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ON THE BOURGIN — YANG THEOREM FOR MULTI-VALUED MAPS IN THE NON-SYMMETRIC CASE, II

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1. Introduction. The Borsuk — Ulam theorem, which states that for every continuous map $f: S^n \to R^n$ there exists a point $x \in S^n$ such that $f(x)=f(-x)$ has many generalizations proceeding in various directions. In some of these generalizations the sphere is replaced by more general space and mappings by compact fields or by multi-valued maps. In particular, the author has proved the following theorem (see [4]).

Theorem (1.1). *Let $X \subset R^{n+k+1}$ $(k \geqq 0)$ be a Borsuk's set (by Borsuk's set it is meant a subset $X \subset R^{n+k+1}$ which is compact and the origin lies in a bounded component of $R^{n+k+1} \setminus X$) and $\varphi: X \to R^n$ be a multi-valued admissible mapping. Then the covering dimension of the set*

$$A_\varphi = \{x \in X: \exists \lambda > 0 \text{ such that } -\lambda x \in X \text{ and } \varphi(x) \cap \varphi(-\lambda x) \neq \emptyset\}$$

is not less than k.

Recall that an u. s. c. map $\varphi: X \to Y$ is admissible if it admits a selector $\psi: X \to Y$ which is a composition of acyclic maps (see [1, 3]). On the other hand, infinite dimensional case of the above theorem for single-valued maps has been proved in [5]. Let E^∞ denotes an infinite dimensional Banach space and let $E^{\infty-k}$ be a linear, closed subspace of E^∞ of codimension k. This theorem can be stated as follows:

Theorem (1.2). *Let X be a closed, bounded subset of E^∞ for which the origin is in a bounded component of $E^\infty \setminus X$ and let $f: X \to E^{\infty-k-1}$ be a compact vector field (i. e. a map of the form $f(x)=x-F(x)$ where $\overline{F(X)}$ is compact). Then the covering dimension of the set*

$$A_f = \{x \in X: \exists \lambda > 0 \text{ such that } -\lambda x \in X \text{ and } f(x) = f(-\lambda x)\}$$

is not less than k.

If $X \subset E^\infty$, a mapping $\varphi: X \to E^\infty$ is said to be an admissible multivalued compact field if the associated displacement mapping Φ from X into E^∞ defined by $\Phi(x)=\{x-y, y \in \varphi(x)\}$ is an admissible multi-valued compact map (comp. [2]).

The aim of this paper is to combine theorems (1.1) and (1.2). Specifically our theorem is:

Theorem (1.3). *Let X be a closed, bounded subset of E^∞ for which the origin lies in a bounded component of $E^\infty \setminus X$ and let $\varphi: X \to E^{\infty-k-1}$ be a multi-valued admissible compact field. Then the covering dimension of the set*

$$A_\varphi = \{x \in X: \exists \lambda > 0 - \lambda x \in X \text{ and } \varphi(x) \cap \varphi(-\lambda x) \neq \emptyset\}$$

is not less than k.

For $k=0$ we have the multi-valued version of Joshi's theorem proved in [7] by W. Segiet.

2. Compactness of A_φ. Analogously as in [5] we prove the following two lemmas.

Lemma (2.1). *The set A_φ is relatively compact i. e. $\bar{A}_\varphi$ is compact.*

Proof. Let $\{x_n\}_{n\in N}$ be a sequence of points in A_φ. Then for each $n\in N$ there exists a positive real number λ_n such that $-\lambda_n x_n \in X$ and

$$\varphi(x_n)\cap\varphi(-\lambda_n x_n)\neq\emptyset.$$

From the definition of a multi-valued admissible compact field we have that $(x_n-\Phi(x_n))\cap(-\lambda_n x_n-\Phi(-\lambda_n x_n))\neq\emptyset$ for $n\in N$, where $\Phi\colon X\to E^\infty$ is the multi-valued admissible compact map associated with the compact field φ. Thus for every $n\in N$ there is a point $y_n\in E^\infty$ such that $y_n\in(x_n-\Phi(x_n))$ and $y_n\in(-\lambda_n x_n-\Phi(-\lambda_n x_n))$. Let's denote $z_n^1:=x_n-y_n$ and $z_n:=-\lambda_n x_n-y_n$ for each $n\in N$ then $(1+\lambda_n)x_n=z_n^1-z_n^2\in\overline{\Phi(X)}-\overline{\Phi(X)}$.

The algebraic difference of compact sets is compact so there is a subsequence $\{(\lambda_{n_l}+1)x_{n_l}\}_{l\in N}=\{z_{n_l}^1-z_{n_l}^2\}_{l\in N}$ of the sequence $\{z_n^1-z_n^2\}_{n\in N}$ which converges to a point $z_0\in E^\infty$. Moreover, from the sequence $\{\lambda_{n_l}\}_{l\in N}$ we can choose a subsequence $\{\lambda_{n_{l_s}}\}_{s\in N}$ which converges to a positive real number λ_0. Therefore we have

$$x_{n_{l_s}}=\frac{z_{n_{l_s}}^1-z_{n_{l_s}}^2}{1+\lambda_{n_{l_s}}}\xrightarrow[s\to\infty]{}\frac{z_0}{1+\lambda_0}\in X.$$

Lemma (2.2). *The set A_φ is closed in X.*

Proof. Let $\{x_n\}_{n\in N}$ be a sequence of points in A_φ which converges to a point $x_0\in X$. There is a sequence $\{y_n\}_{n\in N}$ such that for each $n\in N\, y_n=-\lambda_n x_n$ $\lambda_n>0$ and $\varphi(x_n)\cap\varphi(y_n)\neq\emptyset$. By the lemma (2.1) we can choose a subsequence $\{y_{n_l}\}_{l\in N}$ of $\{y_n\}_{n\in N}$ in such a way that $y_{n_l}\xrightarrow[l\to\infty]{}y_0\in X$ and $\lambda_{n_l}\xrightarrow[l\to\infty]{}\lambda_0>0$. We have $\varphi(x_0)\cap\varphi(y_0)\neq\emptyset$ because φ is an u. s. c. map. Moreover,

$$\begin{array}{ccc}(1+\lambda_{n_l})\cdot x_{n_l}=x_{n_l}+\lambda_{n_l}x_{n_l}=x_{n_l}-(-\lambda_{n_l}x_{n_l}) & = & x_{n_l}-y_{n_l}\\ \downarrow{\scriptstyle l\to\infty} & & \downarrow{\scriptstyle l\to\infty}\\ (1+\lambda_0)\cdot x_0 & & x_0-y_0\end{array}$$

thus $y_0=-\lambda_0 x_0$ and $x_0\in A_\varphi$.

So we have proved that the set A_φ is compact.

3. The main result.

Now we prove the following theorem:

Theorem (3.1). *Let X be a closed, bounded subset of E^∞ for which the origin lies in a bounded component of $E^\infty\setminus X$ and let $\varphi\colon X\to E^{\infty-k-1}$ be a multi-valued admissible compact field. Then the covering dimension of the set*

$$A_\varphi=\{x\in X,\ \exists\lambda>0 \text{ such that } -\lambda x\in X \text{ and } \varphi(x)\cap\varphi(-\lambda x)\neq\emptyset\}$$

is not less than k.

Proof. Assume contrary that $\dim A_\varphi<k$.

Then there exists a single-valued map $g\colon X\to R^k$ with the following properties (see proof of the theorem (3.1) in [4]):

(1) if $\varphi(x)\cap\varphi(-\lambda x)\neq\emptyset$ for some x and $\lambda>0$ then $g(x)\neq g(-\lambda x)$

(2) $g(X)$ is compact.

We can define a map $\langle\varphi, g\rangle\colon X\to E^{\infty-k-1}\oplus R^k\simeq E^{\infty-1}$ as follows

$$\langle\varphi, g\rangle(x)=\{y+g(x),\ y\in\varphi(x)\}.$$

Proposition (1.8) in [2] implies that the map $\langle\varphi, g\rangle$ is the multi-valued admissible compact field and we can use theorem (3.1) for $k=0$ proved in [7]. There exist a ponit $x\in X$ and a positive real number λ such that $\langle\varphi, g(x)\rangle \cap \langle\varphi, g\rangle(-\lambda x)\neq\emptyset$ but this implies $y_1+g(x)=y_2+g(-\lambda x)$ for some $y_1\in\varphi(x)$ and $y_2\in\varphi(-\lambda x)$ thus $y_1=y_2$ and $g(x)=g(-\lambda x)$. Consequently $\varphi(x)\cap\varphi(-\lambda x)\neq\emptyset$ and $g(x)=g(-\lambda x)$ but in view of the condition (1) we obtain a contradiction. Hence we have $\dim A_\varphi\geq k$ and the proof is completed.

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