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APPROXIMATION OF A CONVEX FUNCTION BY ALGEBRAIC POLYNOMIALS IN $L_p[a, b]$ $(1<p<\infty)$

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We prove that the best algebraic approximation of a convex function in $L_p(1<p<\infty)$ is $o(n^{-\frac{2}{p}})$.

1. Notations and main results. We shall use the following symbols: H_n — the set of all algebraic polynomials of a degree at most n; $K^M[a, b]$ — the set of all convex and continuous in $[a, b]$ functions, such that $\max\{f(x); a\le x\le b\}-\min\{f(x); a\le x\le b\} M$. K is the set of all convex and continuous in $[-1, 1]$ functions such that $\max\{f(x); -1\le x\le 1\}=1$; $\min\{f(x); -1\le x\le 1\}=0$. By $E_n(f)_{L_p}$ we denote the best L_p approximation of f by polynomials of a degree n, i. e.

$$E_n(f)_{L_p}=\inf\{\|f-P\|_{L_p};\ P\in H_n\}. \tag{1.1}$$

If $D\subset L_p[a, b]$ then the best L_p approximation of D by polynomials of a degree n is

$$E_n(D)_{L_p}=\sup\{E_n(f)_{L_p};\ f\in D\}. \tag{1.2}$$

Let

$$\omega(f;\ \delta)=\sup\{|f(x)-f(y)|;\ \ |x-y|\le\delta,\ a\le x, y\le b\}$$

be the modulus of continuity of f and

$$\tau_k(f;\delta)_{p',p}=\|\,\omega_k(f,.,\delta(.))_{p'}\|_{L_p}$$

be the K^{th} modulus of L_p continuity, where

$$\omega_k(f, x, \delta(x))_{p'}=\{(1/2\delta(x)\int_{-\delta(x)}^{\delta(x)}|\Delta_v^k f(x)|^{p'}dv\}^{1/p'}$$

and

$$\Delta_v^k f(x)=\begin{cases}\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}f(x+mv) & \text{if}\quad x, x+kv\in[a, b]\\ 0 & \text{otherwise.}\end{cases}$$

We set $\Delta_n(x)=\sqrt{1-x^2}/n+1/n^2$

$$g_a(x)=\begin{cases}0 & \text{for}\quad -1\le x\le a\\ (x-a)/(1-a) & \text{for}\quad a\le x\le 1\end{cases}\qquad h_a(x)=g_{-a}(-x).$$

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The aim of this paper is to estimate the best algebraic approximation (1.1) if f is a convex function, and the best approximation (1.2) of class of convex functions $K^{\mu}[a, b]$. In the case $p=\infty$ Jachson's theorem gives (see. e. g. [1])

$$E_n(f)_c \leq c_1 \omega(f; (b-a)/n) \quad \text{for} \quad f \in K^{\mu}[a, b] \tag{1.3}$$

and

$$E_n(K^{\mu}[a, b])_c \leq c_2 M, \tag{1.4}$$

where c_1 and c_2 are absolute constants. In the case $p=1$ K. Ivanov [2] shows that

$$E_n(f)_{L_1} \leq c_1 M(b-a) n^{-2} \quad \text{for} \quad f \in K^{\mu}[a, b] \tag{1.5}$$

and

$$E_n(K^{\mu}[a, b])_{L_1} \leq c_1 M(b-a) n^{-2}. \tag{1.6}$$

We shall get similar estimates in the case $1<p<\infty$. The following two theorems are proved in this paper.

Theorem 1. *There is an absolute constant $c>0$ such that*

$$E_n(K)_{L_p} \leq c n^{-\frac{2}{p}}. \tag{1.7}$$

From here we get immediately the following

Corollary 1. There is an absolute constant $c>0$ such that

$$E_n(K^M[a, b])_{L_p} \leq c\, M\, (b-a)^{1/p} n^{-2/p}, \quad 1<p<\infty.$$

Theorem 2. *There is an absolute constant $c>0$ such that $E_n(f)_{L_p} \leq c\omega(f; n^{-(p-1)/p}) n^{-2/p}$ for each $f \in K$ and $1<p<\infty$.*

From here we get

Corollary 2. There is $c>0$ such that $E_n(f)_{L_p} \leq c\,(b-a)^{1/p}\, \omega(f; n^{-(p-1)/p}(b-a)) n^{-2/p}$ for each $f \in K^M[a, b]$.

Taking the limit in Corollary 1 and 2 we get (1.4) and (1.3) if $p \to \infty$ and (1.6) and (1.5) if $p \to 1$.

This theorem shows that $E_n(f)_{L_p} = o(n^{-\frac{2}{p}})$ for every $f \in K^M[a, b]$. But we can not characterize this effect. For example, for the function $f(x)=2^{\alpha}-(1-x)^{\alpha}$ $(0<\alpha<1,\ -1 \leq x \leq 1)$ the order of $E_n(f)_{L_p}$ is better than this, given by Theorem 2.

2. Preliminaries. We shall use the following results.

Theorem A. [3] *There is a constant $c(k)>0$ such that $E_n(f)_{L_p} \leq c(k) \tau_k(f; \Delta_n)_{p',p}$; $1 \leq p' \leq p$, for each $f \in L_p[-1, 1]$.*

Theorem B. [4] *There is a constant $c>0$, such that*

$$\tau_k(f; \Delta_n)_{1,p} \leq c(k) n^{-k} \sum_{s=0}^{n} (s+1)^{k-1} E_s(f)_{L_p}.$$

The following lemma is well known.

Lemma A. *For every $f \in K$ and every $\varepsilon>0$ there are numbers $a_1, a_2 \ldots a_N$; $b_1, b_2 \ldots b_M$, where $-1 \leq a_i \leq 1$; $i=1\ 2 \ldots N$; $-1 \leq b_i \leq 1$; $i=1, 2 \ldots, M$, and positive numbers $\alpha_1, \alpha_2 \ldots \alpha_N$; $\beta_1, \beta_2 \ldots \beta_M$, such that*

$$\sum_{i=1}^{N} \alpha_i = f(1), \quad \sum_{i=1}^{M} \beta_i = f(-1) \quad and \quad \| f - \sum_{i=1}^{N} \alpha_i g_{ai} - \sum_{i=1}^{M} \beta_i h_{bi} \| < \varepsilon.$$

Lemma B. [3] *If* $|x|, |y| \leq 1$, $|x-y| \leq \lambda \Delta_n(x)$, $n \geq 2\lambda$, *then* $\Delta_n(x)/(4\lambda+2) \leq \Delta_n(y) \leq (2\lambda+3/2)\Delta_n(x)$.

3. Proof of Theorem 1. To prove this theorem we need the following three lemmas.

Lemma 1. *If* $-1 \leq a \leq 1$, *then there are numbers* b_1 *and* b_2 *such that* $-1 \leq b_1 \leq a \leq b_2 \leq 1$, *and* $x+2\Delta_n(x) > a$ *for* $b_1 < x \leq 1$; $x+2\Delta_n(x) < a$ *for* $-1 \leq x < b_1$; $x-2\Delta_n(x) < a$ *for* $-1 \leq x < b_2$; $x-2\Delta_n(x) > a$ *for* $b_2 < x \leq 1$.

Proof. For $-1 \leq a \leq -1+2/n^2$, $b_1 = -1$ is the only number which meets the requirements. For $-1+2/n^2 < a \leq 1$ we consider the function $F_1(x) = x + 2\Delta_n(x) - a$. $F_1(a) > 0$, $F_1(-1) = -1+2/n^2 - a < 0$. The equation $F_1(x) = 0$ has at most two solutions. Therefore there is unique b_1 $(-1 < b_1 < a)$ such that $F_1(b_1) = 0$ and $F_1(x) < 0$ for $-1 \leq x < b_1$, $F_1(x) > 0$ for $b_1 < x \leq 1$. Thereby all is proved for b_1. The existence of b_2 we verify in the same manner.

Lemma 2. *There is a constant* $c > 0$, *such that*

$$\tau_2(g_a;\ \Delta_n)_{1,p} \leq c(\Delta_n(a))^{(p+1)p}/(1-a)$$

for each $a \in [-1, 1)$.

Proof. From Lemma 1 and the definition of ω_k we see that

$$\omega_2(g_a, x, \Delta_n(x))_1 = 0 \quad \text{for} \quad -1 \leq x \leq b_1 \quad \text{and} \quad b_2 \leq x \leq 1. \tag{3.1}$$

Let $b_1 \leq x \leq a$. Then

$$\Delta_v^2 g_a(x) = \begin{cases} 0 & \text{for} \quad -\Delta_n(x) \leq v \leq (a-x)/2, \\ (x+2v-a)/(1-a) & \text{for} \quad (a-x)/2 \leq v \leq \min(\Delta_n(x), a-x), \\ (a-x)/(1-a) & \text{for} \quad \min(\Delta_n(x), a-x) \leq v \leq \Delta_n(x) \end{cases}$$

and we get

$$\omega_2(g_a, x, \Delta_n(x))_1 \leq (a-x)/(2(1-a)) + (3(a-x)^2)/(8(1-a)\Delta_n(x)) \leq (5/4)((a-x)/(1-a)) \tag{3.2}$$

for $b_1 \leq x \leq a$.

In the same manner

$$\omega_2(g_a, x, \Delta_n(x))_1 \leq (5/4)(\,a-x|/(1-a)) \quad \text{for} \quad a \leq x \leq b_2. \tag{3.3}$$

From Lemma B and Lemma 1 we have

$$\Delta_n(x) \leq 10\Delta_n(a) \quad \text{for} \quad b_1 \leq x \leq b_2. \tag{3.4}$$

From (3.1)—(3.4) and the definition of τ_k we obtain

$$\tau_2(g_a; \Delta_n)_{1,p} \leq (5/4)\sup\{|x-a|/(1-a);\ b_1 \leq x \leq b_2\} \|1\|_{L_p[b_1, b_2]} \leq 25(\Delta_n(a)/(1-a))(b_2-b_1)^{1/p} \leq 1000(\Delta_n(a))^{\frac{p+1}{p}}/(1-a). \tag{3.5}$$

Lemma 3. *There is an absolute constant* $c > 0$, *such that* $\tau_2(g_a, \Delta_n)_{1,p} \leq cn^{-\frac{2}{p}}$ *for each* $a \in [-1, 1)$, $n = 1, 2, \ldots$.

P r o o f. For $-1 \leq a \leq 1-n^{-2}$ from Lemma 1 we have

$$\tau_2(g_a, \Delta_n)_{1,p} \leq c(\Delta_n(a))^{\frac{p+1}{p}}/(1-a) = c\{(1+a)^{\frac{1}{2}}(1-a)^{\frac{1}{2}-\frac{p}{p+1}}n^{-1}+(1-a)^{-\frac{p}{p+1}}n^{-2}\}^{\frac{p+1}{p}}$$

$$\leq c\{n^{-1+\frac{p-1}{p+1}}+n^{\frac{2p}{p+1}-2}\}^{\frac{p+1}{p}} = cn^{-\frac{2}{p}}.$$

Let $1-n^{-2} \leq a < 1$.

$$\omega_2(g_n, x, \Delta_n(x))_1 \leq \begin{cases} 0 & \text{for } -1 \leq x \leq b_1, \\ 4 & \text{for } b_1 < x \leq 1, \end{cases}$$

$$\tau_2(g_a, \Delta_n)_{1,p} \leq 4\|1\|_{L_p[b_1,1]} = 4(1-b_1)^{\frac{1}{p}} \leq c(\Delta_n(a))^{\frac{1}{p}} \leq c(\Delta_n(1))^{\frac{1}{p}} \leq cn^{-\frac{2}{p}}.$$

This completes the proof.

P r o o f o f T h e o r e m 1. From Theorem A for $k=2$, Lemma A, Lemma 3 and Lemma B we obtain (1.7).

Let us consider the function

$$G_n(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 1-n^{-2}, \\ n^2(x-1+n^{-2}) & \text{for } 1-n^{-2} < x \leq 1. \end{cases}$$

$$G_n \in K; \quad \tau_2(G_n; \Delta_n)_{1,p} \geq n^{-\frac{2}{p}}/(8(p+1)^{\frac{1}{p}}).$$

This inequality with Theorem B for $k=2$ shows that we can not improve the estimate in Theorem 1.

Proof of Theorem 2. From Theorem A and Lemma 2 we obtain

$$E_n(g_a)_{L_p} \leq c\tau_2(g_a, \Delta_n)_{1,p} \leq c(\Delta_n(a))^{\frac{p+1}{p}}/(1-a) \leq cn^{-\frac{p+1}{p}}/(1-a). \tag{4.1}$$

Let f be a convex, continuous in $[-1, 1]$ function with the following two properties:

$$f(-1) = \min\{f(x); -1 \leq x \leq 1\} = 0 \tag{4.2}$$

and there are numbers $a, b, -1 < a < b < 1$, such that

$$f(x) = f(1)\, g_a(x) \quad \text{for} \quad b \leq x \leq 1. \tag{4.3}$$

Then the numbers $\alpha_1, \alpha_2, \ldots \alpha_N$; $\beta_1, \beta_2, \ldots \beta_M$ in Lemma A can be chosen such that $\beta_i = 0$, $i = 1, 2, \ldots M$, and

$$f(x) = \sum_{i=1}^{N} \alpha_i\, g_{a_i}(x) \quad \text{for} \quad b \leq x \leq 1.$$

Then

$$f(1)(x-a)/(1-a) = f(x) = \sum_{i=1}^{N} \alpha_i\, g_{a_i}(x) = (\sum_{i=1}^{N} \alpha_i/(1-a_i))x - \sum_{i=1}^{N} \alpha_i a_i/(1-a_i).$$

for $b \leq x \leq 1$. Therefore

$$f(1)/(1-a) = \sum_{i=1}^{N} \alpha_i/(1-a_i) \text{ if } f \text{ satisfies (4.2) and (4.3).} \tag{4.4}$$

From (4.1) and (4.4) we obtain

$$E_n(f)_{L_p} \leq E_n(\sum_{i=1}^{N} \alpha_i g_{ai})_{L_p} + \varepsilon \leq \varepsilon + \sum_{i=1}^{N} \alpha_i E_n(g_{ai})_{L_p} \leq \varepsilon + c \sum_{i=1}^{N} \alpha_i n^{-\frac{p+1}{p}}/(1-a_i)$$

$$= \varepsilon + cn^{-\frac{p+1}{p}} f(1)/(1-a).$$

And since ε is an arbitrary positive number then

$$(4.5) \qquad E_n(f)_{L_p} \leq cn^{-\frac{p+1}{p}} f(1)/(1-a)$$

if f satisfies (4.2) and (4.3).

Let f satisfy only (4.2). We set $\delta_n = n^{-\frac{p-1}{p}}$, $\varepsilon_n = \omega(f, \delta_n) = f(1) - f(1-\delta_n)$. We consider the functions

$$f_2(x) = \begin{cases} f(x) & \text{for } -1 \leq x \leq 1-\delta_n, \\ f'((1-\delta_n)^-)x + f(1-\delta_n) - (1-\delta_n) f'((1-\delta_n)^-) & \text{for } 1-\delta_n \leq x \leq 1, \end{cases}$$

($f'((1-\delta_n)^-)$ is the left derivative of f at the point $1-\delta_n$), $f_1(x) = f(x) - f_2(x)$. f_1 and f_2 are convex functions and $f_1 \in K^{\varepsilon_n}[-1, 1]$. From Corollary 1 we have

$$(4.6) \qquad E_n(f_1)_{L_p} \leq c\varepsilon_n n^{-\frac{2}{p}},$$

f_2 satisfies (4.2) and (4.3) with $a_n = 1 - \delta_n f_2(1)/(f_2(1) - f_2(1-\delta_n))$ and from (4.5) we have

$$(4.7) \qquad E_n(f_2)_{L_p} \leq cn^{-\frac{p+1}{p}} f_2(1)/(1-a_n) = cn^{-\frac{p+1}{p}} \varepsilon_n/\delta_n.$$

From (4.6) and (4.7) we obtain

$$(4.8) \qquad E_n(f)_{L_p} \leq cn^{-\frac{2}{p}} \omega(f, n^{-\frac{p-1}{p}}) \text{ if } f(x) \text{ or } f(-x) \text{ satisfies (4.2).}$$

At last if $f \in k$, then there is $x_0 \in [-1, 1]$ such that $f(x_0) = 0$. We set

$$\tilde{f}(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq x_0, \\ f(x) & \text{for } x_0 < x \leq 1. \end{cases}$$

$$\bar{f}(x) = f(x) - \tilde{f}(x).$$

For f and $\bar{f}$ (4.8) holds true. Then

$$E_n(f)_{L_p} \leq E_n(\bar{f})_{L_p} + E_n(\tilde{f})_{L_p} \leq cn^{-\frac{2}{p}}\{\omega(\bar{f}, n^{-\frac{p-1}{p}}) + \omega(\tilde{f}, n^{-\frac{p-1}{p}})\} \leq cn^{-\frac{2}{p}} \omega(f, n^{-\frac{p-1}{p}}).$$

This completes the proof.

For example for the function $f(x) = |x|$ Theorem 2 gives the order of the best algebraic approximation.

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