr{V}(A)$ in X — the filter of the "$\mathscr{V}$-neighbourhoods" of A. The following conditions are supposed to be satisfied:

(1) if $A \in \mathscr{M}$ and $U \in \mathscr{V}(A)$, then $A \subset U$;

(2) if $A \in \mathscr{M}$ and $U \in \mathscr{V}(A)$, then there exists a $V \in \mathscr{V}(A)$ with $U \in \mathscr{V}(B)$ for any $B \subset V$, $B \in \mathscr{M}$.

Note 1. In the case when $\varnothing \in \mathscr{M}$ the collection $\mathscr{V}(\varnothing)$ must satisfy all the conditions for a filter except eventually the requirement $\varnothing \notin \mathscr{V}(\varnothing)$. In other words, it is permissible to have $\varnothing \in \mathscr{V}(\varnothing)$ and therefore $\mathscr{V}(\varnothing) = \mathscr{P}(X)$.

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A set X with a supertopology on it is called *supertopological space*.

Obviously in the case $\mathcal{M}=\mathcal{J}(X)$ (when the collection $\mathcal{M}$ can be, in fact, identified with the set X itself) the notion of supertopological space coincides with the standard notion of topological space, because then the upper two conditions are nothing but the well-known Hausdorff axioms for topological spaces in the formulation given by Bourbaki [2].

On the other hand, in view of the inclusion $\mathcal{J}(X)\subset\mathcal{M}$, any supertopology $(\mathcal{M},\mathcal{V})$ on X induces a topology on X. Thus every supertopological space is at the same time a topological space. When X is a topological space, any supertopology on X iducing the given topology is said to be *compatible* with it.

A supertopology $(\mathcal{M},\mathcal{V})$ is called *symmetrical* provided the following additional condition is fulfilled:

if $A, B\in\mathcal{M}$ and $A\cap V\neq\emptyset$ for each $V\in\mathcal{V}(B)$, then $B\cap U\neq\emptyset$ for each $U\in\mathcal{V}(A)$.

One can see that the notion of symmetrical supertopological space in the case $\mathcal{M}=\mathcal{P}(X)$ coincides essentially with the standard notion of proximity space. Indeed, let $(\mathcal{P}(X),\mathcal{V})$ be a symmetrical supertopology on X and let us say that A is close to B when $U\cap B\neq\emptyset$ for each $U\in\mathcal{V}(A)$. It turns out that the relation so defined is exactly a proximity on X in the usual sense, and that any proximity on X can be obtained in this way.

So the notions of topological and proximity spaces are special cases of the notion of supertopological space.

Remark 1. The notion of uniform space can also be obtained as a special case of the notion of supertopological space. But for this aim the axioms (1) and (2) must be reformulated in another manner (cf. [4] and also [6]). We shall not discuss this question here.

Let us remind some of the most simple basic properties of supertopological spaces (we omit their proofs which are direct). Suppose we are given a supertopology $(\mathcal{M},\mathcal{V})$ on a space X. Then:

for each $A\in\mathcal{M}$ the filter $\mathcal{V}(A)$ possesses a base consisting of open sets (X considered as a topological space);

if $A, B\in\mathcal{M}$ and $B\subset A$, then $\mathcal{V}(A)\subset\mathcal{V}(B)$;

if $A_i\in\mathcal{M}$, $U_i\in\mathcal{V}(A_i)$, $i=1, 2, \ldots, k$, and if $\bigcap_{i=1}^{k}A_i\in\mathcal{M}$, then $\cap_{i=1}^{k}U_i\in\mathcal{V}(\cap_{i=1}^{k}A_i)$;

if $X\in\mathcal{M}$ and $\emptyset\in\mathcal{M}$, and if the supertopology is symmetrical, then $\mathcal{V}(\emptyset)=\mathcal{P}(X)$.

Note 2. Obviously any $\mathcal{V}$-neighbourhood of a set $A\in\mathcal{M}$ is also a neighbourhood of A in the usual sense (i. e. in the sense of Bourbaki), but the opposite is not true: $\mathcal{V}(A)$ is in general only a part of the collection of all topological neighbourhoods of A.

A supertopology $(\mathcal{M},\mathcal{V})$ on X is called *separated* provided the following condition is satisfied:

if $A, B\in\mathcal{M}$ and there exists a $U\in\mathcal{V}(A)$ with $U\cap B=\emptyset$, then there exist $V\in\mathcal{V}(A)$ and $W\in\mathcal{V}(B)$ with $V\cap W=\emptyset$.

Obviously any separated supertopology is symmetrical.

A supertopology $(\mathcal{M},\mathcal{V})$ is said to be *additive* provided it has the property:

if $A_i\in\mathcal{M}$, $U_i\in\mathcal{V}(A_i)$, $i=1, 2, \ldots, k$ and if $\bigcup_{i=1}^{k}A_i\in\mathcal{M}$, then $\cup_{i=1}^{k}U_i\in\mathcal{V}(\cup_{i=1}^{k}A_i)$.

One readily sees that any symmetrical supertopology of the sort $(\mathscr{P}(X), \mathscr{V})$ is additve.

In the class of all supertopologies on a given set X an o r d e r can be introduced in the following manner: $(\mathscr{M}_1, \mathscr{V}_1) \geqq (\mathscr{M}_2, \mathscr{V}_2)$ provided $\mathscr{M}_1 \subset \mathscr{M}_2$ and $\mathscr{V}_1(A) \supset \mathscr{V}_2(A)$ for each $A \in \mathscr{M}_1$.

One verifies right away that in the corresponding cases this order is no other but the usual order in the class of the topologies, respectively the proximities, on the set X.

Also the following definition seems quite natural. If we are given two supertopological spaces X and Y — the first one with the supertopology $(\mathscr{M}_X, \mathscr{V}_X)$, the second with $(\mathscr{M}_Y, \mathscr{V}_Y)$ — a mapping $f\colon X \to Y$ is called *continuous with respect to these supertopologies* provided that: a) $f(\mathscr{M}_X) \subset \mathscr{M}_Y$; b) for any $A \in \mathscr{M}_X$ and any $V \in \mathscr{V}_Y(f(A))$ there exists a $U \in \mathscr{V}_X(A)$ with $f(U) \subset V$.

Clearly this concept of continuity also coincides in the corresponding cases with the usual notions of topological, respectively proximal, continuity.

Finally if $(\mathscr{M}, \mathscr{V})$ is a supertopology on a set X and if $A \subset X$, then a supertopology $(\mathscr{M}_A, \mathscr{V}_A)$ on A is defined by $\mathscr{M}_A = \{B \subset A \mid B \in \mathscr{M}\}$ and $\mathscr{V}_A(B) = \{U \cap A \mid U \in \mathscr{V}(B)\}$ for $B \in \mathscr{M}_A$. We say the supertopology $(\mathscr{M}_A, \mathscr{V}_A)$ is *induced* by $(\mathscr{M}, \mathscr{V})$ on A.

2. Compactly determined extensions. An extension (Y, φ) of a topological space X is called *compactly determined* extension of X provided for any $y \in Y$ there exists such a set $A \subset X$ that $y \in \overline{\varphi(A)}$ and $\overline{\varphi(A)}$ is compact.

Clearly all locally compact extensions, in particular the compactifications, as well as all metrizable, and — more generally — all first-countable extensions of a space X are its compactly determined extensions.

The main purpose of this section is to show that the compactly determined Hausdorff extensions of a given topological space X are closely connected with a class of supertopologies on X which we call b-supertopologies.

2.1. A supertopology $(\mathscr{M}, \mathscr{V})$ on a space X is called *b-supertopology* provided it is separated, additive and satisfies the condition:

(i) if $A \in \mathscr{M}$ an $\mathrm{d}B \subset A$, then $B \in \mathscr{M}$.

Let us remark that in the case $X \in \mathscr{M}$ any b-supertopology $(\mathscr{M}, \mathscr{V})$ on X is, as one readily sees, nothing but a symmetrical supertopology of the sort $(\mathscr{P}(X), \mathscr{V})$, i. e. a proximity on X.

On the other hand, on any Hausdorff topological space X a b-supertopology $(\mathscr{M}, \mathscr{V})$, compatible with the given topology, is always defined in the following manner: $\mathscr{M} = \mathscr{J}(X) \cup \{\emptyset\}$, for each $x \in X$ the filter $\mathscr{V}(\{x\})$ coincides with the filter of the neighbourhoods of the point x in X, and $\mathscr{V}(\emptyset) = \mathscr{P}(X)$. This supertopology is called *trivial b-supertopology* on X.

From now on in this subsection X will be a given Hausdorff space and $(\mathscr{M}, \mathscr{V})$ will be a given b-supertopology on it which is compatible with its topology. We will see that this supertopology generates in a standard manner a compactly determined extension of X.

A non-empty collection $\mathscr{A}$ of non-empty subsets of X, belonging to $\mathscr{M}$, is said to have *the finite $\mathscr{V}$-intersection property* (w i t h r e s p e c t t o t h e b-s u p e r t o p o l o g y $(\mathscr{M}, \mathscr{V})$) provided from $A_i \in \mathscr{A}$, $U_i \in \mathscr{V}(A_i)$, $i = 1, 2, \ldots, k$ (where k is an arbitrary natural number) it follows $\bigcap_{i=1}^{k} U_i \neq \emptyset$.

Any collection having the finite $\mathscr{V}$-intersection property is easily seen to be contained in a maximal one.

Let X^* be the set of all maximal collections in X having the finite $\mathscr{V}$-intersection property with respect to the b-supertopology $(\mathscr{M}, \mathscr{V})$. The elements of X^* will be denoted by ξ, η, ζ, etc.

Lemma 1. *If $\xi \in X^*$ and $A \in \xi$, then $B \in \xi$ for any $B \supset A$, $B \in \mathscr{M}$.*

Proof. The collection $\xi \cup \{B\}$ has obviously the finite $\mathscr{V}$-intersection property and therefore $B \in \xi$ because of the maximality of the collection ξ.

Lemma 2. *If $\xi \in X^*$, $A \in \xi$ and $B \subset A$, then either $B \in \xi$ or $A \setminus B \in \xi$.*

Proof. We can assume that $B \neq \emptyset$ and $A \setminus B \neq \emptyset$, otherwise the required assertion is evident. Suppose $B \notin \xi$ and $A \setminus B \notin \xi$. Then because of the maximality of ξ there exist $A_i' \in \xi$, $U_i' \in \mathscr{V}(A_i')$ $(i=1, 2, \ldots, k)$ and $U' \in \mathscr{V}(B)$ with $(\cap_{i=1}^{k} U_i') \cap U' = \emptyset$, and also $A_j'' \in \xi$, $U_j'' \in \mathscr{V}(A_j')$ $(j=1, 2, \ldots, s)$ and $U'' \in \mathscr{V}(A \setminus B)$ with $(\cap_{j=1}^{s} U_j'') \cap U'' = \emptyset$. Hence $(\cap_{i=1}^{k} U_i') \cap (\cap_{j=1}^{s} U_j'') \cap (U' \cup U'') = \emptyset$. But this contradicts the finite $\mathscr{V}$-intersection property of the collection ξ because, in view of the additivity of the supertopology $(\mathscr{M}, \mathscr{V})$, we have $U' \cup U'' \in \mathscr{V}(A)$.

Lemma 3. *If $\xi \in X^*$, $A, B \in \xi$ and $V \in \mathscr{V}(B)$, then $A \cap V \neq \emptyset$.*

Proof. If $A \cap V = \emptyset$, then it follows from the separateness of the considered supertopology that $U \cap W = \emptyset$ for some $U \in \mathscr{V}(A)$ and $W \in \mathscr{V}(B)$, a contradiction with the finite $\mathscr{V}$-intersection property of the collection ξ.

Lemma 4. *If $\xi \in X^*$, $A, B \in \xi$ and $V \in \mathscr{V}(B)$, then $A \cap V \in \xi$.*

Proof. Let $A' = A \cap V$. Since $(A \setminus A') \cap V = \emptyset$ it follows from Lemma 3 that $A \setminus A' \notin \xi$. Then, by Lemma 2, $A' \in \xi$.

Lemma 5. *If $\xi \in X^*$, $A_i \in \xi$ and $U_i \in \mathscr{V}(A_i)$, $i = 1, 2, \ldots, k$, then there exists a $B \in \xi$ with $\cap_{i=1}^{k} U_i \in \mathscr{V}(B)$.*

Proof. Let $A_1, A_2 \in \xi$, $U_1 \in \mathscr{V}(A_1)$, $U_2 \in \mathscr{V}(A_2)$. By the condition (2) in the definition of supertopology (Section 1) there exists a $V \in \mathscr{V}(A_1)$ with $U_1 \in \mathscr{V}(C)$ for any $C \subset V$, $C \in \mathscr{M}$. Then $U_1 \in \mathscr{V}(B)$ for $B = A_2 \cap V$. On the other hand, $U_2 \in \mathscr{V}(B)$ and therefore $U_1 \cap U_2 \in \mathscr{V}(B)$. Moreover, $B \in \xi$ by Lemma 4. Thus the lemma is proved for $k=2$. In the general case the assertion is obtained by mathematical induction.

Lemma 6 *If $\xi \in X^*$, $A \in \mathscr{M}$, $A \neq \emptyset$ and if $A \notin \xi$, then there exist $U \in \mathscr{V}(A)$, $B \in \xi$, $V \in \mathscr{V}(B)$ with $U \cap V = \emptyset$.*

Proof. Since $A \notin \xi$ it follows from the maximality of the collection ξ that there exist $U \in \mathscr{V}(A)$, $B_i \in \xi$, $V_i \in \mathscr{V}(B_i)$, $i = 1, 2, \ldots, k$, with $U \cap (\cap_{i=1}^{k} V_i) = \emptyset$. But by Lemma 5 $\cap_{i=1}^{k} V_i \in \mathscr{V}(B)$ for some $B \in \xi$. Thus the lemma is proved by letting $V = \cap_{i=1}^{k} V_i$.

Now we are ready to introduce a convenient for our purpose topology on X^*. For any set $\Gamma \subset X^*$ let us denote by $\overline{\Gamma}$ the set of all elements ξ of X^* possessing the following property:

if $A \in \xi$ and $U \in \mathscr{V}(A)$, then there exist $\eta \in \Gamma$ and $B \in \eta$ with $U \in \mathscr{V}(B)$.

Lemma 7. *The above defined operator $\Gamma \to \overline{\Gamma}$ is a closure operator.*

Proof. One verifies immediately the properties $\overline{\Phi} = \Phi$, $\Gamma \subset \overline{\Gamma}$, $\overline{\overline{\Gamma}} = \overline{\Gamma}$, and it remains to prove the equality

$$\overline{\Gamma' \cup \Gamma''} = \overline{\Gamma'} \cup \overline{\Gamma''} \tag{3}$$

for any $\Gamma', \Gamma'' \subset X^*$. Clearly $\overline{\Gamma'} \cup \overline{\Gamma''} \subset \overline{\Gamma' \cup \Gamma''}$. Let us suppose that $\xi \in \overline{\Gamma' \subset \Gamma''}$, but $\xi \notin \overline{\Gamma'}$ and $\xi \notin \overline{\Gamma''}$. Then there exist such $A', A'' \in \xi$, $U' \in \mathscr{V}(A')$,

$U'' \in \mathscr{V}(A'')$ that $U' \notin \mathscr{V}(B)$ for any $B \in \eta$ when $\eta \in \Gamma'$ and also $U'' \notin \mathscr{V}(C)$ for any $C \in \zeta$ when $\zeta \in \Gamma''$. But by Lemma 5 $U' \cap U'' \in \mathscr{V}(A)$ for some $A \in \xi$. Considering that $\xi \in \overline{\Gamma' \cup \Gamma''}$ one infers that $U' \cap U'' \in \mathscr{V}(D)$ for some $D \in \lambda$, where $\lambda \in \Gamma' \cup \Gamma''$. So $U' \in \mathscr{V}(D)$, $U'' \in \mathscr{V}(D)$ and at the same time either $\lambda \in \Gamma'$ or $\lambda \in \Gamma''$ which contradicts the choice of U' and U''. Hence $\overline{\Gamma' \cup \Gamma''} \subset \overline{\Gamma'} \cup \overline{\Gamma''}$ and (3) holds.

From now on in this section X^* will be always considered as a topological space with topology just introduced on it. This topology can be described in another way.

For any open subset U of the space X let

$$\Omega_U = \{\xi \in X^* \mid U \in \mathscr{V}(A) \text{ for some } A \in \xi\}. \tag{4}$$

L e m m a 8. *For any two open subsets U and V of X*

$$\Omega_{U \cap V} = \Omega_U \cap \Omega_V \tag{5}$$

holds.

P r o o f. If $\xi \in \Omega_{U \cap V}$, then $U \cap V \in \mathscr{V}(A)$ for some $A \in \xi$. Therefore $U \in \mathscr{V}(A)$ and $V \in \mathscr{V}(A)$, hence $\xi \in \Omega_U$ and $\xi \in \Omega_V$. So $\Omega_{U \cap V} \subset \Omega_U \cap \Omega_V$. Conversely, if $\xi \in \Omega_U \cap \Omega_V$, then there exist $A, B \in \xi$ with $U \in \mathscr{V}(A)$ and $V \in \mathscr{V}(B)$. Then $U \cap V \in \mathscr{V}(C)$ for some $C \in \xi$ and so $\xi \in \Omega_{U \cap V}$. Hence $\Omega_U \cap \Omega_V \subset \Omega_{U \cap V}$. Thus (5) is proved.

Let us mention also the obvious equalities $\Omega_X = X^*$, $\Omega_\varnothing = \varnothing$. (In particular $\Omega_U \cap \Omega_V = \varnothing$ when $U \cap V = \varnothing$.)

L e m m a 9. *The collection*

$$\{\Omega_U \mid U \text{ is open in } X\} \tag{6}$$

is a base for the topology on X^*.

P r o o f. First of all let us see that for any open subset U of X the set Ω_U is open in X^*. Assume $\xi \notin X^* \setminus \Omega_U$. Then $\xi \in \Omega_U$ and hence $U \in \mathscr{V}(A)$ for some $A \in \xi$. Suppose $\xi \in \overline{X^* \setminus \Omega_U}$, then $U \in \mathscr{V}(B)$ for some B with $B \in \eta$ and $\eta \in X^* \setminus \Omega_U$. On the other hand, $B \in \eta$ and $U \in \mathscr{V}(B)$ imply that $\eta \in \Omega_U$, a contradiction. So $\xi \notin \overline{X^* \setminus \Omega_U}$. It follows that the set $X^* \setminus \Omega_U$ is closed, i. e., that Ω_U is open.

Next assume that Γ is an open subset of X^* and that $\xi \in \Gamma$. Since $\xi \notin X^* \setminus \Gamma$ and therefore $\xi \notin \overline{X^* \setminus \Gamma}$, there exist such an $A \in \xi$ and such an open $U \in \mathscr{V}(A)$ that $U \notin \mathscr{V}(B)$ for any $B \in \eta$ when $\eta \in X^* \setminus \Gamma$. Clearly $\xi \in \Omega_U$ and we will show that $\Omega_U \subset \Gamma$. Indeed, if $\zeta \in \Omega_U$, then $U \in \mathscr{V}(B)$ for some $B \in \zeta$. Hence $\zeta \notin X^* \setminus \Gamma$, i. e. $\zeta \in \Gamma$. So $\Omega_U \subset \Gamma$, and thus it is proved that the collection (6) is a base for the topological space X^*.

N o t e 3. As one can observe, the assumption that the set U is open is not used in this proof. In fact, the operator Ω, defined by (4), can be introduced in the same manner for arbitrary (not only open) subsets U of X. However, one sees (using the fact that for any $A \in \mathscr{M}$ the filter $\mathscr{V}(A)$ has a base of open sets) that in this case the equality $\Omega_U = \Omega_{\mathrm{Int}\, U}$ always holds.

L e m m a 10. *If $\xi', \xi'' \in X^*$ and $\xi' \neq \xi''$, then there exist $A' \in \xi'$, $A'' \in \xi''$, $U' \in \mathscr{V}(A')$, $U'' \in \mathscr{V}(A'')$ with $U' \cap U'' = \varnothing$.*

P r o o f. Since $\xi' \neq \xi''$, there exists $A' \in \mathscr{M}$ with $A' \in \xi'$ and $A' \notin \xi''$. By Lemma 6 there are $A'' \in \xi''$, $U' \in \mathscr{V}(A')$, $U'' \in \mathscr{V}(A'')$ with $U' \cap U'' = \varnothing$.

C o r o l l a r y. *The topological space X^* is a Hausdorff space.*

Indeed, under the denotations of Lemma 10, $\xi' \in \Omega_{U'}$, $\xi'' \in \Omega_{U''}$ and $\Omega_{U'} \cap \Omega_{U''} = \emptyset$.

Lemma 11. *For any* $x \in X$ *the collection*

$$\alpha(x) = \{A \in \mathcal{M} \mid x \in U \text{ for every } U \in \mathcal{V}(A)\} \tag{7}$$

is an element of the space X^*.

Proof. The collection $\alpha(x)$ has obviously the finite $\mathcal{V}$-intersection property, and it must be shown that it is a maximal one. Take a $C \in \mathcal{M}$, $C \neq \emptyset$ with $C \notin \alpha(x)$. Since $C \notin \alpha(x)$, there is a $V \in \mathcal{V}(C)$ with $x \notin V$. Then by the separateness of the given supertopology there exist $U \in \mathcal{V}(x)$ and $W \in \mathcal{V}(C)$ with $U \cap W = \emptyset$. Hence the collection $\alpha(x) \cup \{C\}$ has not the finite $\mathcal{V}$-intersection property. Thus the maximality of the collection $\alpha(x)$ is proved.

According to the last lemma the equality (7) defines a mapping $\alpha: X \to X^*$ Now it is easy to get the following result.

Lemma 12. (X^*, α) *is an extension of the topological space* X.

Proof. Obviously the mapping α is one-to-one. Also the equality

$$\alpha(U) = \Omega_U \cap \alpha(X) \tag{8}$$

is clearly true for any open $U \subset X$. At the same time the formula (8), in view of Lemma 9, shows that α is a homeomorphic embedding and that $\alpha(X)$ is dense in X^*.

The extension (X^*, α) of the space X will be called *standardly generated* by the b-supertopology $(\mathcal{M}, \mathcal{V})$ on X, and the mapping α — *standard embedding* of X into X^*.

Let us mention also the useful formula $\overline{\Omega_U} = \overline{\alpha(U)}$, valid for any open subset U of X (which follows immediately from (8) and from the density of $\alpha(X)$ in X^*).

Lemma 13. *let* $\xi \in X^*$ *and* $A \in \mathcal{M}$, $A \neq \emptyset$. *Then* $A \in \xi$ *if and only if* $\xi \in \overline{\alpha(A)}$.

Proof. Assume $A \in \xi$. Take a $B \in \xi$ and a $U \in \mathcal{V}(B)$. There exists a $V \in \mathcal{V}(B)$ with $U \in \mathcal{V}(C)$ for any $C \subset V$, $C \in \mathcal{M}$. If $A' = A \cap V$, then $U \in \mathcal{V}(A')$. Observe that, by Lemma 3, $A' \neq \emptyset$. If $x \in A'$ then $A' \in \alpha(x)$ and $\alpha(x) \in \alpha(A)$. Hence $\xi \in \overline{\alpha(A)}$.

Conversely, suppose $\xi \in \overline{\alpha(A)}$ and $A \notin \xi$. Then by Lemma 6 there exist $U \in \mathcal{V}(A)$, $B \in \xi$, $V \in \mathcal{V}(B)$ with $U \cap V = \emptyset$. On the other hand, since $\xi \in \overline{\alpha(A)}$, there exists an $x \in A$ with $V \in \mathcal{V}(C)$ for some $C \in \alpha(x)$. Hence $x \in V$ and then $x \in A \cap V \subset U \cap V = \emptyset$, a contradiction which shows that $A \in \xi$.

Lemma 14. *If* $X \in \mathcal{M}$, *then for any open subset* U *of* X

$$X^* \setminus \Omega_U = \overline{\alpha(X \setminus U)} \tag{9}$$

holds.

Proof. Let U be an open set in X and let $\xi \in X^* \setminus \Omega_U$. Next let us suppose that $X \setminus U \notin \xi$. ($X \setminus U \in \mathcal{M}$ because $X \in \mathcal{M}$.) Then there exist $B \in \xi$, $V \in \mathcal{V}(B)$ and $W \in \mathcal{V}(X \setminus U)$ with $V \cap W = \emptyset$. Therefore $V \subset X \setminus W \subset U$, so $U \in \mathcal{V}(B)$ and hence $\xi \in \Omega_U$, a contradiction. It follows that $X \setminus U \in \xi$ and then, by Lemma 13, $\xi \in \overline{\alpha(X \setminus U)}$.

Conversely, let us suppose that $\xi \in \overline{\alpha(X \setminus U)}$ and $\xi \notin X^* \setminus \Omega_U$. Then $\xi \in \Omega_U$ which means that $U \in \mathcal{V}(B)$ for some $B \in \xi$. By the definition of the closure operator in X^* there exist a point $x \in X \setminus U$ and a $C \in \alpha(x)$ with $U \in \mathscr{V}(C)$. Hence $x \in U$, a contradiction. Therefore $\xi \in X^* \setminus \Omega_U$. Thus (9) is proved.

Corollary. *In the case $X \in \mathscr{M}$ the collection* $\{\overline{\alpha(F)} \mid F$ is closed in $X\}$ *is a base for the closed sets in the space X^*.*

Using a widespread terminology, this corollary can be expressed by saying that if $X \in \mathscr{M}$, then (X^*, α) is a strict extension of the space X.

Now we can obtain the following

Lemma 15. *In the case $X \in \mathscr{M}$ the space X^* is compact.*

Proof. Let us suppose that the collection $\{\Phi_s \mid s \in S\}$ consisting of closed subsets of the space X^* has the finite intersection property. For each $s \in S$ we have $\Phi_s = \bigcap \{\overline{\alpha(F)} \mid F \in \mathscr{F}_s\}$, where $\mathscr{F}_s$ is a collection of closed subsets F, of the space X. Consider the collection $\mathscr{F} = \bigcup \{\mathscr{F}_s \mid s \in S\}$. Let $F_1^{s_1}, F_2^{s_1}, \ldots$ $F_{k_1}^{s_1}, F_1^{s_2}, F_2^{s_2}, \ldots, F_{k_2}^{s_2}, \ldots, F_1^{s_p}, F_2^{s_p}, \ldots, F_{k_p}^{s_p}$ be a finite system of elements of $\mathscr{F}$. Here $s_1, s_2, \ldots, s_p$ are p different indices from S and $F_j^{s_i} \in \mathscr{F}_{s_i}$ for any i and j. We have $\eta \in \bigcap_{i=1}^{p} \Phi_{s_i}$ for some $\eta \in X^*$. It follows that $\eta \in \overline{\alpha(F_j^{s_i})}$ and hence $F_j^{s_i} \in \eta$ for $j = 1, 2, \ldots, k_i$: $i = 1, 2, \ldots, p$. Therefore, if $U_j^{s_i} \in \mathscr{V}(F_j^{s_i})$, then $\bigcap \{U_j^{s_i} \mid j = 1, 2, \ldots, k_i;\ i = 1, 2, \ldots, p\} \neq \emptyset$. Thus we see that the collection $\mathscr{F}$ has the finite $\mathscr{V}$-intersection property. Hence there exists a $\xi \in X^*$ with $\mathscr{F} \subset \xi$. Since $F \in \xi$ for each $F \in \mathscr{F}$, it follows that $\xi \in \overline{\alpha(F)}$ for $F \in \mathscr{F}$. Thus $\xi \in \Phi_s$ for every $s \in S$ and $\bigcap \{\Phi_s \mid s \in S\} \neq \emptyset$. So the space X^* is compact.

Lemma 16. *Let $A \in \mathscr{M}$, $A \neq \emptyset$ and let $(\mathscr{M}_A, \mathscr{V}_A)$ be the b-supertopology induced on A by $(\mathscr{M}, \mathscr{V})$. If (A^*, α_A) is the standardly generated by $(\mathscr{M}_A, \mathscr{V}_A)$ extension of the topological space A (considered as a subspace of the space X) then (A^*, α_A) and $(\overline{\alpha(A)}, \alpha|_A)$ are equivalent extensions of A.*

Proof. Let us recall (cf. Section 1) that $\mathscr{M}_A = \mathscr{P}(A)$ and

$$\mathscr{V}_A(B) = \{U \cap A \mid U \in \mathscr{V}(B)\}$$

for $B \in \mathscr{M}_A$. The verification of the fact that $(\mathscr{M}_A, \mathscr{V}_A)$ is actually a b-supertopology on A is immediate.

If $\xi \in A^*$, then ξ is obviously a collection with the finite $\mathscr{V}$-intersection property in X. Hence $\xi \subset \eta$ for some point η of X^*. Clearly $A \in \eta$ (because $A \in \xi$). Let us suppose that $\xi \subset \eta_1$, $\xi \subset \eta_2$, where $\eta_1, \eta_2 \in X^*$ and $\eta_1 \neq \eta_2$. Then by Lemma 10 there exist $C_1 \in \eta_1$, $C_2 \in \eta_2$, $U_1 \in \mathscr{V}(C_1)$, $U_2 \in \mathscr{V}(C_2)$, with $U_1 \cap U_2 = \emptyset$. Take a $V_1 \in \mathscr{V}(C_1)$ with $U_1 \in \mathscr{V}(D)$ for $D \subset V_1$, $D \in \mathscr{M}$. Let us see that $A \cap V_1 \in \xi$. In order to show this it suffices, according to Lemma 6, to see that if $C \in \xi$, $U' \in \mathscr{V}_A(C)$ and $W' \in \mathscr{V}_A(A \cap V_1)$, then $U' \cap W' \neq \emptyset$. We have $U' = U \cap A$, $W' = W \cap A$, where $U \in \mathscr{V}(C)$, $W \in \mathscr{V}(A \cap V_1)$. Clearly $C \in \eta_1$ and, by Lemma 4, $A \cap V_1 \in \eta_1$. Therefore it follows from Lemma 5 that $U \cap W \in \mathscr{V}(D)$ for some $D \in \eta_1$. Then, by Lemma 3, $(U \cap W) \cap A \neq \emptyset$ or $U' \cap W' = \emptyset$. And so $A \cap V_1 \in \xi$. On the other hand, $U_1 \in \mathscr{V}(A \cap V_1)$ and hence $U_1 \cap A \in \mathscr{V}_A(A \cap V_1)$. Analogously one sees that there exists a $V_2 \in \mathscr{V}(C_2)$ with $A \cap V_2 \in \xi$ and $U_2 \cap A \in \mathscr{V}_A(A \cap V_2)$. So $\emptyset \neq (U_1 \cap A) \cap (U_2 \cap A) \subset U_1 \cap U_2 = \emptyset$, a contradiction. Thus it is shown that for any $\xi \in A^*$ there exists only one point $\eta \in X^*$ with $\xi \subset \eta$. In addition $\eta \in \overline{\alpha(A)}$ because $A \in \eta$.

Thus a mapping $\lambda\colon A^* \to \overline{\alpha(A)}$ is defined by means of the condition $\xi \subset \lambda(\xi)$. We will show that λ is a homeomorphism.

First of all the mapping λ is one-to-one. Indeed, assume that $\xi_1, \xi_2 \in A^*$ and $\lambda(\xi_1)=\lambda(\xi_2)$, i. e. that $\xi_1 \subset \eta$ and $\xi_2 \subset \eta$, where $\eta \in X^*$. Suppose $\xi_1 \neq \xi_2$, then there are $B_1 \in \xi_1$, $B_2 \in \xi_2$, $U_1 \in \mathscr{V}(B_1)$, $U_2 \in \mathscr{V}(B_2)$ with $(U_1 \cap A) \cap (U_2 \cap A) = \emptyset$. But $B_1, B_2 \in \eta$, therefore $U_1 \cap U_2 \in \mathscr{V}(C)$ for some $C \in \eta$, hence $(U_1 \cap U_2) \cap A \neq \emptyset$. This contradiction shows that $\xi_1 = \xi_2$. So λ is one-to-one.

Passing to the verification of the continuity of λ, we will denote by Ω^A the corresponding to the space A operator, analogous to the defined by (4) operator Ω. In other words,

$$\Omega^A_U = \{\xi \in A^* \mid U \in \mathscr{V}_A(B) \text{ for some } B \in \xi\}$$

for every set $U \subset A$ open in A.

Now assume that $\xi \in A^*$, $\eta = \lambda(\xi)$ and $\eta \in \Omega_U$, where U is an open subset of X. There exists a $B \in \eta$ with $U \in \mathscr{V}(B)$. Take a $V \in \mathscr{V}(B)$ with $U \in \mathscr{V}(C)$ for $C \subset V$, $C \in \mathscr{M}$, and next a $W \in \mathscr{V}(B)$ with $V \in \mathscr{V}(D)$ for $D \subset W$, $D \in \mathscr{M}$. As shown above, $W \cap A \in \xi$. Since $V \in \mathscr{V}(A \cap W)$, we have also $V \cap A \in \mathscr{V}_A(A \cap W)$. Hence $\xi \in \Omega^A_{V \cap A}$. For any $\xi' \in \Omega^A_{V \cap A}$ there exists a $C \in \xi'$ with $V \cap A \in \mathscr{V}_A(C)$. As $C \subset V \cap A$, it follows $U \in \mathscr{V}(C)$. On the other hand, $\xi' \subset \lambda(\xi')$, hence $C \in \lambda(\xi')$ and therefore $\lambda(\xi') \in \Omega_U$. Thus $\lambda(\Omega^A_{V \cap A}) \subset \Omega_U$. So the mapping λ is continuous and, as the space A^* is compact, λ is a homeomorphism.

Further, if $x \in A$, then $\alpha_A(x) = \{B \subset A \mid x \in V \text{ for every } V \in \mathscr{V}_A(B)\}$. (Here α_A is, of course, the standard embedding of A into A^*.) Therefore, in view of (7), $\alpha_A(x) \subset \alpha(x)$, hence $\alpha_A(x) = \alpha(x) = \alpha|_A(x)$, i. e. $\alpha|_A = \lambda\alpha_A$.

Thus, in order to finish the proof of the equivalence of the two considered extensions of the space A, it remains to see that $\lambda(A^*) = \overline{\alpha(A)}$. But this is evident because the space A^* is compact and $\lambda(A^*)$ is dense in $\overline{\alpha(A)}$.

As a corollary from Lemmas 15 and 16 we obtain

Lemma 17. *For any $A \in \mathscr{M}$ the set $\overline{\alpha(A)}$ is compact.*

At last Lemmas 13 and 17 give us

Lemma 18. *(X^*, α) is a compactly determined extension of the space X.*

Thus we have proved

Proposition 1. *Any b-supertopology which is compatible with the given topology on a Hausdorff space X generates (in a standard manner) a compactly determined extension (X^*, α) of X.*

In the sequel we will say that a b-supertopology $(\mathscr{M}, \mathscr{V})$ on a space X *generates* the extension (Y, φ) of X provided this extension is equivalent to the standardly generated by $(\mathscr{M}, \mathscr{V})$ extension (X^*, α) of X.

2.2. Now we will see that, at least in case of regular spaces, the method for construction of compactly determined extensions given in the preceding subsection in universal. (Regular and completely regular spaces are always supposed to be T_1-spaces.)

At first let us observe that on any regular space X a b-supertopology $(\mathscr{M}, \mathscr{V})$, compatible with its topology, can be defined in the following manner: $\mathscr{M}$ is the collection of all sets $A \subset X$ whose closure $\bar{A}$ is compat, and for any $A \in \mathscr{M}$ the filter $\mathscr{V}(A)$ consists of all $U \subset X$ with $\operatorname{Int} U \supset \bar{A}$. This supertopology is called the *standard b-supertopology* on the regular space X.

Next, when (Y, φ) is an extension of the space X and when a supertopology $(\mathscr{M}, \mathscr{V})$ on Y is given, we will speak (identifying X with $\varphi(X)$) about the supertopology induced on X by $(\mathscr{M}, \mathscr{V})$.

Proposition 2. *Let* (Y, φ) *be a regular compactly determined extension of a given (regular) space* X. *Then the b-supertopology* $(\mathcal{M}^*, \mathcal{V}^*)$ *induced on* X *by the standard b-supertopology on* Y *generates* (Y, φ).

Proof. We have

$$\mathcal{M}^* = \{A \subset X \mid \overline{\varphi(A)} \text{ is compact }\} \tag{10}$$

and, for any $A \in \mathcal{M}^*$,

$$\mathcal{V}^*(A) = \{U \subset X \mid U = \varphi^{-1}(U^*),\ U^* \subset Y,\ \operatorname{Int} U^* \supset \overline{\varphi(A)}\}. \tag{11}$$

We can assume, for convenience, that $X \subset Y$ and that φ is the identical embedding. We will denote by $\bar{A}$ the closure in Y of a set $A \subset Y$, also in the case when A is a subset of X. Then $\mathcal{M}^* = \{A \subset X \mid \bar{A}$ is compact $\}$ and $\mathcal{V}^*(A) = \{U \subset X \mid U = U^* \cap X,\ U^* \subset Y,\ \operatorname{Int} U^* \supset \bar{A}\}$ for $A \in \mathcal{M}^*$. It must be shown that the extension (Y, φ) is equivalent to the standardly generated by $(\mathcal{M}^*, \mathcal{V}^*)$ extension of X.

For any $y \in Y$ let

$$\lambda(y) = \{A \in \mathcal{M}^* \mid y \in \bar{A}\}. \tag{12}$$

Evidently $\lambda(y)$ is a non-empty collection of sets with the finite $\mathcal{V}^*$-intersection property in X. Therefore $\lambda(y) \subset \eta$ for some $\eta \in X^*$. Let us see that in fact $\lambda(y) = \eta$. Indeed, suppose that there exists a $B \subset X$ with $B \in \eta$ and $B \notin \lambda(y)$. Since $y \notin \bar{B}$, there exist open sets U^* and V^* in Y with $y \in U^*$, $\bar{B} \subset V^*$ and $U^* \cap V^* = \emptyset$. Take a set $A \in \lambda(y)$. Then $y \in \bar{A} \cap U^* \subset \overline{A \cap U^*}$ which means that $A \cap U^* \in \lambda(y)$ and therefore $A \cap U^* \in \eta$. On the other hand, $V^* \cap X \in \mathcal{V}^*(B)$ and consequently $\emptyset \neq (A \cap U^*) \cap (V^* \cap X) \subset U^* \cap V^* = \emptyset$, a contradiction. Hence $\lambda(y) = \eta$.

Thus the equality (12) defines a mapping $\lambda\colon Y \to X^*$. Clearly, in view of (7), $\lambda(x) = \alpha(x)$ for any $x \in X$.

Let us show that the mapping λ is a homeomorphism. It is clear that λ is one-to-one; this follows immediately from the regularity of Y. In order to establish the continuity of λ, take a point $y_0 \in Y$ and assume that $\lambda(y_0) \in \Omega_U$, where U is an open set in X. Then $U \in \mathcal{V}^*(A)$ for some $A \in \lambda(y_0)$. This means that $U = U^* \cap X$, where U^* is an open set in Y and $U^* \supset \bar{A} \ni y_0$. Now take a point $y \in U^*$. There exists an open subset V^* of Y with $y \in V^*$ and $\bar{V}^* \subset U$. For any $B \in \lambda(y)$ we have $y \in \bar{B} \cap V^* \subset \overline{B \cap V^*}$, hence $B \cap V^* \in \lambda(y)$. On the other hand, $U^* \supset \overline{B} \supset \overline{V^*}$, so that $U \in \mathcal{V}^*(B \cap V^*)$ and consequently $\lambda(y) \in \Omega_U$. Thus $\lambda(U^*) \subset \Omega_U$ which proves the continuity of λ.

Next, λ is a mapping onto. Indeed, take a point $\eta \in X^*$. If $A \in \eta$, then $\eta \in \overline{\alpha(A)} = \overline{\lambda(A)}$. But $\lambda(\bar{A})$ is compact and therefore $\lambda(\bar{A}) \subset \overline{\lambda(A)} \subset \overline{\lambda(\bar{A})} = \lambda(\bar{A})$. Thus $\overline{\lambda(A)} = \lambda(\bar{A})$ and hence $y \in \lambda(\bar{A})$. It follows that $X^* = \lambda(Y)$.

Finally, let us see that the inverse mapping $\lambda^{-1}\colon X^* \to Y$ is continuous. Take a point $\eta_0 \in X^*$. Let $\eta^0 = \lambda(y_0)$, where $y_0 \in Y$, and let U^* be an open neighbourhood of y_0 in Y. In view of the regularity of the space Y we can assume $U^* = \operatorname{Int} \overline{U^*}$. There exists an open set V^* in Y with $y_0 \in V^*$ and $\bar{V}^* \subset U^*$. If $A \in \eta_0$, then $y_0 \in \bar{A}$. Hence $y_0 \in \bar{A} \cap V^* \subset \overline{A \cap V^*}$, and so $A \cap V^* \in \eta_0$. Since $U^* \supset \overline{A \cap V^*}$, we have $U \in \mathcal{V}^*(A \cap V^*)$ for $U = U^* \cap X$ and therefore $\eta_0 \in \Omega_U$. We will show that $\Omega_U \subset \lambda(U^*)$. Indeed, let $\eta \in \Omega_U$ and let $\eta = \lambda(y)$, where $y \in Y$ There exists a $B \in \eta$ with $U \in \mathscr{V}^*(B)$. Then $U = U^* \cap X$, where U^* is open in X and $W^* \supset \overline{B} \in y$. But, in view of the density of X in Y,

$$\overline{W^*} = \overline{W^* \cap X} = \overline{U} = \overline{U^* \cap X} = \overline{U^*}.$$

Therefore $W^* \subset \operatorname{Int} \overline{W^*} = \operatorname{Int} \overline{U^*} = U^*$, and consequently $y \in U^*$. Thus $\Omega_U \subset \lambda(U^*)$, hence $\lambda^{-1}(\Omega_U) \subset U^*$. So λ^{-1} is continuous and this completes the proof of the proposition.

The b-supertopology $(\mathscr{M}^*, \mathscr{V}^*)$ defined by the equalities (10) and (11) will be called *canonically connected* with the regular compactly determined extension (Y, φ) of X. In general it is not the only one that generates this extension. Nevertheless, among the b-supertopologies generating (Y, φ) the canonically connected one plays a special role.

Proposition 3. *Let X be a regular space, (Y, φ) be a given regular compactly determined extension of X, and $(\mathscr{M}^*, \mathscr{V}^*)$ be the b-supertopology canonically connected with (Y, φ). If $(\mathscr{M}', \mathscr{V}')$ be another b-supertopology on X generating the extension (Y, φ), then*: a) $\mathscr{M}' \subset \mathscr{M}^*$; b) $\mathscr{V}'(A) = \mathscr{V}^*(A)$ *for each $A \in \mathscr{M}'$.*

Proof. The extensions of X standardly generated by $(\mathscr{M}^*, \mathscr{V}^*)$ and $(\mathscr{M}', \mathscr{V}')$ will be denoted by (X^*, α) and (X', α'), respectively. According to the assumption these two extensions are equivalent to each other. This means that there exists a homeomorphism $\lambda: X' \to X^*$ with $\lambda(X') = X^*$ and $\lambda\alpha' = \alpha$.

On the other hand, by virtue of the equivalence between the extensions (Y, φ) and (X^*, α) of X,

$$\mathscr{M}^* = \{A \subset X \mid \overline{\alpha(A)} \text{ is compact}\}$$

nd, for any $A \in \mathscr{M}^*$,

$$\mathscr{V}^*(A) = \{U \subset X \mid U = \alpha^{-1}(U^*),\ U^* \subset X^*,\ \operatorname{Int} U^* \supset \overline{\alpha(A)}\}.$$

For $A \in \mathscr{M}'$ the set $\overline{\alpha'(A)}$ is compact, hence the set $\lambda(\overline{\alpha'(A)})$ is compact, too. Therefore, since $A \subset X$,

$$\overline{\alpha(A)} = \overline{\lambda\alpha'(A)} \subset \overline{\lambda(\overline{\alpha'(A)})} = \lambda(\overline{\alpha'(A)}) \subset \overline{\lambda\alpha'(A)} = \overline{\alpha(A)},$$

and so

$$\overline{\alpha(A)} = \lambda(\overline{\alpha'(A)}). \tag{13}$$

Thus the set $\overline{\alpha(A)}$ is compact, hence $A \in \mathscr{M}^*$. So $\mathscr{M}' \subset \mathscr{M}^*$.

Now let $A \in \mathscr{M}'$ and $U \in \mathscr{V}'(A)$. For every $\xi \in \overline{\alpha(A)}$ there is a point $\eta \in X'$ with $\xi = \lambda(\eta)$. According to (13) $\eta \in \overline{\alpha'(A)}$ and consequently $A \in \eta$. Therefore $\eta \in \Omega'_U$. (The definition of the operator Ω' is clear without explanation.) Thus

$$\overline{\alpha(A)} \subset \lambda(\Omega'_U).$$

On the other hand, $\lambda(\Omega'_U)$ is open in X^* and

$$\alpha^{-1}(\lambda(\Omega'_U)) = \alpha'^{-1}(\Omega'_U) = U.$$

Hence $U \in \mathscr{V}^*(A)$. So $\mathscr{V}'(A) \subset \mathscr{V}^*(A)$.

Conversely, let $A \in \mathcal{M}'$ and $U \in \mathcal{V}^*(A)$. We can assume that $U = \alpha^{-1}(U^*)$, where U^* is an open subset ot X^* and $U^* \supset \overline{\alpha(A)}$. If $U' = \lambda^{-1}(U^*)$, then, by (13)

$$\overline{\alpha'(A)} = \lambda^{-1}(\overline{\alpha(A)}) \subset \lambda^{-1}(U^*) = U'.$$

For every $\eta \in \overline{\alpha'(A)}$ take an open set W' in X with $\eta \in \Omega'_{W'} \subset U'$. Then $W' \in \mathcal{V}'(B)$, for some $B \in \eta$ and we can choose an open set W'' in X in such a manner that $W'' \in \mathcal{V}'(B)$, and therefore $\eta \in \Omega'_{W''}$, and that $W' \in \mathcal{V}'(C)$ for any $C \subset W''$, $C \in \mathcal{M}'$. Then, in view of the compactness of $\overline{\alpha'(A)}$ there exist in X finitely many open sets $W'_1, W'_2, \ldots, W'_k$ and $W''_1, W''_2, \ldots, W''_k$ such that

$$\alpha'(A) \subset \bigcup_{i=1}^{k} \Omega'_{W''_i},$$

$\Omega'_{W_i} \subset U'$ and $W'_i \in \mathcal{V}'(C)$ for any $C \subset W''_i$, $C \in \mathcal{M}'$ $(i = 1, 2, \ldots, k)$. Then

$$A \subset \bigcup_{i=1}^{k} \alpha'^{-1}(\Omega'_{W''_i}) = \bigcup_{i=1}^{k} W''_i$$

and consequently $A = \bigcup_{i=1}^{k}(A \cap W''_i)$. But $W'_i \in \mathcal{V}'(A \cap W''_i)$ for $i = 1, 2, \ldots, k$ and, because of the additivity of the supertopology, one concludes that $\bigcup_{i=1}^{k} W'_i \in \mathcal{V}'(A)$. On the other hand,

$$U = \alpha^{-1}(U^*) = \alpha^{-1}(\lambda(U')) = \alpha'^{-1}(U') \supset \alpha'^{-1}\left(\bigcup_{i=1}^{k} \Omega_{W'_i}\right) = \bigcup_{i=1}^{k} \alpha'^{-1}(\Omega'_{W'_i}) = \bigcup_{i=1}^{k} W'_i.$$

Hence $U \in \mathcal{V}'(A)$. Thus $\mathcal{V}^*(A) \subset \mathcal{V}'(A)$ and finally $\mathcal{V}^*(A) = \mathcal{V}'(A)$.

Propositions 1, 2 and 3 yield the following result.

Theorem 1. *Any b-supertopology on a Hausdorff space which is compatible with the topology on X generates (in a standard manner) a compactly determined extension of X. When X is regular, then, conversely, any of its regular compactly determined extension is generated by some b-supertopology on X. Among the b-supertopologies on X generating a given regular compactly determined extension (Y, φ) of X there exists one $(\mathcal{M}^*, \mathcal{V}^*)$ with the following property: if the b-supertopology $(\mathcal{M}, \mathcal{V})$ on X generates (Y, φ), then $\mathcal{M} \subset \mathcal{M}^*$ and $\mathcal{V}(A) = \mathcal{V}^*(A)$ for each $A \in \mathcal{M}$.*

In conclusion let us note that, as it follows from Proposition 2, the standard b-supertopology on a regular space X generates the trivial extension of X.

On the other hand, as noted, the notion of proximity on a space X coincides in reality with the notion of b-supertopology provided that $X \in \mathcal{M}$. Therefore Lemma 15 and Proposition 2 yield immediately the following famous result.

Theorem (Yu. M. Smirnov [10]). *There exists a one-to-one correspondence between the class of all (Hausdorff) compactifications of a Tychonoff space X and the class of all compatible with its topology proximities on X.*

2.3. The question arises to characterize those b-supertopologies on a given (regular) space X which generate (in the standard manner) regular extensions of X. For this purpose we give the following definition.

A supertopology $(\mathcal{M}, \mathcal{V})$ on a set X is called *strictly separated* if it satisfies the condition:

for any $A \in \mathcal{M}$ and any $U \in \mathcal{V}(A)$ there exists a $V \in \mathcal{V}(A)$ such that from $B \in \mathcal{M}$ and $B \cap U = \emptyset$ it follows $W \cap V = \emptyset$ for some $W \in \mathcal{V}(B)$.

Clearly any strictly separated supertopology is separated.

It is not difficult to establish the following statement announced here without proof.

Proposition 4. *Let X be a regular space and let $(\mathcal{M}, \mathcal{V})$ be a b-supertopology on X compatible with its topology. The compactly determined extension of X generated by $(\mathcal{M}, \mathcal{V})$ is regular if and only if the supertopology $(\mathcal{M}, \mathcal{V})$ is strictly separated.*

2.4. In the class of extensions of a given topological space one usually introduces an order in the following quite natural way.

Let (Y_1, φ_1) and (Y_2, φ_2) be two extensions of the topological space X. Then $(Y_1, \varphi_1) \geq (Y_2, \varphi_2)$, provided there exists a continuous mapping $\lambda: Y_1 \to Y_2$ with $\lambda\varphi_1 = \varphi_2$.

It is clear that thereby an order-relation is defined and that the extensions (Y_1, φ_1) and (Y_2, φ_2), supposed to be Hausdorff, are equivalent to each other if and only if $(Y_1, \varphi_1) \geq (Y_2, \varphi_2)$ and $(Y_2, \varphi_2) \geq (Y_1, \varphi_1)$ simultaneously.

The just defined order, considered in the class of the regular compactly determined extensions of a space X turns out to be closely connected with the order in the class of supertopologies on X (defined in Section 1).

Proposition 5. *Let X be a regular space and let $(\mathcal{M}_1, \mathcal{V}_1)$ and $(\mathcal{M}_2, \mathcal{V}_2)$ be two b-supertopologies on X generating the extensions (Y_1, φ_1) and (Y_2, φ_2), respectively. If $(\mathcal{M}_1, \mathcal{V}_1) \geq (\mathcal{M}_2, \mathcal{V}_2)$, then $(Y_1, \varphi_1) \geq (Y_2, \varphi_2)$.*

Proof. We identify the extensions (Y_1, φ_1) and (Y_2, φ_2) of X with the standardly generated ones by $(\mathcal{M}_1, \mathcal{V}_1)$ and $(\mathcal{M}_2, \mathcal{V}_2)$, respectively, costructed by means of the method from subsection 2.1 (so that, in particular, φ_1 and φ_2 are the corresponding standard embedding).

Suppose $(\mathcal{M}_1, \mathcal{V}_1) \geq (\mathcal{M}_2, \mathcal{V}_2)$. This means that $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\mathcal{V}_1(A) \supset \mathcal{V}_2(A)$ for any $A \in \mathcal{M}_1$. Let $\xi \in Y_1$. Then ξ is a maximal system (conisting of non-empty sets belonging to $\mathcal{M}_1$) with the finite $\mathcal{V}_1$-intersection property. Clearly ξ is also a system with finite $\mathcal{V}_2$-intersection property. Therefore there exists a point $\eta \in Y_2$ with $\xi \subset \eta$. Thereby the point η is uniquely determined. In order to show this, let us see that the following auxiliary statement is true:

$$\text{(14)} \quad \text{if } \xi \in Y_1,\ \eta \in Y_2,\ \xi \subset \eta \text{ and } A \in \xi,\ B \in \eta,\ V \in \mathcal{V}_2(B), \text{ then } A \cap V \in \xi.$$

Indeed, if $A \cap V \notin \xi$, then, by Lemma 2, $A' = A \setminus (A \cap V) \in \xi$. Therefore $A' \in \eta$ and from $B \in \eta$ and $V \in \mathcal{V}_2(B)$ it follows by Lemma 3 that $A' \cap V \neq \emptyset$, a contradiction.

Now let us suppose that $\xi \in Y_1$, $\xi \subset \eta_1$, $\xi \subset \eta_2$, where $\eta_1, \eta_2 \in Y_2$, and that $\eta_1 \neq \eta_2$. Then there exist $B_1 \in \eta_1$, $B_2 \in \eta_2$, $V_1 \in \mathcal{V}_2(B_1)$, $V_2 \in \mathcal{V}_2(B_2)$ with $V_1 \cap V_2 = \emptyset$. For $i = 1, 2$ take $W_i \in \mathcal{V}_2(B_i)$ with $V_i \in \mathcal{V}_2(C)$ for any $C \subset W_i$, $C \in \mathcal{M}_2$. If $A_0 \in \xi$, it follows from (14) that $A_0 \cap W_1 \in \xi$ and $A_0 \cap W_2 \in \xi$. But $V_1 \in \mathcal{V}_2(A_0 \cap W_1)$, $V_2 \in \mathcal{V}_2(A_0 \cap W_2)$, hence $V_1 \in \mathcal{V}_1(A_0 \cap W_1)$, $V_2 \in \mathcal{V}_1(A_0 \cap W_2)$, and consequently $V_1 \cap V_2 \neq \emptyset$. The so achieved contradiction shows that for any $\xi \in Y_1$ there exists a unique $\eta \in Y_2$ with $\xi \subset \eta$. This allows to define a mapping $\lambda: Y_1 \to Y_2$ by means of the condition $\xi \subset \lambda(\xi)$ for $\xi \in Y_1$. Clearly $\lambda\varphi_1 = \varphi_2$. It remains to establish the continuity of λ.

Take a point $\xi \in Y_1$. Let $\lambda(\xi)=\eta$ and $\eta \in \Omega^2_U$, where U is an open set in X. (We use the denotations Ω^1 and Ω^2 defined in an obvious manner.) There exists an open set V in X with $\eta \in \Omega^2_V$ and $\overline{\Omega^2_V} \subset \Omega^2_U$. Next, take a $B \in \eta$ with $V \in \mathscr{V}_2(B)$ and a $W \in \mathscr{V}_2(B)$ with $V \in \mathscr{V}_2(C)$ for any $C \subset W$, $C \in \mathscr{M}_2$. If now $A_0 \in \xi$, then $A_0 \cap W \in \xi$ by (14). On the other hand, $V \in \mathscr{V}_2(A_0 \cap W)$ and therefore $V \in \mathscr{V}_1(A_0 \cap W)$, hence $\xi \in \Omega^1_V$. We will see that $\lambda(\Omega^1_V) \subset \Omega^2_U$.

Indeed, for any $\xi' \in \Omega^1_V$ one can take a $B' \in \xi'$ with $V \in \mathscr{V}_1(B)$. If $\lambda(\xi') \subset \eta'$, then $\xi' \subset \eta'$, hence $B' \in \eta'$ and therefore, by Lemma 13, $\eta' \in \overline{\varphi_2(B')} \subset \overline{\varphi_2(V)} = \overline{\Omega^2_V} \subset \Omega^2_U$. So we obtain $\lambda(\Omega^1_V) \subset \Omega^2_U$ which shows that the mapping λ is continuous. Thus it is proved that $(Y_1, \varphi_1) \geqq (Y_2, \varphi_2)$.

Proposition 6. *Let (Y_1, φ_1) and (Y_2, φ_2) be two regular compactly determined extensions of a (regular) space X and let $(\mathscr{M}_1, \mathscr{V}_1)$ and $(\mathscr{M}_2, \mathscr{V}_2)$, respectively, be the b-supertopologies on X canonically connected with them. If $(Y_1, \varphi_1) \geqq (Y_2, \varphi_2)$, then $(\mathscr{M}_1, \mathscr{V}_1) \geqq (\mathscr{M}_2, \mathscr{V}_2)$.*

Proof. We again identify (Y_1, φ_1) and (Y_2, φ_2) with the extensions of X standardly generated by $(\mathscr{M}_1, \mathscr{V}_1)$ and $(\mathscr{M}_2, \mathscr{V}_2)$, respectively.

Suppose $(Y_1, \varphi_1) \geqq (Y_2, \varphi_1)$. Then there exists a continuous mapping $\lambda : Y_1 \to Y_2$ with $\lambda\varphi_1 = \varphi_2$. Take a set $A \in \mathscr{M}_1$. The set $\overline{\varphi_1(A)}$ is compact, hence $\lambda(\overline{\varphi_1(A)})$ is compact, too. Therefore

$$\lambda(\overline{\varphi_1(A)}) = \overline{\lambda(\overline{\varphi_1(A)})} \supset \overline{\lambda(\varphi_1(A)} = \overline{\varphi_2(A)}.$$

It follows that the set $\overline{\varphi_2(A)}$ is compact, hence $A \in \mathscr{M}_2$. Thus $\mathscr{M}_1 \subset \mathscr{M}_2$.

Next, let $A \in \mathscr{M}_1$ and $U \in \mathscr{V}_2(A)$. We can assume that U is an open set in X and that $U = \varphi_2^{-1}(U_2^*)$, where U_2^* is open in Y_2 and $U_2^* \supset \overline{\varphi_2(A)}$. For the set $U_1^* = \lambda^{-1}(U_2^*)$ which is open in Y_1 we have $\varphi_1^{-1}(U_1^*) = \varphi_1^{-1}\lambda^{-1}(U_2^*) = \varphi_2^{-1}(U_2^*) = U$ and $U_1^* = \lambda^{-1}(U_2^*) \supset \lambda^{-1}(\overline{\varphi_2(A)}) \supset \lambda^{-1}\varphi_2(A) = \varphi_1(A)$, hence $U_1 \in \mathscr{V}_1(A)$. So $\mathscr{V}_2(A) \subset \mathscr{V}_1(A)$. Thus it is proved that $(\mathscr{M}_1, \mathscr{V}_1) \geqq (\mathscr{M}_2, \mathscr{V}_2)$.

From Propositions 5 and 6 we, of course, get right away the well-known result [10] that the one-to-one correspondence between the compactifications of a space X and the proximities on it is an order-isomorphism.

2.5. The notion of supertopology turns out to be useful in the question of extension of continuous mappings. One establishes without difficulties

Proposition 7. *Let X and Y be regular spaces. The mapping $f: X \to Y$ is continuous if and only if it is continuous with respect to the standard b-supertopologies on X and Y.*

The main result here is the following

Theorem 2. *Let X and Y be regular spaces, (X^*, α_X) and (Y^*, α_Y), respectively, be their regular compactly determined extensions. Then the continuous mapping $f: X \to Y$ can be extended (evidently in a unique manner) to a continuous mapping $f^*: X^* \to Y^*$ (in the sense that $f^*\alpha_X = \alpha_Y f$) if and only if f is continuous with respect to the b-supertopologies $(\mathscr{M}_X, \mathscr{V}_X)$ and $(\mathscr{M}_Y, \mathscr{V}_Y)$ canonically connected with the extensions (X^*, α_X) and (Y^*, α_Y).*

Proof. If f possesses a continuous extension $f^*: X^* \to Y^*$, then by Proposition 7 the mapping f^* is continuous with respect to the standard b-supertopologies on X^* and Y^*. Therefore f (as restriction of f^* to X) is continuous with respect to the b-supertopologies $(\mathscr{M}_X, \mathscr{V}_X)$ and $(\mathscr{M}_Y, \mathscr{V}_Y)$.

Now suppose, coversely, that f is continuous with respect to the supertopologies $(\mathscr{M}_X, \mathscr{V}_X)$ and $(\mathscr{M}_Y, \mathscr{V}_Y)$. We can assume that the extensions (X^*, α_X)

and (Y^*, α_Y) are just the standardly generated ones by $(\mathcal{M}_X, \mathcal{V}_X)$ and $(\mathcal{M}_Y, \mathcal{V}_Y)$, respectively (and, in particular, that α_X and α_Y are the corresponding standard embeddings). For any $\xi \in X^*$ consider the collection $f(\xi)=\{f(A) \mid A \in \xi\}$. Clearly it has the finite $\mathcal{V}_Y$-intersection property. Indeed, if $A_i \in \xi$, $V_i \in \mathcal{V}_Y(f(A_i))$ for $i=1, 2, \ldots, k$, then there exist $U_i \in \mathcal{V}_X(A_i)$ with $f(U_i) \subset V_i$ $(i=1, 2, \ldots, k)$. Therefore

$$\bigcap_{i=1}^{k} V_i \supset \bigcap_{i=1}^{k} f(U_i) \supset f(\bigcap_{i=1}^{k} U_i) \neq \emptyset.$$

So $f(\xi) \subset \eta$ for some $\eta \in Y^*$. We will see that thereby η is uniquely determined. Suppose $f(\xi) \subset \eta_1$ and $f(\xi) \subset \eta_2$, where $\eta_1, \eta_2 \in Y^*$ and $\eta_1 \neq \eta_2$. Then there exist $B_1 \in \eta_1$, $B_2 \in \eta_2$, $V_1 \in \mathcal{V}_Y(B_1)$, $V_2 \in \mathcal{V}_Y(B_2)$ with $V_1 \cap V_2 = \emptyset$. For $i=1, 2$ take $W_i \in \mathcal{V}_Y(B_i)$ with $V_i \in \mathcal{V}_Y(C)$ for $C \subset W_i$, $C \in \mathcal{M}_Y$. Let $A \in \xi$. If $A \cap f^{-1}(W_1) \notin \xi$, then $A \setminus f^{-1}(W_1) \in \xi$, hence $f(A \setminus f^{-1}(W_1)) \in f(\xi)$. It follows $f(A) \setminus W_1 \in \eta_1$ because $f(A) \setminus W_1 \supset f(A \setminus f^{-1}(W_1))$. Therefore $(f(A) \setminus W_1) \cap W_1 \neq \emptyset$ which is impossible. Hence $A \cap f^{-1}(W_1) \in \xi$. On the other hand, $V_1 \in \mathcal{V}_Y(f(A) \cap W_1)$ and $f(A \cap f^{-1}(W_1)) \subset f(A) \cap W_1$, hence $f^{-1}(V_1) \in \mathcal{V}_X(A \cap f^{-1}(W_1))$. Analogously $A \cap f^{-1}(W_2) \in \xi$ and $f^{-1}(V_2) \in \mathcal{V}_X(A \cap f^{-1}(W_2))$. Therefore $\emptyset \neq f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset$. This contradiction shows that the inclusion $f(\xi) \subset \eta$ determines η. Consequently the condition $f(\xi) \subset f^*(\xi)$ defines a mapping $f^*\colon X^* \to Y^*$. One sees immediately that

$$f^*(\alpha_X(x)) = \alpha_Y(f(x)) \tag{15}$$

for any $x \in X$. Indeed, recall that $\alpha_X(x) = \{A \in \mathcal{M}_X \mid U \ni x \text{ for every } U \in \mathcal{V}_X(A)\}$. For $A \in \alpha_X(x)$ and $V \in \mathcal{V}_Y(f(A))$ we have $f^{-1}(V) \in \mathcal{V}_X(A)$, hence $x \in f^{-1}(V)$ and $f(x) \in V$. Therefore $f(A) \in \alpha_Y(f(x))$, hence $f(\alpha_X(x)) \subset \alpha_Y(f(x))$. This shows that (15) holds.

Let us see that f^* is continuous. Using the selfexplaining denotations Ω^X and Ω^Y, suppose that $\xi \in X^*$, $f^*(\xi) = \eta$ and $\eta \in \Omega^Y_V$, where V is an open set in Y. Take an open $V_1 \subset Y$ with $\eta \in \Omega^Y_{V_1}$ and $\overline{\Omega^Y_{V_1}} \subset \Omega^Y_V$. There exist $B \in \eta$ with $V_1 \in \mathcal{V}_Y(B)$ and $W \in \mathcal{V}_Y(B)$ with $V_1 \in \mathcal{V}_Y(C)$ for any $C \subset W$, $C \in \mathcal{M}_Y$. Let $A \in \xi$. Reasoning as above, one concludes that $A \cap f^{-1}(W) \in \xi$, $f(A) \cap W \in \eta$ and $V_1 \in \mathcal{V}_Y(f(A) \cap W)$, hence $U = f^{-1}(V_1) \in \mathcal{V}_X(A \cap f^{-1}(W))$. Therefore $\xi \in \Omega^X_U$. Now take a $\xi_1 \in \Omega^X_U$. Then $U \in \mathcal{V}_X(D)$ for some $D \in \xi_1$. Since $f(D) \in f^*(\xi_1)$, we infer that $f^*(\xi_1) \in \overline{\alpha_Y(f(D))} \subset \overline{\alpha_Y(V_1)} = \overline{\Omega^Y_{V_1}} \subset \Omega^Y_V$. So $f^*(\Omega^X_U) \subset \Omega^Y_V$. This shows that f^* is continuous.

Evidently the above proved theorem yields the well-known result [10] about the extendibility of continuous mappings on compactifications.

3. Locally compact extensions. The results from Section 2 allow a great simplification when one considers the important class of locally compact extensions. (The results of this section were proved first by another method [3; 5] in collaboration with G. Dimov.)

The locally compact extensions of a topological space were described by Leader [9] on the basis of the notion of local proximity space introduced by himself. However the description of these extensions proposed here seems to be simpler and having a proper importance.

In this section X will be always a Tychonoff space. A supertopology $(\mathcal{M}, \mathcal{V})$ on X is called *lc-supertopology* provided it is symmetrical and the following conditions are fulfilled:

(i) if $A \in \mathcal{M}$ and $B \subset A$, then $B \in \mathcal{M}$;
(ii) if $A, B \in \mathcal{M}$, then $A \cup B \in \mathcal{M}$;
(iii) for any $A \in \mathcal{M}$ there exists a $U \in \mathcal{V}(A)$ with $U \in \mathcal{M}$.

A trivial example of *lc*-supertopology is any symmetrical supertopology of the sort $(\mathcal{P}(X), \mathcal{V})$, i. e. any proximity on X. Obviously this is the case whenever a *lc*-supertopology $(\mathcal{M}, \mathcal{V})$ on X is such that $X \in \mathcal{M}$.

On the other hand, when the space X is locally compact, one easily verifies that the standard *b*-supertopology on X (when X is considered as regular space) is a *lc*-supertopology. Therefore we can speak about the *standard lc-supertopology* on a given locally compact space.

Lemma 19. *Any lc-supertopology $(\mathcal{M}, \mathcal{V})$ on a space X is a b-supertopology.*

Proof. It must be shown that the supertopology $(\mathcal{M}, \mathcal{V})$ is separated and additive. Let $A, B \in \mathcal{M}$, $U \in \mathcal{V}(A)$ and $B \cap U = \emptyset$. Take a $V \in \mathcal{V}(A)$ with $U \in \mathcal{V}(C)$ for any $C \subset V, C \in \mathcal{M}$. By the condition (iii) there exists a $U_0 \in \mathcal{V}(A)$ with $U_0 \in \mathcal{M}$. Then $U \in \mathcal{V}(U_0 \cap V)$. By virtue of the symmetry of the supertopology $(\mathcal{M}, \mathcal{V})$ it follows from $B \cap U = \emptyset$ that $W \cap (U_0 \cap V) = \emptyset$ for some $W \in \mathcal{V}(B)$. Since $U_0 \cap V \in \mathcal{V}(A)$, thus it is shown that the supertopology $(\mathcal{M}, \mathcal{V})$ is separated.

In order to establish the additivity of $(\mathcal{M}, \mathcal{V})$ it suffices to see that $A_1, A_2 \in \mathcal{M}$, $U_1 \in \mathcal{V}(A_1)$, $U_2 \in \mathcal{V}(A_2)$ imply $U_1 \cup U_1 \in \mathcal{V}(A_1 \cup A_2)$. Take a $U_0 \in \mathcal{V}(A_1 \cup A_2)$ with $U_0 \in \mathcal{M}$. If $U' = U_1 \cup U_0$ and $U'' = U_2 \cap U_0$, then $U' \in \mathcal{V}(A_1)$ and $U'' \in \mathcal{V}(A_2)$. Let $B_1 = U_0 \setminus U'$, $B_2 = U_0 \setminus U''$. Since $B_1 \in \mathcal{M}$ and $U' \cap B_1 = \emptyset$, there exists a $V' \in \mathcal{V}(B_1)$ with $V' \cap A_1 = \emptyset$. Analogously $V'' \cap A_2 = \emptyset$ for some $V'' \in \mathcal{V}(B_2)$. For $V = V' \cap V''$ we have $V \in \mathcal{V}(B_1 \cap B_2)$ and $V \cap (A_1 \cup A_2) = \emptyset$. Therefore there exists a $U \in \mathcal{V}(A_1 \cup A_2)$ with $U \cap (B_1 \cap B_2) = \emptyset$. Then for $U^* = U_0 \cap U$ we have $U^* \in \mathcal{V}(A_1 \cup A_2)$, $U^* \subset U_0$ and $U^* \cap (B_1 \cap B_2) = \emptyset$. Hence

$$U^* \subset U_0 \setminus (B_1 \cap B_2) = U' \cup U'' \subset U_1 \cup U_2,$$

and consequently $U_1 \cup U_2 \in \mathcal{V}(A_1 \cup A_2)$. So the supertopology $(\mathcal{M}, \mathcal{V})$ is additive.

Now we get immediately

Lemma 20. *Any lc-supertopology $(\mathcal{M}, \mathcal{V})$ on X which is compatible with the topology on X generates (in the standard manner for b-supertopologies) a locally compact extension (X^*, α) of X.*

Proof. Indeed, let $\xi \in X^*$ and $A \in \xi$. Take an open $U \in \mathcal{V}(A)$ with $U \in \mathcal{M}$. Then $U \in \xi$ and $\xi \in \Omega_U$, i. e. Ω_U is a neighbourhood of ξ in X^*. At the same time by Lemma 17 the set $\overline{\Omega_U} = \overline{\alpha(U)}$ is compact.

When (Y, φ) is a locally compact (and therefore compactly determined) extension of the space X, then, as it is easily seen, the *b*-supertopology $(\mathcal{M}^*, \mathcal{V}^*)$ canonically connected with (Y, φ) (defined by means of (10) and (11)) is a *lc*-supertopology on X. Therefore we can speak of the *lc*-supertopology on X canonically connected with a locally compact extension of X. We will see that it is in fact the unique *lc*-supertopology on X generating (Y, φ).

Lemma 21. *Let (Y, φ) be a locally compact extension of the space X and let $(\mathcal{M}^*, \mathcal{V}^*)$ be the lc-supertopology on X canonically connected with (Y, φ). Then any lc-supertopology $(\mathcal{M}, \mathcal{V})$ on X generating (Y, φ) coincides with $(\mathcal{M}^*, \mathcal{V}^*)$.*

Proof. Let $(\mathcal{M}, \mathcal{V})$ be a *lc*-supertopology which is compatible with the topology on X and which (considered as a *b*-supertopology) generates (Y, φ). This means that the standardly generated by $(\mathcal{M}, \mathcal{V})$ extension (X^*, α) of X is equivalent to (Y, φ).

From Proposition 3 we know that $\mathcal{M}\subset\mathcal{M}^*$. Take an $A\in\mathcal{M}^*$. Every point of the compact set $\overline{\alpha(A)}$ has a neighbourhood of the form Ω_U, where U is an open set in X and $U\in\mathcal{M}$. Therefore there exist finitely many open sets $U_1, U_2, \ldots, U_k$ in X such that $U_i\in\mathcal{M}\,(i=1, 2, \ldots, k)$ and

$$\overline{\alpha(A)}\subset \bigcup_{i=1}^{k} \Omega_{U_i}.$$

Then

$$A=\alpha^{-1}\alpha(A)\subset \bigcup_{i=1}^{k} \alpha^{-1}(\Omega_{U_i})= \bigcup_{i=1}^{k} U_i.$$

But $\bigcup_{i=1}^{k} U_i\in\mathcal{M}$ because of the condition (ii). So $A\in\mathcal{M}$. Thus $\mathcal{M}=\mathcal{M}^*$. By virtue of Proposition 3 we now get $(\mathcal{M}, \mathcal{V})=(\mathcal{M}^*, \mathcal{V}^*)$.

The last two lemmas, together with Propositions 2, 5 and 6 give us the following result.

Theorem 3. *There exists a one-to-one correspondence between the class of all locally compact extensions of a Tychonoff space X (defined up to equivalence) and the class of all (compatible with the topology on X) lc-supertopologies on X. This correspondence is an order-isomorphism with respect to the usual orders in these classes.*

Remark 2. One can observe certain relations between some of the notions introduced in this paper and several concepts from the theory of nearness structure in the sense of H. Herrlich [8; 1]. So the systems having the finite $\mathcal{V}$-intersection property with respect to a given *b*-supertopology $(\mathcal{M}, \mathcal{V})$ on a space X in the case $X\in\mathcal{M}$, i. e. in the case $\mathcal{M}=\mathcal{P}(X)$, form a special nearness structure on X, and the maximal systems of this kind are then the clusters. Therefore in the case $X\in\mathcal{M}$ Theorem 1 (which in this case is in fact the theorem of Smirnov) can be obtained also by means of the results from [8], provided it is shown (which is not difficult) that the introduced here (in section 2) topology on the space X^* coincides with the topology introduced in [8] on the same space considered as a set of clusters.

On the other hand, one can now see that a nearness structure on a space X is induced by a compactification of X if and only if it is induced by some (compatible with the topology on X) *b*-supertopology of the sort $(\mathcal{P}(X), \mathcal{V})$ in the following sense: a collection of subsets of X is near provided it has the finite $\mathcal{V}$-intersection property with respect to this *b*-supertopology.

Further, as follows from Theorem 3, a nearness structure on a space X is induced by a locally compact extension of X if and only if it is induced by some (compatible with the given topology) *lc*-supertoplogy $(\mathcal{M}, \mathcal{V})$ on X in the following sense: a collection of subsets of X is near provided it corefines a collection having the finite $\mathcal{V}$-intersection property with respect to $(\mathcal{M}, \mathcal{V})$. (A collection $\mathcal{A}$ corefines the collection $\mathcal{B}$ when for each $A\in\mathcal{A}$ there exists a $B\in\mathcal{B}$ with $B\subset A$.) It is to be noted that a direct proof of the fact that thereby in reality a nearness structure on X is defined seems to be difficult.

4. Čech-complete extensions. As a Čech-complete extension of a space X is, in a sense, an intersection of a countable collection of locally compact extensions of X, it is clear that such an extension can be determined by means of a countable sequence of lc-supertopologies on X. Different sequences of this sort, of course, can in general yield the same extension. But the case is relatively simple when this extension is compactly determined.

Here some results about Čech-complete extensions are announced whose proofs can be let to the reader.

First of all a sequence $\{(\mathcal{M}_n, \mathcal{V}_n) \mid n=1, 2, \ldots\}$ of lc-supertopologies on X will be called *Čech sequence* provided: a) $\mathcal{M}_{n+1} \subset \mathcal{M}_n$ for $n=1, 2, \ldots$; b) $\mathcal{V}_n(A) = \mathcal{V}_1(A)$ for $A \in \mathcal{M}_n$, $n=2, 3, \ldots$.

Any Čech sequence $\{(\mathcal{M}_n, \mathcal{V}_n) \mid n=1, 2, \ldots\}$ induces a Čech-complete extension of X in the following manner. Let (Y_n, α_n) be the locally compact extension of X corresponding to $(\mathcal{M}_n, \mathcal{V}_n)$. One can assume, up to homeomorphism, that $X \subset Y_{n+1} \subset Y_n (n=1, 2, \ldots)$. If now $Y = \bigcap_{n=1}^{\infty} Y_n$ and $\varphi: X \to Y$ is the identical embedding, then (Y, φ) is a Čech-complete extension of X.

P r o p o s i t i o n 8. *Let X be a Tychonoff space. An extension of X is Čech-complete if and only if it is induced by some Čech sequence of lc-supertopologies on X. If a Čech-complete extension (Y, φ) of X is compactly determined and if $(\mathcal{M}, \mathcal{V})$ is the b-supertopology canonically connected with it, then for any Čech sequence $\{(\mathcal{M}_n, \mathcal{V}_n) \mid n=1, 2, \ldots\}$ inducing (Y, φ) the following holds: $\bigcap_{n=1}^{\infty} \mathcal{M}_n = \mathcal{M}$ and $\mathcal{V}_1|_{\mathcal{M}} = \mathcal{V}$.*

The situation is simpler in the case when all Čech-complete extensions of a space X are compactly determined. That this case can actually happen is seen by the following

P r o p o s i t i o n 9. *If X is a pseudocompact space, then any of its Čech-complete extension is a compactly determined extension of X.*

This Proposition can be established by means of the following result (cf. [7, p. 270]): *if a dense subspace X of a compact space Y is pseudocompact, $Y \setminus X$ does not contain non-empty G_δ-subsets of Y.*

Next, a supertopology $(\mathcal{M}, \mathcal{V})$ on a space X is called *δ-lc-supertopology* provided there exists some Čech sequence $\{(\mathcal{M}_n, \mathcal{V}_n) \mid n=1, 2, \ldots\}$ of lc-supertopologies on X with $\bigcap_{n=1}^{\infty} \mathcal{M}_n = \mathcal{M}$ and $\mathcal{V}_1|_{\mathcal{M}} = \mathcal{V}$.

T h e o r e m 4. *Let X be a Tychonoff space such that all its Čech-complete extensions are compactly determined. There exists a one-to-one correspondence (which moreover is an order-isomorphism) between the class of all Čech-complete extensions of X (defined up to equivalence) and the class of all (compatible with the given topology) δ-lc-supertopologies on X. The δ-lc-supertopology corresponding to a given Čech-complete extension (Y, φ) of X coincides with the b-supertopology on X canonically connected with (Y, φ).*

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