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ON FOCAL LOCUS OF SUBMANIFOLDS OF NATURALLY REDUCTIVE COMPACT RIEMANNIAN HOMOGENEOUS SPACES

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The purpose of this paper is to discuss focal points of submanifolds of naturally reductive compact Riemannian homogeneous spaces. Focal points have been studied in detail in [1, 6, 8]. In this paper we have estimated location of focal points along geodesic and obtained condition under which a point is a focal point along geodesic.

1. Introduction. Let M be a C^∞, n-dimensional compact Riemannian manifold with metric $\langle , \rangle$. Let L be a compact connected submanifold embedded in M. The orthogonal complement N_pL of the tangent space T_pL in T_pM will be called the normal space of L at p for $p \in L$. The union $N = \cup \{N_pL \mid p \in L\}$ of these normal spaces which is a subbundle of class C^∞ of the tangent bundle TM is called the normal bundle of the submanifold L.

Let Y: $[0, \beta] \to M$, $\beta \in R$ is a normal geodesic with $Y(0) = z \in L$, $\dot{Y}(0) = v \in N_pL$.

Let $E_1, \dots, E_{n-1}$: $[0, \beta] \to TM$ be a parallel vector fields along Y such that $(\dot{Y}(\tau), E_1(\tau), \dots, E_{n-1}(\tau))$ is an orthonormal base for $T_{Y(\tau)}M$.

Let $Y(\tau)$, $\tau \in [0, \beta]$ be a Jacobi field along Y. If T_zL is spanned by $E_1(0), \dots, E_k(0)$, $Y(\tau)$ is an L-Jacobifield if [1], [8]

(i) $\langle Y(\tau), \dot{Y}(\tau)\rangle \equiv 0$,

(ii) $Y(0) \in T_zL$,

(iii) $\nabla \dot{Y}(0) Y(0) + \sigma_{\dot{Y}(0)} Y(0) \in N_zL$,

where $\sigma_{\dot{Y}(0)}: T_zL \to T_zL$ is a Weingarten map.

A point $Y(\tau_0)$ on Y, for $\tau_0 \in (0, \beta]$, is called a focal point of L with respect to Y if there is a L-Jacobifield on Y that is not identically zero and vanishes at $\tau = \tau_0$. The set of such focal points is called focal locus of L in M.

In this paper we will give estimates of the location of focal points by using the Jacobifield of the form given in [7].

2. Naturally reductive Riemannian homogeneous spaces. Riemannian manifold M is called homogeneous if the isometry group of $\langle , \rangle$ acts transitively on M. If G is a group acting transitively by isometries on M, we can write $M = G/H$ where H is the isotropy subgroup of G at fixed point p_0. We denote the Lie algebra of G, H by $\mathfrak{g}$, $\mathfrak{h}$. M is called reductive homogeneous if there exists a complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ so that $Ad(H)$ leaves $\mathfrak{m}$ invariant. Since H is compact this is always the case but notice that $\mathfrak{m}$ is not necessarily unique. $\mathfrak{m}$ can be identified with $T_{p_0}M$. The metric on $T_{p_0}M$ thus induces a metric on $\mathfrak{m}$ again denoted by $\langle , \rangle$. M being reductive implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

If $X, Y \in \mathfrak{m}$, then we denote by $[X, Y]_{\mathfrak{h}}$, $[X, Y]_{\mathfrak{m}}$ the $\mathfrak{h}$ and $\mathfrak{m}$ component of $[X, Y]$. If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, then M is symmetric.

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M is called naturally reductive if

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0,$$

for all $X, Y, Z \in \mathfrak{m}$,
or in other words, if $[X, \cdot]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew symmetric for all $X \in \mathfrak{m}$.

We will denote the Levi-civita connection and the curvature tensor of $\langle . \rangle$ by ∇ and R. If M is naturally reductive, the connection ∇ can be described as follows:
If X^* is a Killing vector field and $v \in T_{p_0}M \cong \mathfrak{m}$, then

$$\nabla_v X^* = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h}, \\ \frac{1}{2}[X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m}. \end{cases}$$

The curvature tensor at p_0 is given by $R(X, v)v = -[v, [v, X]\,\mathfrak{h}] - \frac{1}{4}[v, [v, X]\mathfrak{m}]\mathfrak{m}$.

One knows that on a naturally reductive space there exists a connection D with torsion T and curvature B so that T and B are D parallel.
D has the same geodesics as ∇ so that

$$\nabla X^Y = D_X Y - \frac{1}{2} T(X, Y),$$

and D, T, and B at p_0 can be expressed in terms of the Lie bracket:

$$D_v X^* = \begin{cases} [X, v] & \text{if } X \in \mathfrak{h}, \\ [X, v]_{\mathfrak{m}} & \text{if } X \in \mathfrak{m} \end{cases}$$

$$T(X, Y) = -[X, Y]_{\mathfrak{m}}$$

$$B(X, Y)Z = -[[X, Y]\mathfrak{h}, Z],$$

where $X, Y, Z \in \mathfrak{m}$. Notice that $R = B - \frac{1}{4} T^2$.

Each $a \in G$ operates on M by left multiplication $L_a : G/H \to G/H$. The maps L_a are isometries.

The geodesics in a naturally reductive space are images of one parameter groups in G.

Let $\exp_G$ be the exponential map of G. Then for $v \in \mathfrak{m}$ $L_{\exp_G \tau v} \cdot p_0$ is the geodesic through p_0. The derivative of $L\exp_G \tau v$ at p_0 is parellel translation along $L_{\exp_G \tau v} \cdot p_0$ with respect to the connection D. Since ∇ and D have the same geodesics they also have the same Jacobi fields. But the Jacobi equation with respect to D along $\mathrm{Y}(\tau) = L_{\exp_G \tau v} \cdot p_0$, $\dot{\mathrm{Y}}(0) = v$:

$$D^2_{\dot{\mathrm{Y}}} \quad - T(\dot{\mathrm{Y}}, D_{\dot{\mathrm{Y}}} X) + B(X, \dot{\mathrm{Y}})\dot{\mathrm{Y}} = 0,$$

is much simpler since T and B are D parallel. If we write X as $X(\tau) = d(L_{\exp_G \tau v})p_0$ $(Y(\tau))$, then the Jacobi equation reads:

$$Y'' - T(Y') + B(Y) = 0,$$

where

$$T(Y) = T(v, Y) = -[v, Y]_{\mathfrak{m}},$$

$$B(Y) = B(Y, v)v = -[v, [v, Y]_{\mathfrak{h}}].$$

This is a differential equation in the vector space $\mathfrak{m}$ with constant coefficients, T is skew symmetric and B is symmetric. The solutions of this equation are obtained by substituting $Y(\tau) = A(\tau) \cdot e^{m\tau}$, where m is a complex number and $A(\tau)$ a complex vector valued polynomial. The real and complex parts of these solutions then give a basis of the Jacobi fields along Y.

We shall use the following theorem given by W. Ziller [7].

Theorem: *If one solves the Jacobi equation on a compact naturally reductive Riemannian homogeneous space in the form $Y(\tau) = A(\tau) \cdot e^{m\tau}$ with $A(\tau)$ a polynomial with D parallel complex vector fields as coefficients and a complex number, one has:*
(i) *m is imaginary or 0;*
(ii) *if m is imaginary and $\neq 0$, then $A(\tau)$ is a constant polynomial so that the corresponding Jacobi fields are of the form:*

$$Y(\tau) = \operatorname{Re} A \cos a\tau - \operatorname{Im} A \sin a\tau$$

$$Y(\tau) = \operatorname{Re} A \sin a\tau + \operatorname{Im} A \cos a\tau$$

with $m = i \cdot a$ and $A(\tau) = A$ a D parallel vector field with $(m^2 \mathrm{Id} - mT + B)A = 0$;
(iii) *if $m = 0$, then $A(\tau) = A_1\tau + A_0$ with A_1 and A_0 D parallel (real) vector fields are the only possible Jacobi fields, where $B(A_1) = 0$ and $B(A_0) = T(A_1)$.*

3. Focal locus of submanifold. Our first aim is to specify the form of Jacobi field given in the above theorem so as to give an L-Jacobifield. Since parallel vector fields $E_1, \ldots, E_{n-1}$ with $\dot{Y}(\tau)$ form an orthonormal base for $T_{Y(\tau)}M$, $Y(\tau)$ defined along Y can be expressed as

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau) \cos a\tau - \beta_i(\tau) \sin a\tau)\ E_i(\tau)$$

$$Y(\tau) = \sum_{i=1}^{n-1} (\alpha_i(\tau) \sin a\tau + \beta_i(\tau) \cos a\tau)\ E_i(\tau).$$

Where $\alpha_i(\tau)$ and $\beta_i(\tau)$ are real and imaginary part of $A(\tau)$ respectively, where $A(\tau)$ is a constant polynomial corresponding to (ii) of the above theorem.

Condition (i) for $Y(\tau)$ to be L-Jacobifield is satisfied by the structure of $Y(\tau)$. For condition (ii)

$$Y(0) = \sum_{i=1}^{k} \alpha_i(0) E_i(0) \in T_z L$$

and therefore $\alpha_{k+1} = \ldots = \alpha_{n-1} = 0$, if $(E_1, \ldots, E_k)$ is base for $T_z L$, and

$$Y(0) = \sum_{i=1}^{k} \beta_i(0)\, E_i(0) \in T_z L$$

and therefore $\beta_{k+1} = \cdots = \beta_{n-1} = 0$, if

$$(E_1, \cdots, E_k) \text{ is base for } T_z L.$$

For condition (iii), we have

$$\nabla_{\dot{Y}(\tau)} Y(\tau) = \sum_{i=1}^{n-1} (-\alpha_i\, a \sin a\tau - \beta_i\, a \cos a\tau) E_i(\tau)$$

since E_i are parallel. Therefore

$$\nabla_{\dot{Y}(0)} Y(0) = \sum_{i=1}^{n-1} -a\,\beta_i E_i(0)$$

and

$$\sigma_{\dot{Y}(0)} Y(0) = \sigma_{\dot{Y}(0)} \sum_{i=1}^{k} \alpha_i E_i(0) = \sum_{i=1}^{k} \alpha_i \lambda_i E_i(0),$$

where λ_i are eigen values of symmetric linear transformation $\sigma_{\dot{Y}(0)}$ applied on eigen vectors E_i.
Therefore

$$\nabla_{\dot{Y}(0)} Y(0) + \sigma_{\dot{Y}(0)} Y(0) = \sum_{i=1}^{n-1} -a\,\beta_i E_i + \sum_{i=1}^{k} \alpha_i \lambda_i E_i \in N_z L,$$

if

$$-a\,\beta_i + \alpha_i \lambda_i = 0 \text{ for } i = 1, \cdots, k.$$

Similarly for the second form of $Y_{(\tau)}$, we have

$$\nabla_{\dot{Y}(0)} Y(0) + \sigma_{\dot{Y}(0)} Y(0) = \sum_{i=1}^{n-1} a\,\alpha_i E_i + \sum_{i=1}^{k} \beta_i \lambda_i E_i \in N_z L,$$

if

$$a\,\alpha_i + \beta_i \lambda_i = 0 \text{ for } i = 1, \cdots, k.$$

So, the general form of L-Jacobi fields are of the form

$$Y(\tau) = \sum_{i+1}^{k} \beta_i \left(\frac{a}{\lambda_i} \cos a\tau - \sin a\tau\right) E_i - \sum_{i=k+1}^{n-1} \beta_1 \sin a\tau\, E_i$$

and

$$Y(\tau) = \sum_{i=1}^{k} \beta_i \left(-\frac{\lambda_i}{a} \sin a\tau + \cos a\tau\right) E_i - \sum_{i=k+1}^{n-1} \alpha_i \sin a\tau\, E_i.$$

For finding the location of focal point of L along Y we consider the following suitable form of L-Jacobifield from the above structure.

$$Y(\tau) = \beta\left(\frac{a}{\lambda} \cos a\tau - \sin a\tau\right) E(\tau)$$

and

$$Y(\tau) = \beta\left(-\frac{\lambda}{a} \sin a\tau + \cos a\tau\right) E(\tau).$$

Let $Y(\tau_0)$ is a focal point of L along Y. Then $Y(\lambda_0) = 0$, i. e.

$$\beta\left(\frac{a}{\lambda} \cos a\tau_0 - \sin a\tau_0\right) = 0, \quad \beta\left(-\frac{\lambda}{a} \sin a\tau_0 + \cos a\tau_0\right) = 0.$$

Consequently

$$\cot a\tau_0 = \frac{\lambda}{a},$$

in both the forms. Let τ_1 be the smallest positive solution of the above equation. Then the first focal point to L along Y occurs at $Y(\tau_1)$. Let $\tau_1, \tau_2, \cdots, \tau_k$ are the solutions of the above equation s. t. $0<\tau_1<\tau_2<\cdots<\tau_k<\beta$.
Since $\lambda_1, \cdots, \lambda_k$ are eigenvalues of the symmetric linear transformation and therefore they define the principal curvatures of L at z, consequently

$$\rho_1=1/\lambda_1, \cdots, \rho_k=1/\lambda_k$$

will be corresponding principal radii of curvatures. Therefore k focal points will lie along Y at distances $\rho_1=(1/a)\tan a\tau_1, \cdots, \rho_k=(1/a)\tan a\tau_k$.

Theorem: *Let M be a C^∞ n-dimensional compact naturally reductive Riemannian homogeneous space and L be a C^∞ k-dimensional submanifold embedded in M. Let* $Y:[0,\beta]\to M$ *be a normal geodesic with* $Y(0)=Z\in L$, $\dot{Y}(0)=u\in N_zL$.
If

$$Y(\tau)=\sum_{i=1}^{n-1}(\alpha_i(\tau)\cos a\tau-\beta_i(\tau)\sin a\tau)E_i$$

$$Y(\tau)=\sum_{i=1}^{n-1}(\alpha_i(\tau)\sin a\tau+\beta_i(\tau)\cos a\tau)E_i$$

where $\dot{Y}_{(\tau)}, E_1(\tau), \cdots, E_{n-1}(\tau)$ *is an orthonormal base for* $T_{Y(\tau)}M$, *are Jacobi fields defined along* Y. *Then the location of k-focal points of L along* Y *is given by the equation*

$$(1/a)\tan a\tau_i=\rho_i, \quad i=1, \cdots, k.$$

k focal points along Y will generate k sets of focal locus in M.

4. Special form of Jacobi fields and focal points. A vector field X is called a Killing vector field if the operator $A_X=\nabla_X$ is skew symmetric. This is equivalent to saying that the one parameter group φ_s generated by X consists of isometries.

A Killing vector field X restricted to a geodesic Y is a Jacobi field, since $X\circ Y(\tau)=\left.\frac{d}{ds}\right|_{s=0}\varphi_s\circ Y(\tau)$ and for each s, $\varphi_s\circ Y(\tau)$ is a geodesic. Jacobi fields which are restrictions of Killing vector fields are called isotropic Jacobi fields. An isotropic Jacobi field Y along
$Y(\tau)=\exp\tau v$, $v\in\mathfrak{m}$ satisfies [7]

$$Y(0)=\begin{cases}0 & \text{if } X\in\mathfrak{h},\\ X & \text{if } X\in\mathfrak{m}\end{cases}$$

and

$$\nabla Y(0)=\begin{cases}[X, v] & \text{if } X\in\mathfrak{h},\\ \frac{1}{2}[X, v]_{\mathfrak{m}} & \text{if } X\in\mathfrak{m}.\end{cases}$$

Let $E=\{W\in T_{p_0}M\,|\,\langle v, W\rangle=0\}$. If $X\in E\subset\mathfrak{m}$ then also $[X, v]_{\mathfrak{m}}\in E$: $\langle[X, v]_{\mathfrak{m}}, v\rangle=-\langle X, [v, v]_{\mathfrak{m}}\rangle=0$.

Thus the Jacobi fields coming from $X\in\mathfrak{m}$ can be restricted to $X\in E$ and all Jacobi fields with initial condition:

$$(Y(0), \nabla Y(0))=(X, [\tfrac{1}{2}X, v]_{\mathfrak{m}}), \; X\in E,$$

are isotropic. These are already half of all Jacobi fields.

To study Jacobi fields coming from $X\in\mathfrak{j}$ we examine the symmetric endomorphism $B(X)=-[v, [v, X]_{\mathfrak{j}}]$. Since $B(v)=0$, B maps E into itself and let X_i, λ_i be the eigen vectors and eigen values of B/E: $B(X_i)=\lambda_i X_i$ and we set $Z_i=[v, X_i]_{\mathfrak{j}}\in\mathfrak{j}$. Then

$$[Z_i, v]=[[v, X_i]_{\mathfrak{f}}, v]=B(X_i)=\lambda_i X_i.$$

Therefore if $\lambda_i\neq 0$, the Jacobi field X_i corresponding to $Z_i\in\mathfrak{j}$ does vanish identically since $\nabla Y_i(0)=[Z_i, v]\neq 0$.

Let $E=E_0\oplus E_1$ with E_0 the o-eigenspace of B and E_1 the sum of the eigenspaces with $\lambda_i\neq 0$. Then the Jacobi fields with initial condition

$$(Y(0), \nabla Y(0))=(0, X), \ X\in E_1,$$

are isotropic Jacobi fields.

If $X\in E_0$, i. e., $B(X)=0$, then the D parallel vector field $Y(t)=d(L_{\exp_G}\tau v)_{p_0}(X)$ is a Jacobi field. The initial conditions are:

$$Y(0)=X\in E_0,$$

$$\nabla_v Y(0)=D_v Y(0)-\frac{1}{2}T(v, Y(0))=\frac{1}{2}[v, X]_{\mathfrak{m}}.$$

These Jacobi fields together with the two sets of isotropic Jacobi fields previously mentioned would generate all Jacobi fields if they were linearly independent.

Now we set $E_0=E_2\oplus E_3$ with

$$E_2=\{X\in E_0\,|\,[X, v]_{\mathfrak{m}}\in E_1\}$$

and $E_3=E_2^{\perp}$. Thus $E=E_1\oplus E_2\oplus E_3$.

Define the subspaces $V_i\subset E\oplus E$ by:

$$V_1=\{(X, \frac{1}{2}[X, v]_{\mathfrak{m}})\,|\,X\in E_1\oplus E_3\},$$

$$V_2=\{(0, X)\,|\,X\in E_1\},$$

$$V_3=\{(X, \frac{1}{2}[v, X]_{\mathfrak{m}}\,|\,X\in E_2\},$$

$$V_4=\{(X, \frac{1}{2}[v, X]_{\mathfrak{m}})\,|\,X\in E_3\},$$

$$V_5=\{(Z, X+\frac{1}{2}[v, X]_{\mathfrak{m}})\,|\,X\in E_2, \ B(X)=T(X)=[X, v)_{\mathfrak{m}}\}.$$

It has been shown in [7] that $E\oplus E=\oplus_{i=1}^{5}V_i$ and following theorem has been proved:

Theorem: $E\oplus E=\oplus_{i=1}^{5}V_i$. *On a naturally reductive homogeneous space, the Jacobi fields along* $Y(\tau)$ *can be written as linear combination of Jacobi fields with initial conditions in* V_i.

Jacobi fields with initial conditions in V_i can be considered in several forms. First we consider a Jacobi field $Y(\tau)$ orthogonal to $\dot{Y}(\tau)$, corresponding to $X_1\in V_1$, $X_2\in V_2$ in the form

$$Y(\tau)=\alpha X_1+\beta X_2, \ \alpha, \beta\in R.$$

Therefore $Y(0)=\alpha X_1$ and

$$Y'(0)=\frac{1}{2}\alpha[X_1(0), v]_{\mathfrak{m}}+\beta X_2(0).$$

Thus

$$Y'(0)+\sigma_{\dot{Y}(0)}Y(0)=\frac{1}{2}\alpha[X_1(0),\ v]_{\mathfrak{m}}+\beta X_2(0)+\lambda_1\alpha X_1(0),$$

where λ_1 is eigen value of $\sigma_{\dot{Y}}$ applied on eigen vector X_1. Now $\frac{1}{2}\alpha[X_1,\ v]_{\mathfrak{m}}$ can be decomposed tangential and normal to the submanifold L at $Y(0)=z$, therefore

$$Y'(0)+\sigma_{\dot{Y}(0)}Y(0)=\frac{\alpha}{2}[X_1,\ v]_{\mathfrak{m}}|_{T_zL}+\lambda_1\alpha X_1+\beta X_2+\frac{\alpha}{2}[X_1,\ v]_{\mathfrak{m}}|_{N_zL},$$

and right hand-side will be element of N_zL if

$$\beta X_2(0)=-\frac{\alpha}{2}[X_1(0),\ v]_{\mathfrak{m}}|_{T_zL}-\lambda_1\alpha X_1(0).$$

Now the Jacobi field $Y=\alpha X_1+\beta X_2$ is L-Jacobi field if

(1) $$\langle Y(\tau),\ \dot{Y}(\tau)\rangle=0,$$

which is satisfied by the assumption that Jacobi field is always orthogonal to $\dot{Y}$.

(2) $$Y(0)\in T_zL,$$

which is obvious since $Y(0)=\alpha X_1(0)\in T_zL$.

(3) $$Y'(0)+\sigma_{\dot{Y}(0)}Y(0)=\frac{\alpha}{2}[X_1(0),\ v]_{\mathfrak{m}}|_{N_zL}\in N_zL,$$

with $\beta X_2(0)=-\frac{\alpha}{2}[X_1(0),\ v]_{\mathfrak{m}}|_{T_zL}-\lambda_1\alpha X_1(0)$.
Therefore the form of L-Jacobi field is

$$Y(\tau)=\alpha(1-\lambda_1)X_1(\tau)-\frac{\alpha}{2}[X_1(\tau),\ v]_{\mathfrak{m}}|_{T_zL}.$$

If $Y(\tau_0)$ is a focal point along $Y(\tau)$, then $Y(\tau_0)=0$ i. e., $(1-\lambda_1)X_1(\tau_0)=\frac{1}{2}[X_1(\tau_0)v]_{\mathfrak{m}}|_{T_zL}$.
Thus we have

Theorem 4.1. *Let M be a C^∞ homogeneous compact naturally reductive Riemannian space and L be a C^∞ submanifold embedded in M. Let $Y:[0,\ \beta]\to M$ be a normal geodesic with $Y_{(0)}=z\in L$, $\dot{Y}_{(0)}=u\in N_zL$. If a Jacobi field $Y(\tau)$ defined orthogonally along $Y(\tau)$ has the form $Y(\tau)=\alpha X_1(\tau)+\beta X_2(\tau)$, $\alpha,\ \beta\in R$, with initial conditions in $V_i(i=1,\ 2)$, then a point $Y_{(\tau_0)}$ along $Y_{(\tau)}$ is a focal point if following condition is satisfied:*

$$(1-\lambda_1)X_1(\tau_0)=\frac{1}{2}[X_1(\tau_0),\ v]_{\mathfrak{m}}|_{T_zL}.$$

From the definition of V_1, V_2, V_3, it is evident that if the Jacobi field $Y(\tau)$ defined along geodesic $Y_{(\tau)}$ corresponding to $X_2\in V_2$, $X_3\in V_3$ has the form

$$Y(\tau)=\alpha_1X_2(\tau)+\beta_1X_3(\tau),\ \alpha_1,\ \beta_1\in R,$$

then a point $Y_{(\tau_0')}$ is a focal point along Y if

$$(1-\lambda_3)X_3(\tau_0')=\frac{1}{2}[v,\ X_3(\tau_0')]_{\mathfrak{m}}|_{T_zL}.$$

Consider the Jacobi field $Y(\tau)$ corresponding to $X_1 \in V_1$, $X_3 \in V_3$ in the form

$$Y(\tau)=\alpha_2 X_1(\tau)+\beta_2 X_3(\tau),\ \alpha_2,\ \beta_2 \in R.$$

By simple calculation, one can see the form of L-Jacobi field is given by

$$Y(\tau)=\frac{1}{\lambda_3}[(\lambda_3-\lambda_1)\,\alpha_2\,X_1(\tau)-\{\frac{\alpha_2}{2}[X_1(\tau),\ v]_{\mathfrak{m}}+\frac{\beta_2}{2}[v,\ X_3(\tau)]_{\mathfrak{m}}\}|_{T_zL}],$$

where λ_1, λ_3 are the eigen values of $\sigma_{\dot{Y}(0)}$ corresponding to eigen vectors X_1 and X_3 respectively.
Thus we have

Theorem (4.2). *Let M be a compact naturally reductive Riemannian homogeneous space and L be a submanifold embedded in M. If the Jacobi field defined orthogonally along normal geodesic $Y_{(\tau)}$ to L has the form*

$$Y(\tau)=\alpha_2 X_1(\tau)+\beta_2 X_3(\tau)$$

with initial conditions in $V_i (i=1,3)$. Then a point $Y_{(\tau_0'')}$ is a focal point along $Y_{(\tau)}$ if

$$(\lambda_3-\lambda_1 X_1(\tau_0''))=\frac{1}{2}\{[X_1(\tau_0''),\ v]_{\mathfrak{m}}+\frac{\beta_2}{\alpha_2}[v,\ X_3(\tau_0'')]_{\mathfrak{m}}\}|_{T_zL}.$$

Analogous theorem can be obtained by considering the Jacobi fields in different forms with initial conditions in V_i.

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