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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NON-OSCILLATORY SOLUTIONS AND OSCILLATION OF ALL SOLUTIONS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper necessary and sufficient conditions are obtained for the existence of non oscillatory solutions and for oscillation of all solutions of a class of second order functional differential equations with a deviation depending on the solution we seek and on its first derivative.
The oscillation results are applied to a model from the theory of rocket motors.

I. Introduction. In this paper we will find necessary and sufficient conditions for the existence of non-oscillatory solutions and for oscillation of all solutions of the equation

$$(1)\qquad (r(t)y'(t))'+f(t, y(t), y(g(t, y(t), y'(t))), y'(t), y'(g(t, y(t), y'(t))))=0,\ t\geqq t_0\in R$$

in the cases when

$$(2)\qquad \int_{t_0}^{\infty}\frac{dt}{r(t)}<\infty$$

and

$$(3)\qquad \int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty$$

as well as sufficient conditions for oscillation of all solutions of the equation

$$(4)\qquad y''(t)+a(t)y^n(t)+F(t, y(t), y(G(t, y(t), y'(t))), y'(t), y'(G(t, y(t), y'(t))))=0,$$

where n is a positive real number and $n\neq 1$.

Equation (1) includes as a particular case the model [1] for perturbation of the velocity of the spray of fuel in liquid propellant rocket motors

$$x''(t)+(\alpha+\beta p)x'(t)+\alpha\beta p x(t)+\gamma x(t-\Delta)=\delta_1 x(t)x'(t)+\delta_2 x^2(t),$$

where

$$\Delta=\Delta(p)+h_1 p\Delta'(p)x(t)+h_2\Delta'(p)x'(t)+\frac{[h_1 p^2\Delta''(p)+h_3\Delta'(p)]x^2(t)}{2}+h_4 p\Delta''(p)x(t)x'(t)$$
$$+\frac{h_2^2\Delta''(p)\,[x'(t)]^2}{2}$$

$$p=\frac{\rho A_2 L^* c^* v_0}{L^* c^*},\ \alpha=\frac{RT_c}{L^* c^*},\ \beta=\frac{AV_c}{\rho l A_2 L^* c^*},\ \gamma=\frac{RT_c A_2}{lV_c},\ \delta_1=-\frac{A}{l},\ \delta_2=-\frac{\alpha A}{2l},$$

$$h_1=-\frac{AV_c}{A_2 L^* c^*},\ h_2=-\rho l,\ h_3=-\rho A,\ h_4=h_1 h_2,\ A=\frac{(A_1A_2)^2-(A_2A_3)^2}{(A_1A_3)^2}+K,\ l=l_1\frac{A_3}{A_1}+\frac{A_2}{A_3},$$

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V_c — volume of the combustion chamber, T_c — absolute temperature in the combustion chamber, p — pressure in the combustion chamber, ρ — density of the fuel, R — combustion coefficient of the product per unit mass, υ_0 — velocity of the flow in the feed-line, L^* — characteristic length of the motor, c^* — characteristic velocity of the flow, A_1 — cross-section of the entry of the reservoir, A_2 — cross-section of the pipeline, A_3 — cross-section of the exit of the nozzle, l_1 — length of the reservoir, l_2 — length of the pipeline, l_3 — length of the nozzle.

Necessary and sufficient conditions for asymptotical stability of the solutions of the linear equation

$$x''(t)+(\alpha+\beta_p)x'(t)+\alpha\beta p x(t)+\gamma x(t-\Delta)=0$$

have been obtained in [2]. Here we present some new results on the behaviour of the solution of a more general equation

$$x''(t)+(\alpha+\beta p)x'(t)+\alpha\beta p x(t)+\gamma x^{\sigma}(t-\Delta)e^{\alpha\sigma t}x(t)=0, \tag{5}$$

where $\sigma>1$ is even and

$$\Delta=\Delta(p)+h_1 p\Delta'(p)x(t)+h_2\Delta'(p)x'(t). \tag{6}$$

In what follows we assume that the functions r, f and g satisfy conditions (H):
H1. $r(t)\in C^1([t_0, \infty); (0, \infty))$, $r'(t)\geq 0$;
H2. $f(t, u_1, u_2, u_3, u_4)\in C([t_0, \infty)\times R^4$, $u_1 f(t, u_1, u_2, u_3, u_4)>0$ for $u_1\neq 0$, $u_2, u_3, u_4\in R$;
H3. $g(t, \upsilon_1, \upsilon_2)\in C([t_0, \infty)\times R^2)$, $g(t, \upsilon_1, \upsilon_2)\to\infty$ as $t\to\infty$ for any $\upsilon_1, \upsilon_2\in R$ fixed and $g(t, \upsilon_1, \upsilon_2)\leq g(t, \bar\upsilon_1, \bar\upsilon_2)$ for $\upsilon_1\leq\bar\upsilon_1, \upsilon_2\geq\bar\upsilon_2$, $t\geq t_0$; and the functions a, F and G satisfy conditions $(\bar H)$:
$\bar{H}1$. $a(t)\in C([t_0, \infty); (0, \infty))$;
$\bar{H}2$. $F(t, u_1, u_2, u_3, u_4)\in C([t_0, \infty)\times R^4)$, $\frac{F(t, u_1, u_2, u_3, u_4)}{u_1^n}\geq 0$ for $u_1\neq 0$, $u_2, u_3, u_4\in R$, $t\geq t_0\in R$ and $0<n\neq 1$;
$\bar{H}3$. $G(t, \upsilon_1, \upsilon_2)\in C([t_0, \infty)\times R^2)$, $G(t, \upsilon_1, \upsilon_2)\to\infty$ as $t\to\infty$ for any $\upsilon_1, \upsilon_2\in R$ fixed where $R=(-\infty, \infty)$ and $R^k=R\times\cdots\times R$ (k times).

The continuous function $\psi\colon [t_0, \infty)\to R$ is said to be oscillatory if there exists an infinite set $\{\tau_\nu\}_{\nu=1}^{\infty}\subset[t_0, \infty)$ of zeros of $\psi(t)$ such that $\tau_\nu\to\infty$ as $\nu\to\infty$; otherwise the function $\psi(t)$ is said to be non-oscillatory.

Denote $\rho(t)=\int_t^\infty \frac{ds}{r(s)}$, $R(t)=\int_{t_0}^t \frac{ds}{r(s)}$, $\bar g(t)=g(t, y(t), y'(t))$, $g_*(t, \upsilon_1, \upsilon_2)=\min\{t, g(t, \upsilon_1, \upsilon_2)\}$ and $g^*(t, \upsilon_1, \upsilon_2)=\max\{t, g(t, \upsilon_1, \upsilon_2)\}$.

II. Main results.

1. The case $\int_{t_0}^\infty \frac{dt}{r(t)}<\infty$

Lemma 1. *Let conditions (H) and (2) hold and*

$$\inf_{t\geq t_0}\{r(t)\rho(t)\}=q=\text{const}>0. \tag{7}$$

Then for each non-oscillatory solution $y(t)$ of (1) there exists a number $t_1\geq t_0$ such that for $t\geq t_1$, $y't$ is bounded and with constant signs,

$$-r(t)y'(t)\rho(t)\leq y(t)\leq k^+ \quad \textit{when} \quad y(t)>0 \quad \text{and} \quad k^+=\text{const}>0 \tag{8}$$

and

(9) $$k^{-}\leq y(t)\leq -r(t)y'(t)\rho(t) \quad \textit{when} \quad y(t)<0 \quad \textit{and} \quad k^{-}=\text{const}<0.$$

Proof. Let $y(t)$ be a non-oscillatory solution of (1) and, for instance, $y(t)<0$ for $t\geq T_1\geq t_0$ (the proof is similar when $y(t)>0$ for $t\geq T_1\geq t_0$). (1) and H2 imply $(r(t)y'(t))>0$, i. e. $|y'(t)|>0$ for $t\geq T_2\geq T_1$ and

(10) $$r(t)y'(t)\geq r(\bar t)y'(\bar t) \quad \text{for any} \quad t\geq \bar t\geq T_2.$$

Dividing (10) by $r(t)$, integrating from $\bar t$ to t and letting $t\to\infty$ we obtain

$$0>y(t)\geq y(\bar t)+r(\bar t)\,y'(\bar t)\int_{\bar t}^{t}\frac{ds}{r(s)}\xrightarrow[t\to\infty]{} y(\bar t)+r(\bar t)y'(\bar t)\rho(\bar t),$$

i. e. $y(\bar t)\leq -r(\bar t)\,y'(\bar t)\rho(\bar t)$. But $\bar t$ is arbitrary, hence the right-hand side inequality of (9) holds. In order to obtain the left-hand side inequality of (9) and the boundedness of $y'(t)$ we shall consider the cases when $y'(t)>0$ and $y'(t)<0$ separately.

Let $y'(t)>0$ for $t\geq T_2$. Since $y(t)<0$ for $t\geq T_1$, we can find $k^{-}=\text{const}<0$ and $T_3\geq T_2$ such that $y(t)\geq k^{-}$ for $t\geq T_3$ which proves the left-hand side of (9). From (9) and (7) we obtain

$$0<y'(t)\leq -\frac{k^{-}}{r(t)\rho(t)}\leq -\frac{k^{-}}{q} \quad \text{for} \quad t\geq T_4\geq T_3.$$

Let $y'(t)<0$ for $t\geq T_2$. Dividing (10) with $\bar t\geq T_2$ by $r(t)$, integrating from T_2 to t and letting $t\to\infty$, we get

$$y(t)\geq y(T_2)+r(T_2)y'(T_2)\int_{T_2}^{t}\frac{ds}{r(s)}\xrightarrow[t\to\infty]{} y(T_2)+r(T_2)y'(T_2)\rho(T_2)=k^{-}=\text{const},$$

i. e. the left-hand side of (9). It is easy to see that $k^{-}<0$.

Let $\bar t=T_2$ and $r(T_2)\,y'(T_2)=a<0$. Dividing (10) by $r(t)$ and using H1, we have $0>y'(t)\geq\frac{a}{r(t)}\geq\frac{a}{r(T_2)}$ for $t\geq T_2$.

Lemma 1 is thus proved.

Theorem 1. *Let the following conditions hold:*
1. *The conditions of lemma 1 are valid.*
2. *The function $f(t, u_1, u_2, u_3, u_4)$ satisfies either*

(11) $$|f(t, u_1, u_2, u_3, u_4)|\leq|f(t, \bar u_1, \bar u_2, \bar u_3, \bar u_4)| \quad \textit{for} \quad |u_i|\leq|\bar u_i|, \quad u_i\bar u_i\geq 0$$

or

(12) $$|f(t, u_1, u_2, u_3, u_4)|\geq|f(t, \bar u_1, \bar u_2, \bar u_3, \bar u_4)| \quad \textit{for} \quad |u_i|\leq|\bar u_i|, \quad u_i\bar u_i\geq 0$$

for $t\geq t_0$, $i=\overline{1, 4}$.

3. $\int_{t_0}^{\infty}\rho(t)\,|f(t, c, c, c', c')|\,dt=\infty$ *for any* $c\neq 0$, $c'\in R$.

Then all non-oscillatory solutions of (1) tend to zero as $t\to\infty$.

Proof. Let, for instance $y(t)>0$ for $t\geq t_1\geq t_0$ (the proof is similar when $y(t)<0$ for $t\geq t_1\geq t_0$). In virtue of lemma 1 $y(t)$ is monotone and $y(t)$ and $y'(t)$ are bounded for $t\geq t_2\geq t_1$, i. e. there exist constants $L\geq 0$ and $M>0$ such that $\lim_{t\to\infty} y(t)=L$ and $0<|y'(t)|\leq M$ for $t\geq t_2$.

If we suppose that $L>0$, then for each $\varepsilon \in (0, L)$ there exists $t_3 \geqq t_2$ such that $|y(t)-L|<\varepsilon$ for $t \geqq t_3$. Let $m=M$ when $y'(t)>0$ and $m=0$ when $y'(t)<0$. Applying H3, we obtain $\bar{g}(t) \geqq g(t, L-\varepsilon, m) \to \infty$ as $t \to \infty$, hence $g(t, y(t), y'(t)) \geqq t_3$ for $t \geqq t_4 \geqq t_3$. Then $|y(g(t, y(t), y'(t))-L|<\varepsilon$ and $0<|y'(g(t, y(t), y'(t))| \leqq M$ for $t \geqq t_4$.

Let $c=L-\varepsilon$ and $c'=0$ when (11) holds, and $c=L+\varepsilon$ and $c'=M$ when (12) holds. Then

$$f(t, y(t), y(\bar{g}(t)), y'(t), y'(\bar{g}(t))) \geqq f(t, c, c, c', c') \quad \text{for} \quad t \geqq t_4. \tag{13}$$

Multiplying (1) by $\rho(t)$, integrating from t_4 to t, applying (13) and (8) and letting $t \to \infty$, we get

$$0=r(t)y'(t)\rho(t)+y(t)-r(t_4)y'(t_4)\rho(t_4)-y(t_4)$$

$$+\int_{t_4}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))\rho(s)ds \geqq -r(t_4)y'(t_4)\rho(t_4)-y(t_4)$$

$$+\int_{t_4}^{t} \rho(s)f(s, c, c, c', c')ds \xrightarrow[t\to\infty]{} -r(t_4)y'(t_4)\rho(t_4)-y(t_4)+\int_{t_4}^{\infty} \rho(s)f(s, c, c, c', c')ds,$$

i. e. $\int_{t_4}^{\infty} \rho(s)f(s, c, c, c', c')ds<\infty$ which contradicts condition 3 of theorem 1. Thus $L=0$ and theorem 1 is proved.

Now we shall obtain necessary and sufficient conditions for the existence of a non-oscillatory solution $y(t)$ of (1) such that $\lim_{t\to\infty} y(t)=\text{const}\neq 0$ and $\lim_{t\to\infty} \frac{y(t)}{\rho(t)}=\text{const}\neq 0$.

Theorem 2. *Let the following conditions hold:*

1. *Conditions 1 and 2 of theorem 1 are fulfilled.*

2. $r(t) \geqq 1$ *for* $t \geqq t_0$ *and the functions* $\frac{1}{r(\cdot)}$, $f(t, ., ., ., .)$ *and* $g(t, ., .)$ *are Lipschitz continuous with Lipschitz constants* ν, ξ_0 *and* $\eta_0>0$, *respectively.*

3. $\sup_{t\geqq t_0} |f(t, c, c, c', c')|<\infty$ *and* $\int_{t_0}^{\infty} |f(t, c, c, c', c')|dt<\infty$ *for some* $c\neq 0$ *and some* $c' \in R$.

4. $g(t, \upsilon_1, \upsilon_2) \leqq t$ *for any* $\upsilon_1, \upsilon_2 \in R$.

Then there exists a non-oscillatory solution of (1) with a non-zero limit as $t\to\infty$ *iff*

$$\int_{t_0}^{\infty} \rho(t)|f(t, c, c, c', c')|dt<\infty \tag{14}$$

where the constants c and c' are the same as in condition 3.

Proof. Necessity. Let $y(t)$ be a non-oscillatory solution of (1) with $\lim_{t\to\infty} y(t)\neq 0$. If we suppose that condition 3 of theorem 1 holds, then by this theorem we conclude that all non-oscillatory solutions of (1) tend to zero as $t\to\infty$ which is a contradiction.

Sufficiency. Let (14) hold and $c>0$ (the proof is similar when $c<0$). Denote $\delta=c$ and $c'=\delta$ when (11) holds and $\delta=2c$ and $c'=0$ when (12) holds. Applying (2), condition 3 of theorem 2 and (14), we can find $t_1 \geqq t_0$ such that $\rho(t)<1$ for $t \geqq t_1$,

$$\int_{t_1}^{\infty} f(t, c, c, c', c')dt \leqq \delta, \quad \int_{t_1}^{\infty} \rho(t)f(t, c, c, c', c')dt \leqq \frac{\delta}{2} \tag{15}$$

and by H3 we can find $t_2 \geq t_0$ so that $g\left(t, \frac{\delta}{2}, c'\right) \geq t_0$ for $t \geq t_2$. Let $T_1 = \max\{t_1, t_2\}$, $T_* = \inf_{t \geq T_1} g\left(t, \frac{\delta}{2}, c'\right)$ and $T_0 = \min\{T_1, T_*\}$.

Denote by C^1 the space of all continuously differentiable functions $y\colon [T_0, \infty) \to R$ with the topology defined by the family of semi-norms $\|y\|_\tau = \sup_{[T_0, \tau]} \{|y(t)| + |y'(t)|\}$, where $\tau > T_0$ and τ is an integer, by B^1 — the set of all monotone functions $y \in C^1$ for which

$$
(16) \qquad \begin{aligned} &0 \leq |y'(t)| \leq \delta, \quad \frac{\delta}{2} \leq y(t) \leq \delta \quad \text{for } t \geq T_0 \text{ and} \\ &|y'(t) - y'(\bar{t})| \leq \alpha' \, |t - \bar{t}| \quad \text{for } t, \bar{t} \geq T_0, \end{aligned}
$$

where $\alpha' = \nu\delta + f_0$ and $f_0 = \sup_{t \geq T_0} f(t, c, c, c', c')$ and by $A\colon B^1 \to C^1$ the operator defined by the formula

$$
(Ay)(t) = \begin{cases} \frac{\delta}{2} + \rho(t) \int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds + \int_t^\infty \rho(s) f(s, y(s), y(\bar{g}(s)), y'(s), \\ \qquad y'(\bar{g}(s)))ds, \quad t \geq T_1 \\ \frac{\delta}{2} + \int_{T_1}^{\infty} \rho(s) f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds, \quad t \in [T_0, T_1]. \end{cases}
$$

It is easy to see that C^1 is a Fréchet space and B^1 is bounded, convex and closed. Let $y \in B^1$. Then the function $(Ay)(t)$ is continuous in $[T_0, \infty)$. From (16) and H3 it follows that $\bar{g}(s) \geq g\left(s, \frac{\delta}{2}, c'\right) \geq T_0$ for $t \geq T_1$ and then from H2, (16) and condition 2 of theorem 1 we get

$$
(17) \qquad 0 < f(t, y(t), y(\bar{g}(t)), y'(t), y'(\bar{g}(t))) \leq f(t, c, c, c', c') \quad \text{for } t \geq T_1.
$$

In view of the properties of $\rho(t)$ and (15) — (17) we obtain

$$
\begin{aligned} \frac{\delta}{2} \leq (Ay)(t) &\leq \frac{\delta}{2} + \int_{T_1}^{\infty} \rho(s) f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds \\ &\leq \frac{\delta}{2} + \int_{T_1}^{\infty} \rho(s) f(s, c, c, c', c')ds \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \text{for } t \geq T_0. \end{aligned}
$$

Since

$$
[(Ay)(t)]' = \begin{cases} -\frac{1}{r(t)} \int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds, \quad t \geq T_1 \\ 0, \; t \in [T_0, T_1] \end{cases}
$$

then in view of (17), (15) and the fact that $r(t) \geq 1$ for $t \geq t_0$ we obtain

$$
0 \leq |[(Ay)(t)]'| \leq \int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds \leq \int_{T_1}^{\infty} f(s, c, c, c', c')ds \leq \delta \quad \text{for } t \geq T_0.
$$

For $t, \bar{t} \in [T_0, T_1]$ we have $|[(Ay)(t) - (Ay)(\bar{t})]'| = 0$ and for $t > \bar{t} \geq T_1$ we obtain

$$
|[(Ay)(t)]' - [(Ay)(\bar{t})]'| = \left| -\frac{1}{r(t)} \int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds \right.
$$

$$+\frac{1}{r(\bar{t})}\int_{T_1}^{\bar{t}} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds|$$

$$\leq\left|\frac{1}{r(\bar{t})}-\frac{1}{r(t)}\right|\int_{T_1}^{\bar{t}} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds+\frac{1}{r(t)}\int_{\bar{t}}^{t} f(s, y(s), y(\bar{g}(s)), y'(s),$$

$$y'(\bar{g}(s)))ds\leq\nu|t-\bar{t}|\int_{T_1}^{\infty} f(s, c, c, c', c')ds+\int_{\bar{t}}^{t} f(s, c, c, c', c')ds\leq(\nu\delta+f_0)|t-\bar{t}|=\alpha'|t-\bar{t}|$$

while $$|[(Ay)(t)]'-[(Ay)(\bar{t})]'|=\frac{1}{r(t)}\int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds$$

$$\leq\int_{\bar{t}}^{t} f(s, c, c, c', c')ds\leq\alpha'|t-\bar{t}|$$

for $t\geq T_1\geq\bar{t}\geq T_0$
since (17) and condition 2 of theorem 2 hold. Consequently, $A(B^1)\subset B^1$ and the functions belonging to $A(B^1)$ are equicontinuous on $[T_0, \infty)$, therefore on the compact subintervals $[T_0, \tau]$ of $[T_0, \infty)$ as well.

Now we shall show that A is continuous. Namely, if $\{y_n\}_{n=1}^{\infty}\subset B^1$ converges to $y_0\in B$ in the topology of C^1, then for $t\in[T_0, \tau]\subset[T_0, T_1]$ we obtain

$$|(Ay_n)(t)-(Ay_0)(t)|\leq\int_{T_1}^{\infty}\rho(s)|f(s, y_n(s), y_n(\bar{g}^n(s)), y_n'(s), y_n'(\bar{g}^n(s)))$$

$$-f(s, y_0(s), y_0(\bar{g}^0(s)), y_0'(s), y_0'(\bar{g}^0(s)))|ds=\int_{T_1}^{\infty}\sigma(s)F_n(s)ds$$

and $|[(Ay_n)(t)]'-[(Ay_0)(t)]'|=0$, and for $t\in[T_1, \tau]$ where $\tau>T_1$ we get

$$|(Ay_n)(t)-(Ay_0)(t)|\leq|\rho(t)\int_{T_1}^{t}f(s, y_n(s), y_n(\bar{g}^n(s)), y_n'(s), y_n'(\bar{g}^n(s)))ds$$

$$+\int_{t}^{\infty}\rho(s)f(s, y_n(s), y_n(\bar{g}^n(s)), y_n'(s), y_n'(\bar{g}^n(s)))ds-\rho(t)\int_{T_1}^{t}f(s, y_0(s), y_0(\bar{g}^0(s)), y_0'(s), y_0'(\bar{g}^0(s)))ds$$

$$-\int_{t}^{\infty}\rho(s)f(s, y_0(s), y_0(\bar{g}^0(s)), y_0'(s), y_0'(\bar{g}^0(s)))ds|\leq\int_{T_1}^{\infty}\rho(s)F_n(s)ds$$

and

$$|[(Ay_n)(t)]'-[(Ay_0)(t)]'|\leq\frac{1}{r(t)}\int_{T_1}^{t}|f(s, y_n(s), y_n(\bar{g}^n(s)), y_n'(s), y_n'(\bar{g}^n(s)))$$

$$-f(s, y_0(s), y_0(\bar{g}^0(s)), y_0'(s), y_0'(\bar{g}^0(s)))|ds\leq\int_{T_1}^{t}F_n(s)ds<\int_{T_1}^{\infty}F_n(s)ds,$$

where $F_n(s)=|f(s, y_n(s), y_n(\bar{g}^n(s)), y_n'(s), y_n'(\bar{g}^n(s)))-f(s, y_0(s), y_0(\bar{g}^0(s)), y_0'(s), y_0'(\bar{g}^0(s)))|$ and $\bar{g}^j(s)=g(s, y_j(s), y_j'(s))$ for $j=0, n$.

In order to estimate $F_n(s)$ we shall use (16), (17) and conditions 2 and 4 of theorem 2. For $s\geq T_1$ we have

$$F_n(s)\leq 2f(s, c, c, c', c') \tag{18}$$

and

$$F_n(s)\leq\xi_0\{|y_n(s)-y_0(s)|+|y_n(\bar{g}^n(s))-y_0(\bar{g}^0(s))|+|y_n'(s)-y_n'(s)|$$

$$+|y_n'(\bar g^n(s))-y_0'(\bar g^0(s))|\}\leq\xi_0\{\|y_n-y_0\|_\tau+|y_n(\bar g^n(s))-y_n(\bar g^0(s))|$$

$$(19)\qquad +|y_n(\bar g^0(s))-y_0(\bar g^0(s))|+|y_n'(\bar g^n(s))-y_n'(\bar g^0(s))|+|y_n'(\bar g^0(s))-y_0'(\bar g^0(s))|\}$$

$$\leq\xi_0\{2\|y_n-y_0\|_\tau+(\delta+\alpha')|g(s,y_n(s),y_n'(s))-g(s,y_0(s),y_0'(s))|\}$$

$$\leq\xi_0\{2\|y_n-y_0\|_\tau+\eta(\delta+\alpha')[|y_n(s)-y_0(s)|+|y_n'(s)-y_0'(s)|]\}$$

$$\leq\xi_0[2+\eta(\delta+\alpha')]\|y_n-y_0\|_\tau\xrightarrow[n\to\infty]{}0.$$

Therefore, $\rho(s)F_n(s)\leq 2\rho(s)f(s,c,c,c',c')$ for $s\geq T_1$ and $\rho(s)F_n(s)\xrightarrow[n\to\infty]{}0$ uniformly since (2) and (19) hold. Then by Lebesgue's theorem for dominated convergence we obtain $\int_{T_1}^{\infty}F_n(s)ds\xrightarrow[n\to\infty]{}0$ and $\int_{T_1}^{\infty}\rho(s)F_n(s)ds\xrightarrow[n\to\infty]{}0$, hence

$$(20)\quad \lim_{n\to\infty}[\sup_{[T_0,\tau]}|(Ay_n)(t)-(Ay_0)(t)|]=0 \text{ and } \lim_{n\to\infty}[\sup_{[T_0,\tau]}|[(Ay_n)(t)]'-[(Ay_0)(t)]'|]=0.$$

From (20) we conclude that $\|Ay_n-Ay_0\|_\tau\xrightarrow[n\to\infty]{}0$, thus A is continuous. Applying Schauder — Tychonoff fixed point theorem [3] we find $y\in B^1$ such that $y=Ay$. Thus the function $y=y(t)$ is a solution of (1) and since $y'(t)\leq 0$ for $t\geq T_1$ and $y(t)\geq\frac{\delta}{2}$ for $t\geq T_0$, we conclude that $\lim_{t\to\infty}y(t)=\text{const}\neq 0$.

Theorem 2 is thus proved.

Theorem 3. *Let conditions 1 and 2 of theorem 2 hold,* $\frac{1}{\rho(t)}$ *is Lipschitz continuous with Lipschitz constant* $\mu>0$, $g(t,\upsilon_1,\upsilon_2)\leq t$ *for any* $\upsilon_1,\upsilon_2\in R$ *and*

$$\sup_{t\geq t_0}|f(t,b\rho(t),b\rho(g(t,\beta,\beta,\beta',\beta')),b',b')|<\infty$$

for any $b,b',\beta,\beta'\in R$ *fixed.*

Then there exists a non-oscillatory solution $y(t)$ *of (1) such that* $\lim_{t\to\infty}\frac{y(t)}{\rho(t)}=\text{const}\neq 0$ *iff*

$$(21)\quad \int_{t_0}^{\infty}|f(t,c\rho(t),c\rho(g(t,\gamma,\gamma')),c',c')|dt<\infty \text{ for some } c\neq 0 \text{ and some } c',\gamma,\gamma'\in R.$$

Proof. Necessity. Let $y(t)$ be a non-oscillatory solution of (1) and $\lim_{t\to\infty}\frac{y(t)}{\rho(t)}=a=\text{const}>0$ (the proof is similar when $a<0$). Then for each $\varepsilon\in(0,a)$ there exists $t_1\geq t_0$ such that $\rho(t)\leq 1$ and

$$(22)\qquad 0<(a-\varepsilon)\rho(t)<y(t)<(a+\varepsilon)\rho(t)\leq a+\varepsilon \text{ for } t\geq t_1.$$

Then by lemma 1 we can find $t_2\geq t_1$ and $d'=\text{const}>0$ such that

$$(23)\qquad 0<|y'(t)|\leq d' \text{ for } t\geq t_2.$$

Let, for instance, $0<y'(t)\leq d'$ (the proof is similar when $-d'\leq y'(t)<0$). From H3, (22) and (23) we get

$$(24)\qquad t_2\leq g(t,0,d')\leq g(t,y(t),y'(t))\leq g(t,a+\varepsilon,0) \text{ for } t\geq t_3\geq t_2$$

and since $\rho(.)$ is decreasing, we obtain

$$(25)\qquad \rho(g(t,a+\varepsilon,0))\leq\rho(g(t,y(t),y'(t)))\leq\rho(g(t,0,d')) \text{ for } t\geq t_3.$$

From (22), applying (24) and (25), we observe that

(26) $(a-\varepsilon)\rho(g(t, a+\varepsilon, 0))\leqq(a-\varepsilon)\rho(\bar{g}(t))\leqq y(\bar{g}(t))\leqq(a+\varepsilon)\rho(\bar{g}(t))\leqq(a+\varepsilon)\rho(\bar{g}(t, 0, d'))$

for $t\geqq t_3$ and from (23) and (24) we have

(27) $$0\leqq y'(\bar{g}(t))\leqq d' \text{ for } t\geqq t_3.$$

Let $c=a-\varepsilon$, $c'=0$, $\gamma=a+\varepsilon$, $\gamma'=0$ when (11) holds and $c=a+\varepsilon$, $c'=d'$, $\gamma=0$, $\gamma'=d'$ when (12) holds. Then from (22), (23), (26) and (27) we obtain the estimate

(28) $f(t, y(t), y(\bar{g}(t)), y'(t), y'(\bar{g}(t)))\geqq f(t, c\rho(t), c\rho(g(t, \gamma, \gamma')), c', c')>0$ for $t\geqq t_3$.

Integrating (1) from t_3 to t, applying (8), (22) and (28) and letting $t\to\infty$, we have

$$0=r(t)y'(t)-r(t_3)y'(t_3)+\int_{t_3}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds$$

$$\geqq-\frac{y(t)}{\rho(t)}-r(t_3)y'(t_3)+\int_{t_3}^{t} f(s, c\rho(s), c\rho(g(s, \gamma, \gamma')), c', c')ds$$

$$\geqq-(a+\varepsilon)-r(t_3)y'(t_3)+\int_{t_3}^{t} f(s, c\rho(s), c\rho(g(s, \gamma, \gamma')), c', c')ds\xrightarrow[t\to\infty]{}$$

$$\xrightarrow[t\to\infty]{}-[a+\varepsilon+r(t_3)y'(t_3)]+\int_{t_3}^{\infty} f(s, c\rho(s), c\rho(g(s, \gamma, \gamma')), c', c')ds$$

i. e. (21) holds.

Sufficiency. Let (21) hold for $c>0$ (the proof is similar when $c<0$). Denote $\delta=\frac{c}{2}$, $c'=2\delta$ when (11) holds and $\delta=c$, $c'=0$ when (12) holds. In view of (2) and (21) we can find $t_1\geqq t_0$ so large that

(29) $$\rho(t_1)\leqq 1, \int_{t_1}^{\infty} f(t, c\rho(t), c\rho(g(t, \gamma, \gamma')), c', c')dt\leqq\delta$$

and by H3 we can find $t_2\geqq t_1$ so large that $g(t, \delta, c')\geqq t_0$ for $t\geqq t_2$. Let $T_1=\max\{t_1, t_2\}$, $T_*=\inf_{t\geqq T_1} g(t, \delta, c')$ and $T_0=\min\{T_1, T_*\}$.

Denote by C_ρ^1 the space of continuously differentiable functions $y: [T_0, \infty)\to R$ with the topology defined by the family of semi-norms $\|y\|_\tau=\sup_{[T_0, \tau]}\{\frac{|y(t)|}{\rho(t)}+|y'(t)|\}$ where $\tau\in(T_0, \infty)$ is an integer, by B_ρ^1 — the set of all $y\in C_\rho^1$ for which $\delta\rho(t)\leqq y(t)\leqq 2\delta\rho(t)$, $0\leqq|y'(t)|\leqq 2\delta$ for $t\geqq T_0$ and $|\frac{y(t)}{\rho(t)}-\frac{y(\bar{t})}{\rho(\bar{t})}|\leqq\alpha|t-\bar{t}|$, $|y'(t)-y'(\bar{t})|\leqq\alpha'|t-\bar{t}|$ for $t, \bar{t}\geqq T_0$, where $\alpha=\mu\delta+2f_0$, $\alpha'=2\nu\delta+f_0$ and $f_0=\sup_{t\geqq T_0} f(t, c\rho(t), c\rho(g(t, \gamma, \gamma')), c', c')$ and by $\Phi: B_\rho^1\to C_\rho^1$ the operator defined by the formula

$$(\Phi y)(t)=\begin{cases}\delta\rho(t)+\rho(t)\int_{T_1}^{t} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds+\int_{t}^{\infty}\rho(s)f(s, y(s), \bar{y}(g(s)), y'(s),\\ y(\bar{g}(s)))ds, \quad t\geqq T_1\\ \delta\rho(t)+\int_{t_1}^{\infty}\rho(s)f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds, \quad t\in[T_0, T_1].\end{cases}$$

Further the proof is analogous to that of theorem 2.

Theorem 3 is proved.

The following theorem guarantees oscillation of all solutions of equation (1).

Theorem 4. *In addition to conditions (H), (7) and (2) assume*:

1. *There exists* $\sigma>1$ *such that* $\frac{|f(t, u_1, u_2, u_3, u_4)|}{|u_2|^\sigma} \le \frac{|f(t, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)|}{|\bar{u}_2|^\sigma}$ *for* $|u_i|\le|\bar{u}_i|$, $u_i\bar{u}_i>0$ $(i=\overline{1,4})$ *and* $t\ge t_0$.

2. $\int_{t_0}^{\infty} (\frac{\rho(g^*(t,\gamma,\gamma'))}{\rho(g(t,\gamma,\gamma'))})^\sigma |f(t, c\rho(t), c\rho(g(t,\gamma,\gamma')), c', c')|\,dt=\infty$ *for any* $c\ne 0$, $c', \gamma, \gamma' \in R$.

Then all solutions of (1) are oscillatory.

Proof. Suppose that there exists a non-oscillatory solution $y(t)$ of (1) and let, for instance, $y(t)>0$ for $t\ge t_1\ge t_0$ (the proof is similar when $y(t)<0$ for $t\ge t_1\ge t_0$). By lemma 1

$$0<|y'(t)|\le k' \text{ for } t\ge t_2\ge t_1 \text{ and some } k'=\text{const}>0. \tag{30}$$

We shall consider the cases $y'(t)>0$ and $y'(t)<0$ for $t\ge t_2$ separately.

Let $y'(t)>0$ for $t\ge t_2$. Since $y(t)>0$ for $t\ge t_1$, we can find $t_3\ge t_2$ and $l=\text{const}>0$ so that $y(t)\ge l$ for $t\ge t_3$. But (8) implies $y(t)\le k^+$ for $t\ge t_2$, hence $\bar{g}(t)\ge g(t, l, k')\ge t_3$ and $y(\bar{g}(t))\ge l$ for $t\ge t_4$ and $y'(\bar{g}(t))>0$ for $t\ge t_4$.

But condition 1 of theorem 4 yields (11). Thus (13) holds for $c=l$, $c'=0$ and $t\ge t_4$.

On the other hand, $g^*(t,\gamma,\gamma')\ge g(t,\gamma,\gamma')$ and $\rho(.)$ decreases. Choose the numbers $t_6\ge t_5\ge t_4$ so large that $\rho(t)<1$ for $t\ge t_5$ and $g(t,\gamma,\gamma')\ge t_5$ for $t\ge t_6$. Then

$$(\frac{\rho(g^*(t,\gamma,\gamma'))}{\rho(g(t,\gamma,\gamma'))})^\sigma f,(t, c\rho(t), c\rho(g(t,\gamma,\gamma')), c', a')\le f(t, c, c, c', c') \text{ for } t\ge t_6$$

and hence

$$\int_{t_6}^{\infty} f(t, c, c, c', c')\,dt=\infty \text{ for any } c\ne 0,\ c'\in R. \tag{31}$$

Integrating (1) from t_6 to t, applying (13) and letting $t\to\infty$, we obtain $\int_{t_6}^{\infty} f(t, k', k', 0, 0)dt<\infty$ which contradicts (31).

Let $y'(t)<0$ for $t\ge t_2$. Consider the derivative

$$\begin{aligned}(-(-r(t)y'(t))^{1-\sigma})' &= (\sigma-1)(-r(t)y'(t))^{-\sigma}(-r(t)y'(t))' \\ &= (\sigma-1)(-r(t)y'(t))^{-\sigma} f(t, y(t), y(\bar{g}(t)), y'(t), y'(\bar{g}(t))) \\ &= (\sigma-1)(-r(t)y'(t))^{-\sigma}\frac{y^\sigma(\bar{g}(t)) f(t, y(t), y(\bar{g}(t)), y'(t), y'(\bar{g}(t)))}{y^\sigma(\bar{g}(t))} \text{ for } t\ge t_3.\end{aligned} \tag{32}$$

From (1) in view of the positiveness of $y(t)$ and H2 it follows that $(r(t)y'(t))<0$ for $t\ge t_1$. Then

$$-r(t)y'(t)\le -r(g^*(t,\gamma,\gamma'))y'(g^*(t,\gamma,\gamma'))\le\frac{y(g^*(t,\gamma,\gamma'))}{\rho(g^*(t,\gamma,\gamma'))} \text{ for } t\ge t_1 \tag{33}$$

since $t\le g^*(t,\gamma,\gamma')$ and (8) holds, and $r(t)y'(t)\le r(t_1)y'(t_1)$ for $t\ge t_1$.

Integrating last inequality from t to ∞, we obtain

$$y(t)\ge a\rho(t) \text{ for } t\ge t_1\ (a=-r(t_1)y'(t_1)>0). \tag{34}$$

On the other hand, $y(t)>0$ and $y'(t)<0$ for $t\geqq t_1$ ensure that $\lim_{t\to\infty} y(t)=k\geqq 0$ exists, hence

$$(35)\qquad |y(t)-k|<\varepsilon \text{ for each } \varepsilon\in(0,k) \text{ and } t\geqq t_2\geqq t_1.$$

From (30), (35) and H3 it follows that

$$(36)\qquad t_2\leqq g(t,k-\varepsilon,k')\leqq g(t,y(t),y'(t))\leqq g(t,k+\varepsilon,0) \text{ for } t\geqq t_3\geqq t_2.$$

From (34), (36) and (30) we deduce

$$(37)\qquad y(\bar g(t))\geqq a\rho(\bar g(t))\geqq a\rho(g(t,\gamma,\gamma'))\,|y'(\bar g(t))|>0 \text{ for } t\geqq t_3$$

where $\gamma=k+\varepsilon$, $\gamma'=0$.

Condition 1 of theorem 4, (34), (30) and (37) yield

$$(38)\qquad \frac{f(t,y(t),y(\bar g(t)),y'(t)y'(\bar g(t)))}{y^\sigma(\bar g(t))}\geqq\frac{f(t,a\rho(t),a\rho(g(t,\gamma,\gamma')),0,0)}{a^\sigma\rho^\sigma(g(t,\gamma,\gamma'))} \text{ for } t\geqq t_3.$$

From (36) and the negativeness of $y'(t)$ we get

$$(39)\qquad y(\bar g(t))\geqq y(g(t,\gamma,\gamma'))\geqq y(g^*(t,\gamma,\gamma')) \text{ for } t\geqq t_3.$$

Then

$$(40)\quad (-(-r(t)y'(t))^{1-\sigma})'\geqq(\sigma-1)\,\frac{\rho^\sigma(g^*,(t,\gamma,\gamma'))}{y^\sigma(g^*(t,\gamma,\gamma'))}\,y^\sigma(g^*,(t,\gamma,\gamma'))\,\frac{f(t,a\rho(t),a\rho(g(t,\gamma,\gamma'),00,)}{a^\sigma\rho^\sigma(g,(t,\gamma,\gamma))}$$

$$=\frac{\sigma-1}{a^\sigma}\Big(\frac{\rho(g^*(t,\gamma,\gamma'))}{\rho(g(t,\gamma,\gamma'))}\Big)^\sigma f(t,a\rho(t),a\rho(g(t,\gamma,\gamma')),0,0) \text{ for } t\geqq t_3$$

since (32), (33), (38) and (39) hold.

Integrating (40) from t_3 to t, we have

$$(41)\qquad \frac{\sigma-1}{a^\sigma}\int_{t_3}^{t}\Big(\frac{\rho(g^*(s,\gamma,\gamma'))}{\rho(g(s,\gamma,\gamma'))}\Big)^\sigma f(s,a\rho(s),a\rho(g(s,\gamma,\gamma')),0,0)$$

$$\leqq(-r(t_3)y'(t_3))^{1-\sigma}-(-r(t)y'(t))^{1-\sigma}<(-r(t_3)y'(t_3))^{1-\sigma}.$$

Letting $t\to\infty$ in (41), we obtain a contradiction with condition 2 of theorem 4.

Theorem 4 is thus proved.

Theorems 3 and 4 imply

Corollary 1. *Let the conditions of theorem 3 and condition 1 of theorem 4 hold and* $\inf_{t\geqq t_0}\frac{\rho(g^*(t,\gamma,\gamma'))}{\rho(g(t,\gamma,\gamma'))}>0$ *for any* $\gamma,\gamma'\in R$.

Then all solutions of (1) oscillate iff

$$\int_{t_0}^{\infty}|(t,c\rho(t),c\rho(g(t,\gamma,\gamma')),c',c')|\,dt=\infty \text{ for any } c\neq 0,\ c',\gamma,\gamma'\in R.$$

2. The case $\int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty$.

Lemma 2. *Let conditions (H) and (3) hold and $y(t)$ be a non-oscillatory solution of (1).*

Then there exists $t_1\geqq t_0$ and $a_1, a_2>0$ such that for $t\geqq t_1$ $y'(t)$ is bounded and

$$(42)\qquad y(t)y'(t)>0 \text{ and } a_1\leqq(y(t)\leqq a_2R(t).$$

P r o o f. Let $y(t)>0$ for $t \geqq t_1 \geqq t_0$ (the proof is similar when $y(t)<0$ for $t \geqq t_1 \geqq t_0$). Then (1) and H2 yield $(r(t)y'(t))'<0$, $|y'(t)|>0$ and

$$r(t)y'(t) \leqq r(t_1)y'(t_1) \quad \text{for} \quad t \geqq t_1. \tag{43}$$

Dividing (43) by $r(t)$ and integrating from t_1 to t, we get

$$y(t) \leqq y(t_1) + r(t_1)y'(t_1) \int_{t_1}^{t} \frac{ds}{r(s)} \quad \text{for} \quad t \geqq t_1. \tag{44}$$

Suppose that $y'(t)<0$ for $t \geqq t_1$. Then (44) implies the contradiction $y(t) \underset{t\to\infty}{\longrightarrow} -\infty$ since (3) holds. Consequently, $y'(t)>0$ and there exists $a_1>0$ such that $y(t) \geqq a_1$ for $t \geqq t_2 \geqq t_1$. Denote $a_2 = r(t_1)y'(t_1)>0$. Then (44) yields $y(t) \leqq a_2 R(t)$ for $t \geqq t_1$.

Dividing (44) by $r(t)$ and applying H_1, we obtain

$$0<y'(t) \leqq \frac{a_2}{r(t)} \leqq \frac{a_2}{r(t_1)}$$

i. e. $y'(t)$ is bounded.

Lemma 2 is proved.

T h e o r e m 5. *In addition to* (H) *and* (3) *assume that:*

1. *Condition 2 of theorem 1 is fulfilled.*

2. $R(.)$, $\frac{1}{r(.)}$ *and* $g(t, ., .)$ *are Lipschitz continuous with Lipschitz constants* μ, ν, η_0, *respectively,* $\sup_{t \geqq t_0} \{R(t)\,|f(t, b, b, b', b')|\}<\infty$ *for any* $b, b' \in R$ *fixed and* $g(t, \upsilon_1, \upsilon_2) \leqq t$ *for any* $\upsilon_1, \upsilon_2 \in R$.

3. $|f(t, u_1, u_2, u_3, u_4) - f(t, \bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)| \leqq \xi(t) \sum_{i=1}^{4} |u_i - \bar{u}_i|$ *where* $\xi(t)>0$ *and* $\sup_{t \geqq t_0} R(t)\xi(t)<\infty$.

Then there exists a bounded non-oscillatory solution of (1) *iff*

$$\int_{t_0}^{\infty} R(t)\,|f(t, c, c, c', c')|\,dt<\infty \quad \text{for some } c \neq 0 \text{ and some } c' \in R. \tag{45}$$

P r o o f. Necessity. Let $y(t)$ be a bounded non-oscillatory solution of (1) and, for instance, $y(t)>0$ for $t \geqq t_1 \leqq t_0$ (the proof is similar when $y(t)<0$ for $t \geqq t_1 \geqq t_0$). By lemma 2 we can find numbers $0<a_1<a_2$, $d'>0$ and $t_2 \geqq t_1$ such that $a_1 \leqq y(t) \leqq a_2$ and $0<y'(t) \leqq d'$ for $t \geqq t_2$. As in the proof of theorem 1 we obtain (13) for $t \geqq t_3 \geqq t_2$ and $c=a_1$, $c'=0$ when (11) holds and $c=a_2$, $c'=d'$ when (12) holds. Multiplying (1) by $R(t)$, integrating from t_3 to t, applying (13) and letting $t\to\infty$, we get

$$R(t)r(t)y'(t) - y(t) - R(t_3)r(t_3)y'(t_3) + y(t_3) + \int_{t_3}^{t} R(s)f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))\,ds$$

$$> -a_2 - R(t_3)r(t_3)y'(t_3) + \int_{t_3}^{t} R(s)f(s, c, c, c', c')ds \underset{t\to\infty}{\longrightarrow}$$

$$\underset{t\to\infty}{\longrightarrow} -(a_2 + R(t_3)\,r(t_3)y'(t_3)) + \int_{t_3}^{\infty} R(s)f(s, c, c, c', c')ds$$

i. e. (45).

S u f f i c i e n c y. Let (45) hold for $c>0$ (the proof is similar when $c<0$). Denote $\delta=c$, $c'=\delta$ when (11) holds and $\delta=2c$, $c'=0$ when (12) holds. In view of (3), (45) and H1 we can find a number $t_1 \geqq t_0$ so that $R(t) \geqq 1$ and $r(t) \geqq 1$ for $t \geqq t_1$,

$\int_{t_1}^{\infty} R(s)f(s, c, c, c', c')ds \leqq \frac{\delta}{2}$ and by H3 we can find $t_2 \geqq t_0$ so large that $g(t, \frac{\delta}{2}, c') \geqq t_0$ for $t \geqq t_2$. Let $T_1 = \max\{t_1, t_2\}$, $T_2 = \inf_{t \geqq T_1} g(t, \frac{\delta}{2}, c')$ and $T_0 = \min\{T_1, T_2\}$, and C^1 and B^1 be the space and the set defined in the proof of theorem 2 and $\alpha' = \frac{\nu\delta}{2} + \bar{f}$ and $\bar{f} = \sup_{t \geqq T_1} R(t)f(t, c, c, c', c')$.

Let $\psi: B^1 \to C^1$ be the operator defined by the formula

$$(\psi y)(t) = \begin{cases} \frac{\delta}{2} + \int_{T_1}^{t} R(s)f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds + R(t)\int_t^{\infty} f(s, y(s), y(\bar{g}(s)), \\ y'(s), y'(\bar{g}(s)))\,ds, \ t \geqq T_1 \\ \frac{\delta}{2} + R(T_1)\int_{T_1}^{\infty} f(s, y(s), y(\bar{g}(s)), y'(s), y'(\bar{g}(s)))ds, \ t \in [T_0, T_1]. \end{cases}$$

Further on we proceed as in the proof of theorem 2.

Theorem 5 is thus proved.

Theorem 6. *In addition to* (H) *and* (3) *assume that:*

1. *Condition 1 of theorem 4 holds.*
2. *There exists* $h_*(t, \upsilon_1, \upsilon_2) \in C^1([t_0, \infty) \times R^2)$ *such that* $h_*(t, \upsilon_1, \upsilon_2) \leqq g_*(t, \upsilon_1, \upsilon_2)$ *for any* $\upsilon_1, \upsilon_2 \in R$ *fixed and* $h_*(t, \upsilon_1, \upsilon_2) \to \infty$ *as* $t \to \infty$, $\frac{\partial h_*(t, \upsilon_1, \upsilon_2)}{\partial t} \geqq 0$.
3. $\int_{t_0}^{\infty} R(h_*(t, \gamma, \gamma'))\,|f(t, c, c, c', c')|\,dt = \infty$ *for any* $c \neq 0$, $c', \gamma, \gamma' \in R$.

Then all bounded solutions of (1) are oscillatory.

The proof is similar to that of theorem 4 and we omit it.

From theorems 5 and 6 we obtain

Corollary 2. *Let the conditions of theorem 5 and conditions 1 and 2 of theorem 6 hold and* $\inf_{t \geqq t_0} \frac{R(h_*(t, \gamma, \gamma'))}{R(t)} > 0$ *for any* $\gamma, \gamma' \in R$.

Then all bounded solutions of (1) are oscillatory iff

$$\int_{t_0}^{\infty} R(t)\,|f(t, c, c, c', c')|\,dt = \infty \quad \text{for any } c \neq 0,\ c' \in R.$$

Remark. We note that analogus results have been proved in [4]—[7] in the cases when $f = y(g(t))F((v(g(t)))^2, t)$ and $g(t) < t$; $f = f(y(g(t)), t)$ and $g(t) < t$; $f = f(y(g(t)), t)$ and $g(t) > t$; $r(t) = 1$, $f = f(t, y(g(t, y(t))))$ and $g(t, \upsilon)$ is of mixed type, respectively. While in [4—7] the authors have used Schauder's fixed point theorem, here we consider Fréchet spaces and apply the Schauder—Tychonoff fixed point theorem.

3. A comparison theorem. For equation (4) and the equation

$$y''(t) + a(t)y^n(t) = 0 \tag{46}$$

we shall prove the following comparison theorem.

Theorem 7. *Let conditions* $(\bar{H})$ *hold and let all solutions of (46) be oscillatory.* Then all solutions of (4) are oscillatory.

P r o o f. Let all solutions of (46) be oscillatory and there exists a solution $y_0(t)$ of (4) which is non-oscillatory. Since $y_0(t)\neq 0$ for $t\geq t_1\geq t_0$, then by $\overline{H}2$ we obtain

(47) $$\mathscr{F}(t)=\frac{F(t,\ y_0(t),\ y_0(\overline{G}(t)),\ y_0'(t),\ y_0'\ (\overline{G}(t)))}{y_0^n(t)}\geq 0 \quad \text{for} \quad t\geq t_1.$$

Let $n>1$. According to theorem 1 [8], the oscillation of all solutions of (46) implies

(48) $$\int_{t_0}^{\infty} ta(t)dt=\infty.$$

From (47) and (48) we observe that $\int_{t_1}^{\infty} tA(t)dt=\infty$ where $A(t)=a(t)+\mathscr{F}(t)$. Applying once more theorem 1 [8], we conclude that all solutions of the equation

(49) $$y''(t)+A(t)y(t)=0$$

are oscillatory. It is easy to see that $y_0(t)$ is a non-oscillatory solution of (49) which is a contradiction.

Let $0<n<1$. According to theorem 1 [9], the oscillation of all solutions of (46) yields

(50) $$\int_{t_0}^{\infty} t^n a(t)dt=\infty.$$

From (46) and (50) we observe that $\int_{t_1}^{\infty} t^n A(t)dt=\infty$. Applying now theorem 1 [9], we obtain that all solutions of (49) are oscillatory which contradicts the assumption that $y_0(t)$ is a non-oscillatory solution of (4). Thus, all solutions of (4) are oscillatory. Theorem 7 is proved.

4. An aplication. Finally, we will illustrate theorem 6 on equation (5) with (6).

C o r o l l a r y 3. *Let α, β, γ, p, h_1 and h_2 be defined as in the introduction and $\sigma>1$ be even. If $\alpha=\beta p$, $\Delta(p)>1/\alpha$, $\Delta'(p)<0$ and the numbers $t_0=t_0(\upsilon)$ and $\varepsilon_0=\varepsilon_0(v)>0$ are defined so that $\varepsilon_0=\Delta(p)+h_2\Delta'(p)\upsilon$ and $t_0>0$ for $\upsilon\geq 0$, and $\varepsilon_0=\Delta(p)$ and $t_0>\frac{1}{\alpha}\ln[-\alpha h_2\Delta'(p)\upsilon]$ for $\upsilon<0$, then all solutions of (5) with (6) are oscillatory for $t\geq T_0=\sigma(t_0+\varepsilon_0)$.*

In fact, by the substitution $x(t)=e^{-\alpha t}y(t)$ equation (5) is transformed into the equation

(51) $$y''(t)+\gamma e^{\alpha\sigma\overline{\Delta}}y(t)y^{\sigma}(t-\overline{\Delta})=0,$$

where $\overline{\Delta}=\Delta(p)+[(h_1p-\alpha h_2)y(t)+h_2y'(t)]\Delta'(p)e^{-\alpha t}=\Delta(p)+h_2\Delta'(p)e^{-\alpha t}y'(t)$ since $h_1p-\alpha h_2=\frac{-AV_c}{A_2L^*c^*}\cdot\frac{\rho A_2L^*c^*\upsilon_0}{V_c}+\frac{RT_c\rho l}{L^*c^*}=\rho[-Av_0+\frac{RT_cl}{L^*c^*}]=0$ when $\alpha=\beta p$, i. e. when $\frac{A\upsilon_0}{l}=\frac{RT_c}{L^*c^*}$.

It is easy to see that (51) is a particular case of (1) when $r(t)=1$, $f(t, u_1, u_2, u_3, u_4)=\gamma e^{\alpha\sigma\overline{\Delta}}u_1u_2^{\sigma}$ and $g(t, \upsilon_2)=t-\overline{\Delta}(t, \upsilon_2)$ and the functions r, f and g satisfy (H), (3) and condition 1 of theorem 4. From the choice of t_0, ε_0 and $\Delta(p)$ it follows that $\overline{\Delta}(t, \upsilon_2)>0$ for $t\geq t_0$ and for any $\upsilon_2\in R$ fixed. Then $g_*(t, \upsilon_2)=g(t, \upsilon_2)$ and the function $h_*(t, \upsilon_2)=g_*(t, \upsilon_2)$ satisfies condition 2 of theorem 6. But (3) yields

$$R(h_*(t, \upsilon_2))=h_*(t, \upsilon_2)-t_0=t-\overline{\Delta}(t, \upsilon_2)-t_0\geq\sigma(t_0+\varepsilon_0)-\varepsilon_0-t_0=(\sigma-1)(t_0+\varepsilon_0)$$

for $t \geqq T_0$ and $|f(t, c, c, c', c')| = \gamma |c| |c|^{\sigma} e^{\alpha\sigma\bar{\Lambda}(t, c')} = \gamma |c|^{\sigma+1} e^{\alpha\sigma\bar{\Lambda}(t, c')}$ for any $c \neq 0$, $c' \in R$ since $\alpha > 0$, $\sigma > 0$ and $\bar{\Delta}(t, c) > 0$.

Then

$$\int_{T_0}^{\infty} R(h_*(t, \gamma'))\, |f(t, c, c, c', c')|\, dt = \int_{T_0}^{\infty} [h_*(t, \gamma') - t_0]\gamma |c|^{\sigma+1} e^{\alpha\sigma\bar{\Lambda}(t, c')}\, dt$$

$$> \gamma |c|^{\sigma+1} (\sigma - 1)(t_0 + \varepsilon_0) \int_{T_0}^{\infty} dt = \infty.$$

Thus, all conditions of theorem 6 hold and in virtue of it the solutions of (51) hence the solutions of (5) as well, are oscillatory.

Corollary 3 is established.

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