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AN EXACT ESTIMATE OF THE APPROXIMATION OF THE FUNCTION X^α WITH BERNSTEIN POLYNOMIALS IN HAUSDORFF METRIC

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In the paper an estimate is obtained for the approximation of the function

$$f(x)=x^\alpha,\ 0\leq x\leq 1,\ 0<\alpha<1$$

with Bernstein polynomials

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}$$

in the Hausdorff metric.

It is proved that the estimate is exact to the order $n^{-1}\ln^{1-\alpha} n$.

In the paper an estimate is obtained for the approximation of the function $f(x)=x^\alpha$, $0\leq x\leq 1$, $0<\alpha<1$ with Bernstein polynomials in the Hausdorff metric. It is proved that the estimate is exact to the order $n^{-1}.\ln^{1-\alpha} n$.

We shall use the notation

$$B_n(f;x)=B_n(x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{kn}(x),$$

where $p_{kn}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ — the Bernstein polynomial for f; according to [3]

$$r(\Delta;f,g)—\max\{\max_{A\in f}\min_{B\in g}\rho(A,B),\ \max_{A\in g}\min_{B\in f}\rho(A,B)\},$$

where $\rho(A,B)=\rho(A(x_1,y_1),\ B(x_2,y_2))=\max\{|x_1-x_2|,\ |y_1-y_2|\}$, defines the Hausdorff distance between the functions $f,\ g\in C_\Delta$.

It is known [4] that for the function $f(x)=x^\alpha$, $0\leq x\leq 1$, $0<\alpha<1$ one has

(1) $$\max\{|f(x)—B_n(f;x)|,\ x\in[0,1]\}=0((1-\alpha)n^{-\alpha}).$$

We prove

Theorem. *If $n\geq \exp(2\alpha^{\frac{1}{\alpha-1}})$ for the function $f(x)=x^\alpha$, $0\leq x\leq 1$, $0<\alpha<1$ one gets*

$$r([0,1];\ B_n(f),f)=0((1-\alpha)n^{-1}.\ln^{1-\alpha} n).$$

Proof. After some elementary transformations we obtain

(2) $$B_{k-1}(x)-B_k(x)=\sum_{\nu=1}^{k-1}\left\{\frac{k-\nu}{k}f\left(\frac{\nu}{k-1}\right)-f\left(\frac{\nu}{k}\right)+\frac{\nu}{k}f\left(\frac{\nu-1}{k-1}\right)\right\}p_{\nu k}(x).$$

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It follows from (2) that for the function $f(x)=x^\alpha$ one has

$$B_{k-1}(x)-B_k(x)=-\sum_{\nu=1}^{k-1}\frac{\nu^\alpha}{k(k-1)^\alpha}\{\varphi(\frac{1}{\nu})-\varphi(\frac{1}{k})\}p_{\nu k}(x), \tag{3}$$

where $\varphi(x)=[1-(1-x)^\alpha]\,.\,x^{-1}$.

1. For n^{-1} in $n\leqq x\leqq 1$ we shall prove that if

$$\left|\begin{array}{l} k\geqq n\geqq \exp\{2\alpha^{\frac{1}{\alpha-1}}\};\\ \frac{kx}{2}\leqq \nu\leqq k,\end{array}\right. \tag{4}$$

then

$$\nu^\alpha\{\varphi(\frac{1}{\nu})-\varphi(\frac{1}{k})\}\leqq\alpha\{\frac{\ln k}{kx}\}^{1-\alpha}(1-\varphi(\frac{1}{k})). \tag{5}$$

For this purpose we use the inequality

$$(2-\alpha)\,\vartheta^{1-\alpha}-(1-\alpha)\,\vartheta^{2-\alpha}\leqq 1,\ \ \vartheta\in[0,1]. \tag{6}$$

It is obviously true as the function on the left hand side is increasing on $[0, 1]$ and reaches its maximal value for $\vartheta=1$. We set $\vartheta=1-u$, $0<u\leqq 1$ and from (6) we get

$$1-(1-u)^\alpha\leqq\alpha u+(1-\alpha)\,u^2. \tag{7}$$

If

$$1\leqq\alpha\,[\frac{\ln k}{kx}]^{1-\alpha}u^{\alpha-1}, \tag{8}$$

then (7) gives

$\varphi(u)\leqq\alpha+\alpha(1-\alpha)[\frac{\ln k}{kx}]^{1-\alpha}u^\alpha$, which is equivalent to

$$(1-\alpha)[1-\alpha(\frac{\ln k}{kx})^{1-\alpha}u^\alpha]\leqq 1-\varphi(u). \tag{9}$$

According to the definition φ is increasing on $[0, 1]$. Therefore the function $1-\varphi(t)$ is decreasing on $[0, 1]$ and for $x\in[0, 1]$ one has

$$1-\varphi(x)\leqq 1-\alpha. \tag{10}$$

From (9) and (10) we obtain $[1-\varphi(t)]\,[1-\alpha(\frac{\ln k}{kx})^{1-\alpha}u^\alpha]\leqq 1-\varphi(u)$ or

$$\varphi(u)-\varphi(t)\leqq\alpha\,(\frac{\ln k}{kx})^{1-\alpha}u^\alpha(1-\varphi(t)). \tag{11}$$

It is easy to see that (11) yields (5) with the substitution $t=1/k$, $u=1/\nu$, if the condition (8) is satisfied. But this proves (5) as the inequality (8) is true under the restrictions (4).

Using similar considerations we prove [4] that for $1\leqq\nu\leqq k-1$ one gets

$$\nu^\alpha\{\varphi(\frac{1}{\nu})-\varphi(\frac{1}{k})\}\leqq 1-\alpha. \tag{12}$$

Further we express (3) as follows

$$B_k(x)-B_{k-1}\ (x)=\frac{1}{k\,(k-1)^{\alpha}}\sum_{|x-\nu/k|\leq 2\delta(x,\,k)}\nu^{\alpha}\ \{\varphi\,(\frac{1}{\nu})-\varphi\,(\frac{1}{k})\}\,p_{\nu k}(x)$$

$$+\frac{1}{k\,(k-1)^{\alpha}}\sum_{|x-\nu/k|>2\delta(x,\,k)}\nu^{\alpha}\,\{\varphi\,(\frac{1}{\nu})-\varphi\,(\frac{1}{k})\}p_{\nu k}(x),$$

where $\delta(x,k)=\sqrt{\frac{x\,(1-x)\ln k}{k}}$.

From (5) and (12) holds

$$(13)\quad B_k(x)-B_{k-1}(x)\leq 2\alpha\,(1-\alpha)\,\frac{\ln^{1-\alpha}k}{k\,(k-1)}\cdot x^{\alpha-1}+\frac{1-\alpha}{k\,(k-1)^{\alpha}}\sum_{|x-\nu/k|>2\delta\,(x,\,k)}p_{\nu k}(x).$$

According to [1]

$$\sum_{|x-\nu/k|>2\delta(x,\,k)}p_{\nu k}(x)\leq 2/k.$$

Then we obtain from (13)

$$B_k(x)-B_{k-1}(x)\leq 2\alpha(1-\alpha)\frac{\ln^{1-\alpha}k}{k\,(k-1)}\,x^{\alpha-1}+\frac{2\,(1-\alpha)}{k^2\,(k-1)^{\alpha}}\leq c_1\alpha\,(1-\alpha)\,\frac{\ln^{1-\alpha}k}{k\,(k-1)}\,x^{\alpha-1}.$$

It is known [1] that the sequence of Bernstein polynomials for a continuous function f converges to f. Therefore

$$(14)\qquad x^{\alpha}-B_n(x)=\sum_{k=n+1}^{\infty}\{B_k(x)-B_{k-1}(x)\}$$

$$=c_1\alpha\,(1-\alpha)\,x^{\alpha-1}\sum_{k=n+1}^{\infty}\frac{\ln^{1-\alpha}k}{k\,(k-1)}\leq c_2\,\alpha(1-\alpha)\,x^{\alpha-1}\,n^{-1}\ln^{1-\alpha}n.$$

Using the obtained estimate (14) we have

$$(15)\qquad [x-c_2\,(1-\alpha)\,\frac{\ln^{1-\alpha}n}{n}]^{\alpha}\leq B_n(x)\leq x^{\alpha}.$$

From (15) and the definition of Hausdorff distance for $n^{-1}\ln n\leq x\leq 1$ and $n\geq\exp\,(2\alpha^{\frac{1}{\alpha-1}})$ one gets $r([n^{-1}\ln n,\,1]\,;\ B_n(f),f)\leq c_3(1-\alpha)\,n^{-1}\ln^{1-\alpha}n$.

2. For $x_{\gamma}\in[n^{-1},\ n^{-1}\ln n]$, $x_{\gamma}=n^{-1}\ln^{1-\gamma}.\,n$, $\gamma\in[0,\,1]$ we assume that the Hausdorff distance is $\delta_{\gamma}=c_4\,(1-\alpha)\,n^{-1}\ln^{1-\beta}n$, $\beta\in[0,\,1]$, $\beta>\gamma$. ($\beta=\gamma+1$ contradict to (1)). Then the definition of Hausdorff distance gives

$$B_n(x_{\gamma})=(x_{\gamma}-\delta_{\gamma})^{\alpha}-\delta_{\gamma}=x_{\gamma}^{\alpha}-\alpha\delta_{\gamma}x_{\gamma}^{\alpha-1}\,.\,[1-\frac{(\alpha-1)\,\delta_{\gamma}}{2!\ x_{\gamma}}+\frac{(\alpha-1)\,(\alpha-2)\,\delta_{\gamma}^2}{3!\ x_{\gamma}^2}+\cdots$$

$$+(-1)^n\cdot\frac{(\alpha-1)\,(\alpha-2)\ldots(\alpha-n+1)\,\delta_{\gamma}^{n-1}}{n!\ x_{\gamma}^{n-1}}+\cdots]-\delta_{\gamma}.$$

The series in the break square is convergent and for its sum $A_{\gamma}(\alpha)$ one has $1\leq A_{\gamma}(\alpha)<1/\alpha$.

We obtain

$$B_n(x_{\gamma})=(x_{\gamma}-\delta_{\gamma})^{\alpha}-\delta_{\gamma}=x_{\gamma}^{\alpha}-\alpha A_{\gamma}(\alpha)\,\delta_{\gamma}\,x_{\gamma}^{\alpha-1}-\delta_{\gamma}$$

or

(16) $$\alpha\delta_\gamma x_\gamma^{\alpha-1}-\delta_\gamma\leqq x_\gamma^\alpha-B_n(x_\gamma)\leqq\delta_\gamma x_\gamma^{\alpha-1}-\delta_\gamma.$$

In view of (1) the values of β and γ must satisfy

(17) $$\alpha A_\gamma(\alpha)\,\delta_\gamma x_\gamma^{\alpha-1}\leqq(1-\alpha)\,c_5\,n^{-\alpha}\ln^\sigma n\leqq(1-\alpha)\,c_6\,n^{-\alpha},$$

where $\sigma=\alpha(1-\gamma)+(\gamma-\beta)$. It is clear that (17) would be true for $\sigma=\alpha+(1-\alpha)\gamma-\beta\leqq0$. Therefore for the Hausdorff distance in the interval $[n^{-1}, n^{-1}\ln n]$ one gets

$$r([n^{-1}, n^{-1}\ln n]\,;\ B_n(f), f)\leqq c_7\,(1-\alpha)n^{-1}\ln^{1-\alpha} n.$$

3. We prove that the obtained estimate is exact to the order.

For $k>n$ from (3) follows that

$$B_k(x)-B_{k-1}(x)=\frac{1}{k(k-1)^\alpha}\sum_{\nu=1}^{k-1}\{\varphi(\frac{1}{\nu})-\varphi\frac{1}{k}\}\,p_{\nu k}(x)\geqq x(1-x)^{k-1}(k-1)^{-\alpha}(1-\varphi(\frac{1}{k})).$$

Hence

$$x^\alpha-B_n(x)=\sum_{k=n+1}^{\infty}[B_k(x)-B_{k-1}(x)]\geqq x\sum_{k=n+1}^{\infty}\frac{(1-x)^{k-1}(1-\varphi(\frac{1}{k}))}{(k-1)^\alpha}$$

$$\geqq x\,(1-\varphi(\frac{1}{n+1}))\,\frac{1}{(2n)^\alpha}\sum_{k=n+1}^{2n}(1-x)^{k-1}\geqq(1-\varphi(\frac{1}{n+1}))\,[(1-x)_n-(1-x)^{2n}](2n)^{-\alpha}.$$

We set $x_\alpha=\alpha^{1/1-\alpha}\,\dfrac{\ln n}{n}$. Then the following will be true:

(18) $$x_\alpha^\alpha-B_n(x_\alpha)\geqq(1-\varphi(\frac{1}{n}))\,[(1-x_\alpha)^n-(1-x_\alpha)^{2n}]\,(2n)^{-\alpha}$$

$$\geqq(1-\varphi(\frac{1}{n}))\,[(1-\frac{1}{n})^n-(1-\frac{1}{n})^{2n}]\,(2n)^{-\alpha}$$

$$\geqq\alpha(1-\alpha)\,(e^{-1}-e^{-2})\,(2n^{-\alpha})\,[(nx_\alpha)^{-1}\ln n]^{1-\alpha}\geqq\alpha(1-\alpha)\,c_8\,(n^{-1}\ln^{1-\alpha} n)\,.\,x_\alpha^{\alpha-1}.$$

From (16) and (18) it follows that the Hausdorff distance in the point x_α must satisfy the inequality

$$\alpha A_\alpha(\alpha)\,\delta_\alpha x_\alpha^{\alpha-1}-\delta_\alpha\geqq\alpha(1-\alpha)\,c_8\,x_\alpha^{\alpha-1}\,n^{-1}\ln n$$

or $\delta_\alpha\geqq c_9(1-\alpha)\,n^{-1}\ln^{1-\alpha} n$.

Thus the theorem is proved.

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