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**AN EXACT ESTIMATE OF THE APPROXIMATION
OF THE FUNCTION X^a WITH
BERNSTEIN POLYNOMIALS IN HAUSDORFF METRIC**

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In the paper an estimate is obtained for the approximation of the function

$$f(x) = x^a, \quad 0 \leq x \leq 1, \quad 0 < a < 1$$

with Bernstein polynomials

$$B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

in the Hausdorff metric.

It is proved that the estimate is exact to the order $n^{-1} \ln^{1-a} n$.

In the paper an estimate is obtained for the approximation of the function $f(x) = x^a, 0 \leq x \leq 1, 0 < a < 1$ with Bernstein polynomials in the Hausdorff metric. It is proved that the estimate is exact to the order $n^{-1} \ln^{1-a} n$.

We shall use the notation

$$B_n(f; x) = B_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{kn}(x),$$

where $p_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ — the Bernstein polynomial for f ; according to [3]

$$r(\Delta; f, g) = \max \left\{ \max_{A \in f} \min_{B \in g} \rho(A, B), \max_{A \in g} \min_{B \in f} \rho(A, B) \right\},$$

where $\rho(A, B) = \rho(A(x_1, y_1), B(x_2, y_2)) = \max \{ |x_1 - x_2|, |y_1 - y_2| \}$, defines the Hausdorff distance between the functions $f, g \in C_\Delta$.

It is known [4] that for the function $f(x) = x^a, 0 \leq x \leq 1, 0 < a < 1$ one has

$$(1) \quad \max \{ |f(x) - B_n(f; x)|, x \in [0, 1] \} = O((1-a)n^{-a}).$$

We prove

Theorem. *If $n \geq \exp(2a^{a-1})$ for the function $f(x) = x^a, 0 \leq x \leq 1, 0 < a < 1$ one gets*

$$r([0, 1]; B_n(f), f) = O((1-a)n^{-1} \ln^{1-a} n).$$

Proof. After some elementary transformations we obtain

$$(2) \quad B_{k-1}(x) - B_k(x) = \sum_{v=1}^{k-1} \left\{ \frac{k-v}{k} f\left(\frac{v}{k-1}\right) - f\left(\frac{v}{k}\right) + \frac{v}{k} f\left(\frac{v-1}{k-1}\right) \right\} p_{vk}(x).$$

It follows from (2) that for the function $f(x) = x^\alpha$ one has

$$(3) \quad B_{k-1}(x) - B_k(x) = - \sum_{v=1}^{k-1} \frac{v^\alpha}{k(k-1)^\alpha} \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} p_{vk}(x),$$

where $\varphi(x) = [1 - (1-x)^\alpha] \cdot x^{-1}$.

1. For n^{-1} in $n \leq x \leq 1$ we shall prove that if

$$(4) \quad \left| \begin{array}{l} k \geq n \geq \exp \left\{ 2\alpha^{\frac{1}{\alpha-1}} \right\}; \\ \frac{kx}{2} \leq v \leq k, \end{array} \right.$$

then

$$(5) \quad v^\alpha \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} \leq \alpha \left\{ \frac{\ln k}{kx} \right\}^{1-\alpha} (1 - \varphi\left(\frac{1}{k}\right)).$$

For this purpose we use the inequality

$$(6) \quad (2-\alpha) \vartheta^{1-\alpha} - (1-\alpha) \vartheta^{2-\alpha} \leq 1, \quad \vartheta \in [0, 1].$$

It is obviously true as the function on the left hand side is increasing on $[0, 1]$ and reaches its maximal value for $\vartheta = 1$. We set $\vartheta = 1 - u$, $0 < u \leq 1$ and from (6) we get

$$(7) \quad 1 - (1-u)^\alpha \leq \alpha u + (1-\alpha) u^2.$$

If

$$(8) \quad 1 \leq \alpha \left[\frac{\ln k}{kx} \right]^{1-\alpha} u^{\alpha-1},$$

then (7) gives

$\varphi(u) \leq \alpha + \alpha(1-\alpha) \left[\frac{\ln k}{kx} \right]^{1-\alpha} u^\alpha$, which is equivalent to

$$(9) \quad (1-\alpha) \left[1 - \alpha \left(\frac{\ln k}{kx} \right)^{1-\alpha} u^\alpha \right] \leq 1 - \varphi(u).$$

According to the definition φ is increasing on $[0, 1]$. Therefore the function $1 - \varphi(t)$ is decreasing on $[0, 1]$ and for $x \in [0, 1]$ one has

$$(10) \quad 1 - \varphi(x) \leq 1 - \alpha.$$

From (9) and (10) we obtain $[1 - \varphi(t)] \left[1 - \alpha \left(\frac{\ln k}{kx} \right)^{1-\alpha} u^\alpha \right] \leq 1 - \varphi(u)$ or

$$(11) \quad \varphi(u) - \varphi(t) \leq \alpha \left(\frac{\ln k}{kx} \right)^{1-\alpha} u^\alpha (1 - \varphi(t)).$$

It is easy to see that (11) yields (5) with the substitution $t = 1/k$, $u = 1/v$, if the condition (8) is satisfied. But this proves (5) as the inequality (8) is true under the restrictions (4).

Using similar considerations we prove [4] that for $1 \leq v \leq k-1$ one gets

$$(12) \quad v^\alpha \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} \leq 1 - \alpha.$$

Further we express (3) as follows

$$B_k(x) - B_{k-1}(x) = \frac{1}{k(k-1)^\alpha} \sum_{|x-v/k| \leq 2\delta(x,k)} v^\alpha \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} p_{v,k}(x) + \frac{1}{k(k-1)^\alpha} \sum_{|x-v/k| > 2\delta(x,k)} v^\alpha \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} p_{v,k}(x),$$

where $\delta(x, k) = \sqrt{\frac{x(1-x) \ln k}{k}}$.

From (5) and (12) holds

$$(13) \quad B_k(x) - B_{k-1}(x) \leq 2\alpha(1-\alpha) \frac{\ln^{1-\alpha} k}{k(k-1)} \cdot x^{\alpha-1} + \frac{1-\alpha}{k(k-1)^\alpha} \sum_{|x-v/k| > 2\delta(x,k)} p_{v,k}(x).$$

According to [1]

$$\sum_{|x-v/k| > 2\delta(x,k)} p_{v,k}(x) \leq 2/k.$$

Then we obtain from (13)

$$B_k(x) - B_{k-1}(x) \leq 2\alpha(1-\alpha) \frac{\ln^{1-\alpha} k}{k(k-1)} x^{\alpha-1} + \frac{2(1-\alpha)}{k^2(k-1)^\alpha} \leq c_1 \alpha(1-\alpha) \frac{\ln^{1-\alpha} k}{k(k-1)} x^{\alpha-1}.$$

It is known [1] that the sequence of Bernstein polynomials for a continuous function f converges to f . Therefore

$$(14) \quad x^\alpha - B_n(x) = \sum_{k=n+1}^\infty \{B_k(x) - B_{k-1}(x)\} = c_1 \alpha(1-\alpha) x^{\alpha-1} \sum_{k=n+1}^\infty \frac{\ln^{1-\alpha} k}{k(k-1)} \leq c_2 \alpha(1-\alpha) x^{\alpha-1} n^{-1} \ln^{1-\alpha} n.$$

Using the obtained estimate (14) we have

$$(15) \quad [x - c_2(1-\alpha) \frac{\ln^{1-\alpha} n}{n}]^\alpha \leq B_n(x) \leq x^\alpha.$$

From (15) and the definition of Hausdorff distance for $n^{-1} \ln n \leq x \leq 1$ and $n \geq \exp(2\alpha^{\frac{1}{1-\alpha}})$ one gets $r([n^{-1} \ln n, 1]; B_n(f), f) \leq c_3(1-\alpha) n^{-1} \ln^{1-\alpha} n$.

2. For $x_\gamma \in [n^{-1}, n^{-1} \ln n]$, $x_\gamma = n^{-1} \ln^{1-\gamma} n$, $\gamma \in [0, 1]$ we assume that the Hausdorff distance is $\delta_\gamma = c_4(1-\alpha) n^{-1} \ln^{1-\beta} n$, $\beta \in [0, 1]$, $\beta > \gamma$. ($\beta = \gamma \pm 1$ contradict to (1)). Then the definition of Hausdorff distance gives

$$B_n(x_\gamma) = (x_\gamma - \delta_\gamma)^\alpha - \delta_\gamma = x_\gamma^\alpha - \alpha \delta_\gamma x_\gamma^{\alpha-1} \cdot [1 - \frac{(\alpha-1) \delta_\gamma}{2! x_\gamma} + \frac{(\alpha-1)(\alpha-2) \delta_\gamma^2}{3! x_\gamma^2} + \dots + (-1)^n \cdot \frac{(\alpha-1)(\alpha-2) \dots (\alpha-n+1) \delta_\gamma^{n-1}}{n! x_\gamma^{n-1}} + \dots] - \delta_\gamma.$$

The series in the break square is convergent and for its sum $A_\gamma(\alpha)$ one has $1 \leq A_\gamma(\alpha) < 1/\alpha$.

We obtain

$$B_n(x_\gamma) = (x_\gamma - \delta_\gamma)^\alpha - \delta_\gamma = x_\gamma^\alpha - \alpha A_\gamma(\alpha) \delta_\gamma x_\gamma^{\alpha-1} - \delta_\gamma$$

or

$$(16) \quad \alpha \delta_\gamma x_\gamma^{\alpha-1} - \delta_\gamma \leq x_\gamma^\alpha - B_n(x_\gamma) \leq \delta_\gamma x_\gamma^{\alpha-1} - \delta_\gamma.$$

In view of (1) the values of β and γ must satisfy

$$(17) \quad \alpha A_\gamma(\alpha) \delta_\gamma x_\gamma^{\alpha-1} \leq (1-\alpha) c_5 n^{-\alpha} \ln^\sigma n \leq (1-\alpha) c_6 n^{-\alpha},$$

where $\sigma = \alpha(1-\gamma) + (\gamma-\beta)$. It is clear that (17) would be true for $\sigma = \alpha + (1-\alpha)\gamma - \beta \leq 0$. Therefore for the Hausdorff distance in the interval $[n^{-1}, n^{-1} \ln n]$ one gets

$$r([n^{-1}, n^{-1} \ln n]; B_n(f), f) \leq c_7 (1-\alpha) n^{-1} \ln^{1-\alpha} n.$$

3. We prove that the obtained estimate is exact to the order.

For $k > n$ from (3) follows that

$$B_k(x) - B_{k-1}(x) = \frac{1}{k(k-1)^\alpha} \sum_{v=1}^{k-1} \left\{ \varphi\left(\frac{1}{v}\right) - \varphi\left(\frac{1}{k}\right) \right\} p_{v,k}(x) \geq x(1-x)^{k-1} (k-1)^{-\alpha} \left(1 - \varphi\left(\frac{1}{k}\right)\right).$$

Hence

$$\begin{aligned} x^\alpha - B_n(x) &= \sum_{k=n+1}^{\infty} [B_k(x) - B_{k-1}(x)] \geq x \sum_{k=n+1}^{\infty} \frac{(1-x)^{k-1} (1 - \varphi(\frac{1}{k}))}{(k-1)^\alpha} \\ &\geq x \left(1 - \varphi\left(\frac{1}{n+1}\right)\right) \frac{1}{(2n)^\alpha} \sum_{k=n+1}^{2n} (1-x)^{k-1} \geq \left(1 - \varphi\left(\frac{1}{n+1}\right)\right) [(1-x)_n - (1-x)^{2n}] (2n)^{-\alpha}. \end{aligned}$$

We set $x_\alpha = \alpha^{1/1-\alpha} \frac{\ln n}{n}$. Then the following will be true:

$$\begin{aligned} (18) \quad x_\alpha^\alpha - B_n(x_\alpha) &\geq \left(1 - \varphi\left(\frac{1}{n}\right)\right) [(1-x_\alpha)^n - (1-x_\alpha)^{2n}] (2n)^{-\alpha} \\ &\geq \left(1 - \varphi\left(\frac{1}{n}\right)\right) \left[\left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n}\right] (2n)^{-\alpha} \\ &\geq \alpha(1-\alpha) (e^{-1} - e^{-2}) (2n^{-\alpha}) [(nx_\alpha)^{-1} \ln n]^{1-\alpha} \geq \alpha(1-\alpha) c_8 (n^{-1} \ln^{1-\alpha} n) \cdot x_\alpha^{\alpha-1}. \end{aligned}$$

From (16) and (18) it follows that the Hausdorff distance in the point x_α must satisfy the inequality

$$\alpha A_\alpha(\alpha) \delta_\alpha x_\alpha^{\alpha-1} - \delta_\alpha \geq \alpha(1-\alpha) c_8 x_\alpha^{\alpha-1} n^{-1} \ln n$$

or $\delta_\alpha \geq c_9 (1-\alpha) n^{-1} \ln^{1-\alpha} n$.

Thus the theorem is proved.

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