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A SEPARATOR THEOREM FOR GRAPHS OF FIXED GENUS

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The separator theorems are connected with the effective use of the divide-and-conquer strategy for solving problems defined on graphs. In the paper a separator theorem is proved showing that every n-vertex graph of genus g can be partititioned into two or more components of roughly equal size by deleting only $O(\sqrt{gn})$ vertices.

1. Introduction. A useful approach for solving a variety of combinatorial problems is the method "divide-and-conquer" [6]. To solve a problem by this method, we decompose the problem into two or more smaller and relatively independent subproblems, solve them recursively by the same method and use the subproblem solutions to build a solution of the original problem. The efficiency of this method strongly depends on the way the original problem is decomposed into subproblems.

Consider now problems on graphs. Let S be a class of graphs closed under the subgraph relation (i. e. if $G_1 \in S$ and G_2 is a subgraph of G_1 then $G_2 \in S$). An $f(n)$-separator theorem for S was defined in [2] as a theorem of the following form:

Theorem A. *There exist constants $\alpha<1$ and $\beta>0$ such that, if G is any n-vertex graph in S, then the vertices of G can be partititioned into three sets A, B, C, so that no edge joins a vertex in A with a vertex in B, $|A|\leq\alpha n$, $|B|\leq\alpha n$, $|C|\leq\beta f(n)$.*

If G is the graph in S on which the problem is defined, then the subgraphs induced by the sets of vertices A and B define subproblems, which are relatively independent of each other. The cost of combining the solutions to the subproblems into a solution to the original problem is a function of the size of C (and thus of $f(n)$).

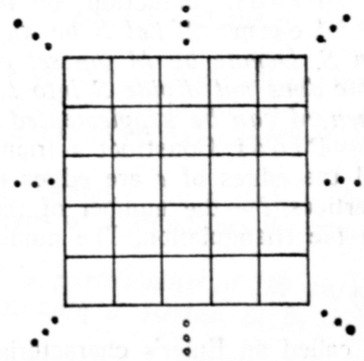

Fig. 1. Infinite two-dimensional square grid

Previously known separator theorems include the following:

(A) Q grid graph is any subgraph of the infinite two-dimensional square grid illustrated in Fig. 1. A $\sqrt{n}$-separator theorem holds for the class of grid graphs.

(B) A one-tape Turing machine graph is a graph representing the computations of a one-tape Turing machine. A $\sqrt{n}$-separator theorem holds for such graphs.

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(C) A planar graph is a graph which can be imbedded in the plane, so that no two edges intersect except at a common endpoint. A $\sqrt{n}$-separator theorem was proved for the class of all planar graphs in [2]. Some improvements were made in [3,4].

It might seem that sparsity is the most important condition for the existing of $o(n)$-separator theorems. However, the following result of Erdös, Graham and Szemeredi [7] rejects this hypothesis.

Theorem 1. *For every rational* $\varepsilon>0$ *there is a positive constant* $c=c(\varepsilon)$ *such that almost all* graphs with* $n=[(2+\varepsilon)k]$ *vertices and ck edges have the property that after the omission of any k vertices, a connected component of at least k vertices remains.*

Hence sparsity per self is not sufficient to obtain useful separator theorems, then we must pay attention to the structural features of the graphs.

The genus of a graph G is the minimum genus of a two-dimensional orientable surface on which G can be imbedded, so that no two edges intersect except at a common endpoint. Since the genus of the sphere is O, the class of all planar graphs is identical to the class of graphs of genus O. As mentioned above, a $\sqrt{n}$-separator theorem holds for the class of all planar graphs. We shall show now that for every positive integer g a $\sqrt{n}$-separator theorem holds for the class of graphs of genus not exceeding g.

2. Proof of the Separator Theorem.

Lemma 1 [1]. *Let G be any graph of genus g. Shrinking any edge in G to a single vertex does not increase the genus g of the graph.*

Corollary 1. Let G be any graph of genus g. Shrinking any connected subgraph of G to a single vertex does not increase the genus g of the graph.

Proof. Induction on the number of the vertices in the subgraph.

Lemma 2. *Let S be any surface of genus* $g>0$ *and c be a closed curve on S. Denote by M the set of the points on S, which do not belong to c. If c does not divide S into two connected regions (i. e. if M is connected), then M can be supplemented to an orientable surface of genus* $g-1$.

Proof. Construct a triangulation of S which contains c (i. e. such that all the edges of c are edges of the triangulation). Let n be the number of the vertices, l — the number of the edges, and m — the number of the triangles of the triangulation. The number

$$E(S)=n-l+m \tag{1}$$

is called an Euler's characteristics of S and

$$E(S)=2-2g. \tag{2}$$

Apply some continuous transformation φ to M, such that the contour of the image of M consists of two closed curves with an empty intersection, corresponding to c in M (Fig. 2). The triangulation of M is transformed by φ to a triangulation of the image $\bar{M}_1$ of M. Wo shall prove the Lemma by complementing $\bar{M}_1$ to a orientable surface of genus $g-1$.

* By "almost all" we mean that the fraction of the graphs possessing the property tends with increasing n to one.

Let u_1 and u_2 be 2 points outside $\bar{M}_1$. For each edge (v, w) from the contour of $\bar{M}_1$ construct a triangle (v, w, u_1), if (v, w) belongs to c_1, or a triangle (v, w, u_2), if (v, w) belongs to c_2. In addition the triangles must be so constructed, that, when added to $\bar{M}_1$, to form an orientable triangulated surface S' (Fig. 3).

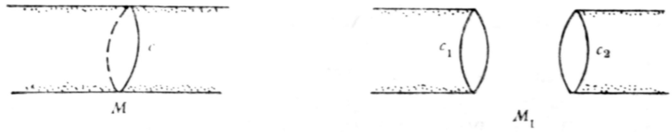

Fig. 2. The transformation of M

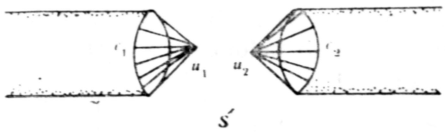

Fig. 3. The surface S'

Let us estimate the Euler's characteristics of S'.

Denote by k the length of c. Since c_1 and c_2 have the same lengths as c then the number of the vertices and the number of the edges of each of the cycles c_1 and c_2 will be k. Then the number of the vertices, edges and triangles in the triangulation of S' will be $n+k+2$, $l+3k$ and $m+2k$, respectively (see Fig. 3). Thus by (1) and (2) it follows $E(S')=E(S)+2$ and the genus of S' is $g-1$.

Now let G be a graph of genus g and c be a cycle in G. We shall say that c divides G if there exists an imbedding of G on some orientable surface of genus g such that the image of c divides S. Then the next statement follows from Lemma 2.

Corollary 2. Let G be any graph of genus $g>0$ and c be a cycle which does not divide G. Then the removal of the vertices of c diminishes the genus g of the graph with at least one.

Definition. *If G is any graph and A, B, C is a partitioning of the vertices of G such that no edge joins a vertex in A with a vertex in B, then A, B, C is a regular partitioning.*

Lemma 3. *Let G be any n-vertex graph of genus g. Suppose that G has a breadth-first spanning tree with a root the vertex t and radius r. Then there exists a regular partitioning A, B, C, of the vertices of G, such that neither A nor B contains more than $2n/3$ vertices and C contains no more than $(4g+2)r+1$ vertices, one of them the root of the tree.*

Proof. Make an induction on g. If $g=0$, then the lemma is true (Lemma 2 in [2]). Suppose that the lemma is true for all graphs of genus not exceeding $g-1$. Imbed G on some surface S of genus g. Add additional edges to make all faces triangles. Each non-tree edge forms a simple cycle with some of the tree edges. We shall call such cycles basic cycles. Every basic cycle has a length at most $2r+1$ if it contains the root of the tree, and at most $2r-1$ otherwise.

If each basic cycle divides S into two connected regions then the lemma can be proved in the same way as Lemma 2 in [2] (Jordan's Curve Theorem [8] is used in that proof).

If a cycle c exists which does not divide S into two connected regions, then by Corollary 2 the removal of the vertices of c reduces the genus of G. Divide the set of all triangles incident to c into two classes L and R, so that the triangles in each of the classes lie at one side regarding c. If some triangle has vertices on both sides of c then the triangle must be in both classes L and R.

Similarly, divide the set of the vertices which belong to triangles in L or R, excepting the vertices on c, into classes L' and R'.

The classes L' and R' will be used below to form a new graph G' from G:

Let $c=(v_1, v_2, \dots, v_k)$. Replace c by two cycles $c^l=(v_1^l, v_2^l, \dots, v_k^l)$ and $c^r=(v_1^r, v_2^r, \dots, v_k^r)$. Each edge (v_i, w) in G, $1\leq i\leq k$, replace by an edge (v_i^l, w), if w belongs to L', or (v_i^r, w), if w belongs to R' (Figure 4). By Corollary 2 the genus of G' does not exceed $g-1$.

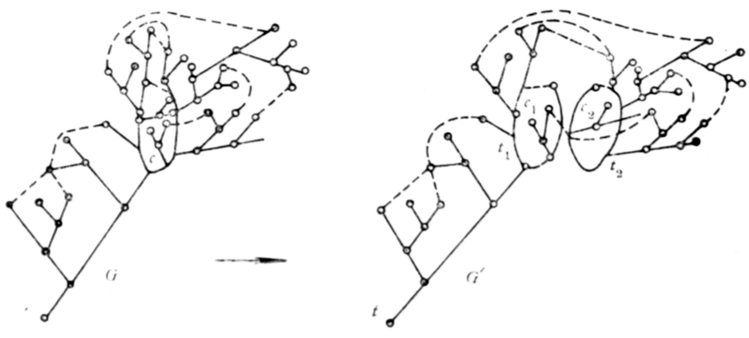

Fig. 4. The graph G'

Suppose that there exists a regular cycle $\bar{c}$ in G, which intersects c. That means that the path c' in G', which corresponds to $\bar{c}$, connects a vertex in c_1 with a vertex in c_2. Denote by M the set of the vertices on c_1, c_2 and c'. Shrink the graph induced by M to a single vertex v'' and let $G''=(V'', E'')$ be the resulting graph. By Corollary 1 the genus of G'' does not exceed the genus of G' and hence it does not exceed $g-1$. Moreover the maximum distance between t and vertices in G'' does not exceed r. Then it follows from the inductive conjecture that there exists a regular partitioning A'', B'', C'' of the vertices of G'', such that $|A''|\leq 2n/3$, $|B''|\leq 2n/3$, $|C''|\leq (4g-2)r+1$. Therefore the partitioning of the vertices of G $A=A''$, $B=B''$ and $C=C''\{V''\}\cup M$ satisfies the lemma.

Suppose now that no regular cycle intersects c. We shall show that this is not possible. Let t_1 and t_2 be any vertices from c_1 and c_2. Define maximum trees T_1 and T_2 with roots t_1 and t_2 and edges corresponding to the edges in T (see Fig. 4). Since T does not contain cycles (as a tree), then T_1 and T_2 do not have a common vertex. Besides there is no edge $(v_1, v_2)\in E'$ such that v_1

is in T_1 and v_2 is in T_2, because to any such edge a regular cycle corresponds in G, intersecting c, which supposedly is not possible.

Then no edge in G' joins a vertex in T_1 with a vertex in T_2, which means that c divides G. That contradicts once again to the conjecture that no regular cycle divides G. This completes the proof.

Lemma 4. *Let G be any n-vertex connected graph of genus g. Suppose that the vertices of G are partitioned into levels according to their distance from some vetex v, and that $L(l)$ denotes the number of vertices on level l. Given any two levels $l_1 \leqq l_2$ such that the number of vertices on levels 0 through l_1-1 does not exceed $2n/3$ and the number of vertices on levels l_2+1 and above does not exceed $2n/3$, it is possible to find a regular partitioning A, B, C of the vertices of G such that $|A| \leqq 2n/3$, $|B| \leqq 2n/3$, $|C| \leqq L(l_1)+L(l_2)+(4g+2)(l_2-l_1-1)$.*

The proof of this lemma is the same as the proof of Lemma 3 in [2].

Theorem 2. *Let G be any n-vertex graph of genus g. There exists a regular partitioning A, B, C of the vertices of G such that $|A| \leqq 2n/3$, $|B| \leqq 2n/3$ and $|C| \leqq \sqrt{2g+1}\sqrt{6n}$.*

Proof. Assume G is connected. Divide the vertices into levels according to their distance from some vertex v. Let $L(l)$ denote the number of vertices on level l. If the maximum distance between v and the other vertices of G is r, then define additional levels -1 and $r+1$ containing no vertices.

For each $\alpha \in (0, 1)$ let l_α denote a level such that

$$\sum_{l=0}^{l_\alpha-1} L(l) < \alpha n, \qquad \sum_{l=0}^{l_\alpha} L(l) \geqq \alpha n.$$

Case 1. There exists a level l such that $l_{1/3} \leqq l \leqq l_{2/3}$ and $L(l) \leqq \sqrt{2g+1}\sqrt{6n}$. Let A be the set of vertices on levels 0 through $l-1$, let B be the set of vertices on levels $l+1$ through r, and let C be the set of vertices on level l. Then the theorem is true.

Case 2. For each $l \in [l_{1/3}, l_{2/3}]$ $L(l) > \sqrt{2g+1}\sqrt{6n}$. Let $\alpha = (\Sigma_{l=l_{1/3}}^{l_{2/3}} L(l))/n$. Since

$$\sum_{l=l_{1/3}}^{l_{2/3}} L(l) = \sum_{l=0}^{l_{2/3}} L(l) - \sum_{l=0}^{l_{1/3}-1} L(l) > 2/3 \,.\, n - 1/3 \,.\, n = 1/3 \,.\, n,$$

then $\alpha > 1/3$. Furthermore

$$\alpha n = \sum_{l=l_{1/3}}^{l_{2/3}} L(l) > \sum_{l=l_{1/3}}^{l_{2/3}} \sqrt{2g+1}\sqrt{6n} = (l_{2/3} - l_{1/3} + 1)\sqrt{2g+1}\sqrt{6n}.$$

Thus

$$l_{2/3} - l_{1/3} + 1 < \alpha / (\sqrt{2g+1} \,.\, \sqrt{6}) \,.\, \sqrt{n}. \tag{3}$$

Let j be a non-negative integer such that

$$\sum_{l=l_{1/3}-j+1}^{l_{2/3}+j-1} L(l) < 2/3 \,.\, n, \qquad \sum_{l=l_{1/3}-j}^{l_{2/3}+j} L(l) \geqq 2/3 \,.\, n.$$

Subcase 2.1. There exists i such that $0 \leqq i \leqq j$ and $L(l_{1/3}-i) + L(l_{2/3}+i) \leqq \sqrt{2g+1}\sqrt{6n}$. Then let C be the set of the vertices on levels $l_{1/3}-i$ and $l_{2/3}+i$, let A be the set of the vertices on levels $l_{1/3}-i+1$ through $l_{2/3}+i-1$ and let B be the set of the remaining vertices. Then the theorem is true.

Subcase 2.2. For each i, $1\leqq i\leqq j$, holds

$$L(l_{1/3}-i)+L(l_{2/3}+i)>\sqrt{2g+1}\sqrt{6n}.$$

Let $\beta=(\sum_{l=l_{1/3}-j}^{l_{2/3}+j} L(l))/n$. Then $\beta\geqq 2/3$. Furthermore

$$\beta n=\sum_{l=l_{1/3}-j}^{l_{2/3}+j} L(l)=\sum_{l=l_{1/3}-j}^{l_{1/3}-1} L(l)+\sum_{l=l_{1/3}}^{l_{2/3}} L(l)+\sum_{l=l_{2/3}+1}^{l_{2/3}+j} L(l)$$

$$=\alpha n+\sum_{i=1}^{j}(L(l_{1/3}-i)+L(l_{2/3}+i))>\alpha n+j\sqrt{2g+1}\sqrt{6n}.$$

Thus $(\beta-\alpha)n>j\sqrt{2g+1}\sqrt{6n}$ and

$$j<(\beta-\alpha)/(\sqrt{2g+1}\sqrt{6})\cdot\sqrt{n} \tag{4}$$

Let $m=\Sigma_{l=0}^{l_{1/3}-j-1} L(l)$. Then $\Sigma_{l=l_{2/3}+j+1}^{r} L(l)=n-m-\beta n$.

Find levels l' and l'' such that

$$l'\leqq l_{1/3}-j-1<l_{2/3}+j+1\leqq l'',$$

$$L(l')+(4g+2)(l_{1/3}-j-1-l')\leqq 2\sqrt{2g+1}\sqrt{m}, \tag{5}$$

$$L(l'')+(4g+2)(l''-(l_{2/3}+j+1))\leqq 2\sqrt{2g+1}\sqrt{n-m-\beta n}. \tag{6}$$

Assume that a suitable level l' does not exist. Then for each l, $0\leqq l\leqq l_{1/3}-j-1$ $L(l)>2\sqrt{2g+1}\sqrt{m}-(4g+2)(l_{1/3}-j-1-l)$. Since $L(0)=1$, this means $1>2\sqrt{2g+1}\sqrt{m}-(4g+2)(l_{1/3}-j-1)$. Thus $l_{1/3}-j-1=\lfloor l_{1/3}-j-1+1/(4g+2)\rfloor\geqq\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor$*, and

$$m=\sum_{l=0}^{l_{1/3}-j-1} L(l)\geqq\sum_{l=l_{1/3}-j-1-\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor}^{l_{1/3}-j-1}[2\sqrt{2g+1}\sqrt{m}-(4g+2)(l_{1/3}-j-1-l)]$$

$$=\sum_{l=0}^{\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor}[2\sqrt{2g+1}\sqrt{m}-(4g+2)l]=\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor 2\sqrt{2g+1}\sqrt{m}$$

$$-(2g+1)\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor(\sqrt{m}/\sqrt{2g+1}\rfloor+1)\geqq\sqrt{m}\sqrt{2g+1}(2\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor$$

$$-\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor+1)=\sqrt{m}\sqrt{2g+1}(\lfloor\sqrt{m}/\sqrt{2g+1}\rfloor+1)>\sqrt{m}\sqrt{2g+1}\sqrt{m}/\sqrt{2g+1}=m.$$

This is a contradiction. A similar contradiction arises after an assumption that a suitable level l'' does not exist.

Add together the inequalities [5] and [6].

$$L(l')+L(l'')+2(l''-l'-1-(l_{2/3}-l_{1/3}+2j+1))\leqq 2\sqrt{2g+1}(\sqrt{m}+\sqrt{n-m-\beta n}). \tag{7}$$

Multiply the inequality (3) by $4g+2$, the inequality (4) by $8g+4$ and add them together. The result is

(8) $$(4g+2)(l_{2/3}-l_{1/3}+2j+1)<(4g+2)[(\alpha/(\sqrt{6}\sqrt{2g+1}) +2(\beta-\alpha)/(\sqrt{2g+1}\sqrt{6})]\sqrt{n}=\frac{\sqrt{2g+1}}{\sqrt{6}}(4\beta-2\alpha)\sqrt{n}.$$

Finally add together (7) and (8).

(9) $$L(l')+L(l'')+(4g+2)(l''-l'-1)<\sqrt{2g+1}\,[2(\sqrt{m}+\sqrt{n-m-\beta n})+\frac{4\beta-2\alpha}{\sqrt{6}}\sqrt{n}].$$

In [4] is proved that

$$2(\sqrt{m}+\sqrt{n-m-\beta n})+\frac{4\beta-2\alpha}{\sqrt{6}}\sqrt{n}\leqq\sqrt{6n}.$$

Thus $L(l')+L(l'')+(4g+2)(l''-l'-1)\leqq\sqrt{2g+1}\sqrt{6n}$ and by Lemma 4 the theorem is true in this subcase. This completes the proof for connected graphs.

The proof in the case when G is not connected is the same as in Theorem 4 in [2].

3. **Sharpness of the separator theorem.** Here we shall prove that Theorem 2 is tight within a constant factor, i. e. that no $o(\sqrt{gn})$ for $n\to\infty, g\to\infty$ separator theorem** exists for the class of graphs of genus g with constants independent of g.

Lemma 5. *Let S be a class of graphs, closed under the subgraph relation. If an $f(n)$-separator theorem holds for S with $\alpha=\alpha_0\in(1/2, 1)$ and $\beta=\beta_0>0$ (see Theorem A), then an $f(n)$-separator theorem holds for S with $\alpha=1/2$ and $\beta=\beta_0/(1-\sqrt{\alpha_0})$.*

The proof of Lemma 5 is the same as the proof of Corollary 3 in [2].

Let the length of a line L and the area of a two-dimensional region S be $l(L)$ and $s(S)$, respectively. Then the following statement can be proved easily·

Lemma 6. *Let t be a regular polygon with area 1and let c be a continuous rectifiable curve on t, dividing it into two regions A and B. Then*

1) *there exists a constants $\varepsilon_0>0$ which does not depend on t and c such that if $l(c)<\varepsilon_0$ then $\max\{s(A),s(B)\}\geqq\frac{2}{3}(s(A)+s(B))$,*

2) *there exits a continuous rectifiable curve c' on the contour K_t of t* *$c'\supset c\cap K_t$ satisfying $l(c')\leqq 2l(c)$.*

Lemma 6 can be used in the following way. Suppose that we have a surface S which is covered by regular polygons and that a curve c is build on S which divides it into two equal parts. Then c can be substituted by a curve lying on the contours of the polygons, so that the new line has a length at most 2 times greater than the length of the original curve and the new curve divides S into two parts of roughly equal sizes.

Now let S be a surface, μ be a map upon S and k be an integer. For each face t in μ add a new vertex in t and edges, connecting the new vertex with all other vertices of the face (Fig. 5(a)).

The result of this operation is a map μ', each face of which is a triangle.

* By $\lfloor x \rfloor$ we denote the greatest integer not exceeding x.

** $f(x, y)=o(h(x, y))$ for $y\to\infty$, $x\to\infty$ means that for each $\varepsilon>0$ there exists x_0, such that if $x>x_0$ the inequality $f(x, y)\leqq\varepsilon h(x, y)$ holds for all sufficiently great y.

Divide each edge of μ' into k parts. Add new edges in each of the triangles to connect the points of division, as shown in Fig. 5(b). Thus for every polygon t in μ a graph $G(k, t)$ is constructed, and to the whole map corresponds a graph, which will be denoted by $G(k, \mu)$. Note that $G(k, t)$ de-

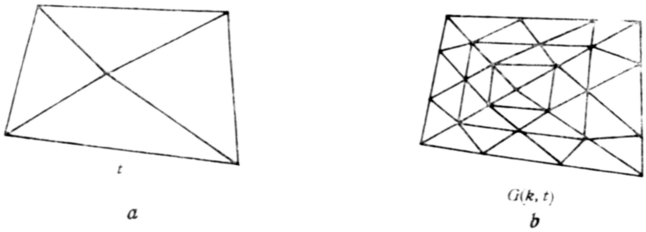

Fig. 5. Adding new vertices and edges in t -or $k=32$

pends only on the number of the vertices of t. Besides the genus of $G(k, \mu)$ is not greater than the genus of S.

The following analogue of Lemma 6 can be proved using the lemma and a similar technique as in [4].

Corollary 3. There exist constants $\varepsilon_1 \in (0, 1)$ and $k_0 > 0$ such that if t is a polygon and $k \geq k_0$ is an integer, A, B, C is a regular partitioning of the vertices of $G(k, t)$, $|C| \leq \varepsilon_1 k$, and K is the set of the contour vertices of $G(k, t)$, then

1) $\max\{|A \setminus K|, |B \setminus K|\} \geq 2/3(|A|+|B|+|C|)$,

2) a set $C' \subset K$ exists such that $C' \supset C \cap K$ and every 2 points of K connected by a path in C are connected by a path in C', and $|C' \setminus \bar{K}| \leq \varepsilon_1^{-1}|C \setminus \bar{K}|$, where $\bar{K} = C \cap K$.

L e m m a 7. *Let S be a surface, μ be a map upon S and k be an integer satisfying the condition $k \geq k_0$, where k_0 is the number from Corollary 3. Let j be the maximum number of vertices of any polygon of μ and let n be the number of vertices of the graph $G = G(k, \mu)$. If A, B, C is a regular partitioning of the vertices of G and $|A| \leq n/2$, $|B| \leq n/2$, then there exists a regular partitioning of the vertices of G into three sets A', B', C', such that $|A'| \leq 3n/4$, $|B'| \leq 3n/4$, $|C'| \leq j \cdot \varepsilon_1^{-1}|C|$ and all vertices of C' are upon the edges of μ.*

Proof. Let t be a polygon in μ and $G_t = G(k, t) = (V_t, E_t)$ be the graph corresponding to t. Let K_t be the set of the contour vertices of G_t (i. e. the vertices on the contour of t), $C_t = C \cap V_t$, $\bar{K}_t = C_t \cap K_t$. Suppose that

$$|C_t| \leq \varepsilon_1 k. \tag{10}$$

After Corollary 3 there exists a set $C'_t \subset K_t$, such that $C'_t \supset \bar{K}_t$ and every two vertices in $\bar{K}_t$, connected by a path in C'_t, are connected by a path in C'_t, and $|C'_t \setminus \bar{K}_t| \leq \varepsilon_1^{-1}|C_t \setminus \bar{K}_t|$.

Denote the sets into which C_t divides V_t by A_t and B_t, $A_t \subset A$, $B_t \subset B$ (it is possible that $A_t = \emptyset$ or $B_t = \emptyset$). Without loss of generality suppose that $|A_t \setminus K_t| \geq |B_t \setminus K_t|$. Then by Corollary 3

$$|A_t \setminus K_t| \geq 2/3\,(|A_t|+|B_t|+|C_t|),$$

and hence $|V_t| \leq 3/2\,|A_t \setminus K_t|$.

Denote $|A'_t| = V_t \setminus C'_t$, $B'_t = \emptyset$. Then $|A'_t| \leq 3/2\,|A_t \setminus K_t|$, $0 = |B'_t| \leq 3/2\,|B_t \setminus K_t|$.

If the inequality (10) holds for all polygons t in μ then let $A' = \cup_{t \in \mu} A'_t$, $B' = \cup_{t \in \mu} B'_t$ and $C' = \cup_{t \in \mu} C'_t$. Then $|A'| = \Sigma_{t \in \mu} |A'_t| \leq 3/2\, \Sigma_{t \in \mu} |A_t \setminus K_t| \leq 3/2 |A| \leq 3/2\, n/2 = 3n/4$. By analogy $|B'| \leq 3/4\,n$ and $|C'| \leq \varepsilon_1^{-1} |C|$.

Suppose now that there exists exactly one polygon τ in μ, which does not satisfy the inequality (10), i. e. such that $|C_\tau| > \varepsilon_1 k$. Then let $C'_\tau = K_\tau$, $D = V_\tau \setminus K_\tau$, $A^* = \bigcup_{\substack{t \in \mu \\ t \neq \tau}} A'_t$, $B^* = \bigcup_{\substack{t \in \mu \\ t \neq \tau}} B'_t$, $C^* = \bigcup_{\substack{t \in \mu \\ t \neq \tau}} C'_t$.

As in the previous case $\max \langle |A^*|, |B^*| \rangle \leq 3n/4$. Without loss of generality suppose that $|A^*| \leq |B^*| \leq 3n/4$. Since the number of the vertices of t can not exceed the sum of the number of the vertices of the other polygons in μ, then $|D| \leq n/2$. Furthermore

$$|A^*|+|D| = n - |B^*| - |C^*| \leq n - (|A^*|+|B^*|+|C^*|)/2 = n - (n - |D|)/2 \leq 3n/4,$$

$$|C'_\tau| = |K_\tau| \leq jk < j \,.\, \varepsilon_1^{-1} |C_\tau|.$$

Then the sets $A' = A^* \cup D$, $B' = B^*$, $C' = C^* \cup C'_t$ satisfy the requirements of the theorem.

In the general case the correctness of the theorem is proved by an induction on the number of the polygons which do not satisfy (10).

Now we shall make use of Theorem 1 mentioned in the Introduction. Since the number of the vertices with degree exceeding $4c$ is not greater than $k/2$, we can give the following (more convenient for our purpose) formulation of Theorem 1. Let ε be an arbitrary positive number and $c = c(\varepsilon)$ be the corresponding constant from the theorem.

Corollary 4. There exists a sequence of graphs $\{\widehat{G}_i\}_{i=1}^{\infty}$ satisfying

1) $\lim_{i \to \infty} \widehat{g}_i = \infty$, $\widehat{g}_i = O(\widehat{n}_i)$, where $\widehat{g}_i$ and $\widehat{n}_i$ are the genus and the number of the vertices of $\widehat{G}_i$.

2) The degree of each vertex in a graph of the sequence is a number in the interval $[3, 4c]$.

3) If $\widehat{A}_i$, $\widehat{B}_i$, $\widehat{C}_i$ is a regular partitioning of the vertices of $\widehat{G}_i$ such that $|\widehat{A}_i| = \Omega(\widehat{n}_i)$ and $|\widehat{B}_i| = \Omega(\widehat{n}_i)$, then $|\widehat{C}_i| = \Omega(\widehat{n}_i)^*$.

Let G_i^* be the dual graph of $\widehat{G}_i$ regarding tbe regular imbedding of $\widehat{G}_i$ on, some surface S_i of genus $\widehat{g}_i$. By the definition of duality the vertices of G_i^* correspond to the faces of the imbedding of $\widehat{G}_i$ and two vertices in G_i^* are joined by an edge if and only if the corresponding faces share a common edge. From the imbedding of $\widehat{G}_i$ an imbedding μ_i of G_i^* on the same surface S_i can easily be obtained. With regard to this imbedding the vertices in $\widehat{G}_i$ correspond to the faces of μ_i and two vertices in $\widehat{G}_i$ are adjacent if and only if the corresponding faces share a common edge.

The next statement is dual to Corollary 4.

Corollary 5. The sequence $\langle G_i^* \rangle_{i=1}^{\infty}$ has the following properties.

1) $\lim_{i\to\infty} g_i^* = \infty$, $g_i^* = O(m_i^*)$, where g_i^* and m_i^* are the genus of G_i^* and the number of the vertices of μ_i.

2) Each face in μ_i is a polygon with no more than $4c$ vertices.

3) If C_i^* is a set of edges dividing S_i into 2 regions A_i^* and B_i^* each containing $\Omega(m_i^*)$ faces, then $|C_i^*| = \Omega(m_i^*)$.

Theorem 3. *No $f_g(n)$-separator theorem exists for the class of the graphs of genus g for $f_g(n) = o(\sqrt{gn})$ for $n\to\infty$ $g\to\infty$ and constants independent of g.*

Proof. Consider the sequence of graphs $\{G_i^*\}_{i=1}^\infty$ and the sequence of maps $\{\mu_i\}_{i=1}^\infty$, defined above. Let $k_1, k_2, \ldots, k_n, \ldots$ be a sequence of integers greater than k_0 such that $\lim_{i\to\infty} k_i = \infty$. Define the sequence of graphs $\{G_i\}_{i=1}^\infty$, such that $G_i = G(k_i, \mu_i)$. Denote by n_i the number of the vertices and by g_i the genus of G_i.

Assume that there exists a regular partitioning A_i, B_i, C_i of the vertices of G_i, such that $|A_i| \le 3n_i/4$, $|B_i| \le 3n_i/4$, $|C_i| = o(\sqrt{g_i n_i})$, where n_i denotes the number of the vertices of G_i. According to Lemma 5 and Lemma 7 suppose (without loss of generality) that all vertices in C_i lie on edges of μ_i.

Any face in μ_i contains between $k_i^2/2 + O(k_i)$ and $4ck_i^2/2 + O(k_i)$ vertices of G_i. Then $n_i = O(m_i^* k_i^2)$, where m_i^* is the number of the faces in μ_i. Since each edge in G_i^* contains $k_i + 1$ vertices of G_i, the vertices of G_i can cover entirely at most $o(\sqrt{g_i n_i})/(k_i+1) = o(\sqrt{g_i m_i^*}) = o(m_i^*)$ edges of G_i^* ($g_i = g_i^* = 0(m_i^*)$ according to Corollary 5).

Denote the set of those edges by $\tilde{C}_i$. The curve upon S_i that corresponds to $\tilde{C}_i$, divides the surface into two regions $\tilde{A}_i$ and $\tilde{B}_i$, containing the vertices of A_i and B_i, respectively. Since $g_i = O(m_i^*) = o(n_i)$, then $|C_i| = o(\sqrt{g_i n_i}) = o(n_i)$ and thus $|A_i| = \Omega(n_i)$ and $|B_i| = \Omega(n_i)$. As $\Omega(n_i) = \Omega(m_i^* k_i^2)$, the number of the faces, that each of the regions $\tilde{A}_i$ and $\tilde{B}_i$ contains, is $\Omega(m_i^*)$. That contradicts Corollary 5. The contradiction shows that it is not possible to find a regular partitioning A_i, B_i, C_i of the vertices of G_i, such that $|A_i| \le 3n_i/4$, $|B_i| \le 3n_i/4$ and $|C_i| = o(\sqrt{g_i n_i})$.

Then the theorem is true, since the sequence $\{G_i\}_{i=1}^\infty$ has the following properties:

1) $\lim_{i\to\infty} g_i = \infty$.

2) the degree of n_i/g_i is k_i^2 and the sequence $\{k_i\}_{i=1}^\infty$ can be chosen arbitrarily.

By analogy of [2, 3, 4] we can prove now a variety of new versions and generalizations of Theorem 2. These theorems will not be included in this paper for lack of space. We shall offer the following result only, which follows directly from Theorem 1 and Theorem 2.

Theorem 4. *There exist constants $c_1 < c_2 < c_3$, such that the genus of almost all graphs with k vertices and $c_3 k$ edges is an integer in the interval $(c_1 k, c_2 k)$.*

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* $f(x) = \Omega(h(x))$ means that there exists a positive constant k such that $f(x) \ge kh(x)$ holds for all sufficiently great x.

It is very important to have an algorithm which for any given graph G can find quickly the sets A, B, C from Theorem 2. In a subsequent paper will be described an algorithm, which finds an appropriate partitioning in $O(n)$ time

For some of the applications of Theorem 2 see [5].

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