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GEOMETRY OF THE TANGENT BUNDLE OVER 2-DIMENSIONAL HYPERBOLIC SPACE

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The 2-dimensional Poincaré model of the hyperbolic space is well-known. Our considerations are based on this model. It is a 2-dimensional riemannian manifold $Ł_2$ with a support

$$Ł_2=\{(x^1, x^2)\in R^2: x^2>0\}.$$

The riemannian metric g in $Ł_2$ is given by

$$[g_{ij}(x^1, x^2)]=\begin{bmatrix} (\frac{k}{x^2})^2 & 0 \\ 0 & (\frac{k}{x^2})^2 \end{bmatrix},$$

where k is arbitrary positive constant ([1, 3, 4]). The significance of the Lobachevski geometry in mathematics and physics is obvious for everybody. Thus it seems to be interesting to investigate the geometry of the tangent bundle over $Ł_2$ with the pseudoriemannian metric g^c (a complete lift of g). The paper is devoted to this problem.

Let $(TŁ_2, g^c)$ be the tangent bundle over $Ł_2$ with the metric g^c, where g^c is a complete lift to $TŁ_2$ of g, [5].

The matrix g^c in the local coordinate system $(\pi^{-1}(U), \bar{x}=(x^1, x^2, y^1, y^2))$ on $TŁ_2$ associated with a local coordinate system $(U, x=(x^1, x^2))$ on $Ł_2$ is of the form

$$[g^c_{AB}(x^1, x^2, y^1, y^2)]=\begin{bmatrix} -\frac{2k^2y^2}{(x^2)^3} & 0 & \frac{k^2}{(x^2)^2} & 0 \\ 0 & -\frac{2k^2y^2}{(x^2)^3} & 0 & \frac{k^2}{(x^2)^2} \\ \frac{k^2}{(x^2)^2} & 0 & 0 & 0 \\ 0 & \frac{k^2}{(x^2)^2} & 0 & 0 \end{bmatrix}.$$

The index of g^c is equal to 2.

1. We will find and investigate the isometry group of $(TŁ_2, g^c)$. This group is determined by 1-parameter transformation groups of $TŁ_2$ generated by the Killing vector fields of this manifold. In order to determine a basis of a Lie algebra of Killing vector fields on $TŁ_2$ we have to find a basis of a Lie algebra of Killing vector fields on $Ł_2$. It leads us to the solution of the following system of partial differential equations:

$$\partial_1 K^1(x^1, x^2) - \frac{1}{x^2} K^2(x^1, x^2) = 0,$$

$$\partial_2 K^1(x^1, x^2) + \partial_1 K^2(x^1, x^2) = 0,$$

$$\partial_2 K^2(x^1, x^2) - \frac{1}{x^2} K^2(x^1, x^2) = 0,$$

vhere $K(x^1, x^2) = (K^1(x^1, x^2), K^2(x^1, x^2))$ in the local coordinate system (U, x). It turns ut after some calculations that the following vector fields form the basis of a Lie lgebra of Killing vector fields on Ł_2:

$$\underset{1}{K}(x^1, x^2) = [\frac{1}{2}((x^1)^2 - (x^2)^2),\ x^1 x^2],$$

$$\underset{2}{K}(x^1, x^2) = [x^1, x^2],$$

$$\underset{3}{K}(x^1, x^2) = [1{,}0].$$

The vector fields $\underset{1}{K}, \underset{2}{K}, \underset{3}{K}$ are complete ones on Ł_2. The vector fields $\underset{1}{K^v}, \underset{2}{K^v}, \underset{3}{K^v}$ vertical lifts of $\underset{1}{K}, \underset{2}{K}, \underset{3}{K}$) and $\underset{1}{K^c}, \underset{2}{K^c}, \underset{3}{K^c}$ (complete lifts of $\underset{1}{K}, \underset{2}{K}, \underset{3}{K}$) form the basis of a Lie algebra of complete Killing vector fields on $T\text{Ł}_2$ (see [5]). These fields in the local coordinate system $(\pi^{-1}(U), \bar{x})$ have the form

$$\underset{1}{K^v}(x^1, x^2, y^1, y^2) = [0, 0, \frac{1}{2}((x^1)^2 - (x^2)^2),\ x^1 x^2],$$

$$\underset{2}{K^v}(x^1, x^2, y^1, y^2) = [0,\ 0,\ x^1,\ x^2],$$

$$\underset{3}{K^v}(x^1, x^2, y^1, y^2) = [0,\ 0,\ 1.0],$$

$$\underset{1}{K^c}(x^1, x^2, y^1, y^2) = [\frac{1}{2}((x^1)^2 - (x^2)^2),\ x^1 x^2,\ x^1 y^1 - x^2 y^2,\ x^2 y^1 + x^1 y^2],$$

$$\underset{2}{K^c}(x^1, x^2, y^1, y^2) = [x^1, x^2, y^1, y^2],$$

$$\underset{3}{K^c}(x^1, x^2, y^1, y^2) = [1,\ 0,\ 0,\ 0].$$

Now, we are able to determine 1-parameter transformation groups $\overset{i}{\varphi}{}^v$, $\overset{i}{\varphi}{}^c$ $(i = 1, 2, 3)$ of $T\text{Ł}_2$ generated by $\underset{i}{K^v}$, $\underset{i}{K^c}$, respectively. We have

$$\overset{1}{\varphi}{}^v(t, (x^1, x^2, y^1, y^2)) = (x^1, x^2, \frac{1}{2}((x^1)^2 - (x^2)^2)\, t + y^1,\ x^1 x^2\, t + y^2),$$

$$\overset{2}{\varphi}{}^v(t, (x^1, x^2, y^1, y^2)) = (x^1, x^2,\ tx^1 + y^1,\ tx^2 + y^2),$$

$$\overset{3}{\varphi}{}^v(t, (x^1, x^2, y^1, y^2)) = (x^1,\ x^2,\ y^1 + t,\ y^2),$$

$$\overset{1}{\varphi}{}^c(t, (x^1, x^2, y^1, y^2)) = (\frac{-2t((x^1)^2 + (x^2)^2) + 4x^1}{t^2((x^1)^2 + (x^2)^2) - 4tx^1 + 4},\quad \frac{4x^2}{t^2((x^1)^2) + (x^2)^2) - 4tx^1 + 4},$$

$$\frac{4y^1\, [(tx^1 - 2)^2 - (tx^2)^2] + 8tx^2 y^2 (tx^1 - 2)}{[t^2((x^1)^2 + (x^2)^2) - 4tx^1 + 4]^2},\quad \frac{8tx^2 y^1 (2 - tx^1) + 4y^2 [(tx^1 - 2)^2 - (tx^2)^2]}{[t^2((x^1)^2 + (x^2)^2) - 4tx^1 + 4]^2}),$$

$$\overset{2}{\varphi}{}^c(t, (x^1, x^2, y^1, y^2)) = (e^t x^1,\ e^t x^2,\ e^t y^1,\ e^t y^2),$$

$$\overset{3}{\varphi}{}^{c}(t,(x^1,x^2,y^1,y^2))=(x^1+t,\ x^2,\ y^1,\ y^2),$$

for $(t,(x^1,\ x^2,\ y^1,\ y^2))\in R\times TŁ_2$.

We denote by H the isometry group of $TŁ_2$ generated by transformations defined by the above formulas. H is the maximal isometry groups which preserves fibres. The special attention has to be devoted with subgroups H_v and H_c generated by 1-parameter transformation group $\overset{1}{\varphi}{}^{v},\ \overset{2}{\varphi}{}^{v},\ \overset{3}{\varphi}{}^{v}$ and $\overset{1}{\varphi}{}^{c},\ \overset{2}{\varphi}{}^{c},\ \overset{3}{\varphi}{}^{c}$, respectively.

The subgroup H_v of H is normal one. It follows from the fact that the Lie algebra of H_v is an ideal of Lie algebra of complecte Killing vector fields on $TŁ_2$.

Theorem 1. *The group H_v acts transitively on fibres of the tangent bundle over $Ł_2$.*

Proof. Let us fix arbitrary points (x^1,x^2,z^1,z^2) and (x^1,x^2,y^1,y^2) of the fibre $\pi^{-1}((x^1,\ x^2))$. The isometry $\overset{3}{\varphi}{}^{v}_{t_1}\circ\overset{2}{\varphi}{}^{v}_{t_2}$ where $t_1=y^1-z^1+\frac{z^2-y^2}{x^2}x^1$ and $t_2=\frac{y^2-z^2}{x^2}$, maps $(x^1,\ x^2,\ z^1,\ z^2)$ on $(x^1,\ x^2,\ y^1,y^2)$.

Let $GL(2,\ R)^+$ be a subgroup of the linear group $GL(2,R)$ consisting of all matrices with positive determinant. The action φ of $GL(2,R)^+$ on $Ł_2$ is given by the formula

$$\varphi\left(\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix},\ z\right)=\frac{\alpha z+\beta}{\gamma z+\delta},$$

where the identification $Ł_2\ni(x^1,\ x^2)\sim z=x^1+lx^2$ is used for convenience. Then the action ψ of H_c on $TŁ_2$ is of the form:

$$\psi:\ GL(2,\ R)^+\times TŁ_2\longrightarrow TŁ_2,$$

$$\psi:(g,(x^1,\ x^2,y^1,y^2))\longrightarrow(\varphi(g,(x^1,x^2)),\ y^i\,\partial_i\,\varphi(g,(x^1,x^2))).$$

This action is not transitive on $TŁ_2$. Moreover, we have

Theorem 2. *$(TŁ_2,\ H)$ is a homogeneous pseudoriemannian manifold.*

Proof. It is sufficient to show that the group H acts transitively on $TŁ_2$. Let us fix arbitrary points $(x^1,x^2,y^1,y^2),(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2)$ of $TŁ_2$. The isometry $\overset{3}{\varphi}{}^{v}_{t_6}\circ\overset{2}{\varphi}{}^{v}_{t_5}\circ\overset{3}{\varphi}{}^{c}_{t_4}\circ\overset{2}{\varphi}{}^{c}_{t_3}\circ\overset{3}{\varphi}{}^{v}_{t_2}\circ\overset{2}{\varphi}{}^{v}_{t_1}$,

where $t_1=\frac{\bar{y}^2-y^2}{x^2}$, $t_2=\bar{y}^1-y^1-\frac{\bar{y}^2-y^2}{x^2}x^1$, $t_3=\ln\frac{\bar{x}^2}{x^2}$, $t_4=\bar{x}^1-\frac{\bar{x}^2}{x^2}x^1$, $t_5=\frac{x^2\bar{y}^2-\bar{x}^2\ \bar{y}^2}{x^2\ \bar{x}^2}$,

$t_6=\bar{y}^1-\bar{y}^1\frac{\bar{x}^2}{x^2}-\bar{x}^1\frac{x^2\bar{y}^2-\bar{x}^2\ \bar{y}^2}{x^2\ \bar{x}^2}$ maps $(x^1,\ x^2,\ y^1,\ y^2)$ on $(\bar{x}^1,\ \bar{x}^2,\ \bar{y}^1,\ \bar{y}^2)$.

2. Let ∇ be a Levi-Civita connection on the pseudoriemannian manifold $(TŁ_2,g^c)$. Non-zero coefficients of this connection in a local coordinate system $(\pi^{-1}(U),x)$ associated with $(U,\ x)$ take values

$$\Gamma^1_{12}=\Gamma^1_{21}=\Gamma^2_{22}=\Gamma^3_{14}=\Gamma^3_{41}=\Gamma^3_{23}=\Gamma^3_{32}=\Gamma^4_{24}=\Gamma^4_{42}=-\frac{1}{x^2}$$

$$\Gamma^2_{11}=\Gamma^4_{13}=\Gamma^4_{31}=\frac{1}{x^2},\ \Gamma^3_{12}=\Gamma^3_{21}=\Gamma^4_{22}=\frac{y^2}{(x^2)^2},\ \Gamma^4_{11}=-\frac{y^2}{(x^2)^2}.$$

It implies that non-zero coefficients R^i_{jkl} of the curvature tensor R of the connection ∇ are of the form

$$R^1_{212}=R^2_{121}=R^3_{214}=R^4_{123}=R^3_{232}=R^4_{141}=R^3_{422}=R^4_{321}=\frac{1}{(x^2)^2},$$

$$R^1_{122}=R^2_{211}=R^3_{124}=R^4_{213}=R^3_{142}=R^4_{231}=R^3_{322}=R^4_{411}=-\frac{1}{(x^2)^2},$$

$$R^3_{212}=R^4_{121}=-\frac{2y^2}{(x^2)^3},\qquad R^3_{122}=R^4_{211}=\frac{2y^2}{(x^2)^3},$$

and non-zero coefficients R_{ij} of Ricci tensor take values $R_{11}=R_{22}=-2/(x^2)^2$. We note that:

The sectional curvature of the pseudoriemannian manifold $(T\L_2, g^c)$ is not constant. The scalar curvature of $(T\L, g^c)$ is equal to zero.

3. Now, we will deal with geodesics on $(T\L_2, g^c)$.

(a) At first, we will find parametric representations of geodesics on $T\L_2$. Geodesics are solutions of the following system of differential equation

$$\ddot{x}^1-\frac{2}{x^2}\dot{x}^1\dot{x}^2=0,$$

$$\ddot{x}^2+\frac{1}{x^2}(\dot{x}^1\dot{x}^1-\dot{x}^2\dot{x}^2)=0,$$

$$\ddot{y}^1+\frac{2y^2}{x^2x^2}\dot{x}^1\dot{x}^2-\frac{2}{x^2}(\dot{x}^1\dot{y}^2+\dot{x}^2\dot{y}^1)=0,$$

$$\ddot{y}^2-\frac{y^2}{x^2x^2}(\dot{x}^1\dot{x}^1-\dot{x}^2\dot{x}^2)+\frac{2}{x^2}(\dot{x}^1\dot{y}^1-\dot{x}^2\dot{y}^2)=0.$$

The solutions of this system are of the form

$$x^1(t)=a+b\,th\frac{\alpha}{k}(t+t_0),$$

$$x^2(t)=b\left(ch\frac{\alpha}{k}(t+t_0)\right)^{-1},$$

$$y_1(t)=m_1\,th\frac{\alpha}{k}(t+t_0)-m_2\,t\left(ch\frac{\alpha}{k}(t+t_0)\right)^{-2}+m_3\,th^2\frac{\alpha}{k}(t+t_0)+m_4, \tag{1}$$

$$y^2(t)=m_1\left(ch\frac{\alpha}{k}(t+t_0)\right)^{-1}+sh\frac{\alpha}{k}(t+t_0)+\left(ch\frac{\alpha}{k}(t+t_0)\right)^{-2}(tm_2+m_3)$$

or

$x^1(t)=\text{const},$

$$x^2(t)=e^{\lambda(t+t_0)/k},\quad y^1(t)=c_1+c_2\,te^{2\lambda(t+t_0)/k},\quad y^2(t)=e^{\lambda(t+t_0)/k}(c_3+c_4t) \tag{2}$$

or

$$x^1(t)=\text{const},\quad x^2(t)=\text{const},\quad y^1(t)=b_3t+b_4,\quad y^2(t)=b_5t+b_6. \tag{3}$$

Making use of the initial conditions $x(0)=(a^1, a^2, a^3, a^4)$ and $\dot{x}(0)=(v^1, v^2, v^3, v^4)$, $v^1\neq 0$, the geodesic x is given by (1), where $\alpha=k\,((v^1)^2+(v^2)^2)^{1/2}\operatorname{sgn} v^1$, $t_0=((v^1)^2+(v^2)^2)^{1/2}\operatorname{arc} sh(-v^2/v^1)\operatorname{sgn} v^1$, $a=a^1+a^2v^2/v^1$, $b=a^2(((v^1)^2+(v^2)^2)(v^1)^{-2})^{1/2}$,

$$m_1=\operatorname{sgn} v^1\left[a^4((v^1)^2+(v^2)^2)^{1/2}+((v^1)^2+(v^2)^2)^{-1/2}\frac{v^2}{v^1}\left(v^4-\frac{v^2v^3}{v^1}\right)\right],$$

$$m_2=a^4((v^1)^2+(v^2)^2)-v^3+\frac{v^2v^4}{v^1},$$

$$m_3=\frac{sgn\, v^1}{(v^1)^2}\,(v^4-\frac{v^2\, v^3}{v^1}),\ m_4=a^3+a^4\,\frac{v^2}{v^1}\,.$$

For initial conditions $x(0)=(a^1,\ a^2,\ a^3,\ a^4)$ and $\dot{x}(0)=(0,\ v^2,\ v^3,\ v^4)$, $v^2\neq 0$, the geodesic x is given by (2), where $\lambda=k\, v^2/a^2$, $t_0=(a^2/v^2)$ ln a^2, const $=a^1$,

$$c_1=a^3,\ c_2=\frac{v^3}{(a^2)^2},\ c_3=\frac{a^4}{a^2},\ c_4=\frac{a^2\, v^4-a^4\, v^2}{(a^2)^2}.$$

If a geodesic x of $T\L_2$ pass through a point $(a^1,\ a^2,\ a^3,\ a^4)$ at isotropy direction $(0,\ 0,\ v^3,\ v^4)$, then it is given by (3) where

$$b_1=a^1,\ b_2=a^2,\ b_3=v^3,\ b_4=a^3,\ b_5=v^4,\ b_6=a^4.$$

Thus we proved

Theorem 3. *Exactly one geodesic passes through a fixed point of the manifold $T\L_2$ at a given direction. This geodesic is represented by (1) or (2) or (3).*

(b) We will investigate properties of geodesics on $T\L_2$. As we know [2], if a curve $t\to\varphi(t)\in T\L_2$ is a geodesic, then the function $t\to g^c\,(\dot{\varphi}(t),\ \dot{\varphi}(t))\in T\L_2$ is a constant one. Now, we want to pass a geodesic through two given points $x=(x^1,\ x^2,\ y^1,\ y^2)$ and $\bar{x}=(\bar{x}^1,\ \bar{x}^2,\ \bar{y}^1,\ \bar{y}^2)$, $x\neq\bar{x}$ of $T\L_2$.

1°. At first, we assume that $x^1=\bar{x}^1$ and $x^2\neq\bar{x}^2$.

Let $\varphi: t\to(\varphi^1(t),\ \varphi^2(t),\ \varphi^3(t),\ \varphi^4(t))$ be a required geodesic. It is known ([6], Th. 9.1, p.58) that the projection of this geodesic onto $\L_2$ is a geodesic which pass through the points $(x^1,\ x^2)$ and $(x^1,\ \bar{x}^2)$. Thus we have

$$\varphi^1(t)=x^1,\ \varphi^2(t)=x^2\, e^{t/k}.$$

From the properties of geodesics we get $\varphi^2(0)=x^2$ and $\varphi^2(t_1)=\bar{x}^2$ for some $t_1\neq 0$. Now we find the coefficients c_1, c_2, c_3 and c_4 in equations (2) such that $\varphi^3(0)=y^1$, $\varphi^4(0)=y^2$ $\varphi^3(t_1)=\bar{y}^1$, $\varphi^4(t_1)=\bar{y}^2$. To this aim we must to solve the system of linear equations

$$c_1=y^1,\ c_3\, x^2=y^2,\ \ c_1+c_2\,(\bar{x}^2)^2\ t_1=\bar{y}^1,\ c_3\,\bar{x}^2+c_4\, t_1\ \bar{x}^2=\bar{y}^2.$$

The determinant of the system is equal to $(-t_1^2\, x^2\,(\bar{x}^2)^3)\neq 0$ and we have exactly one solution.

2°. Now, let $x^1\neq\bar{x}^1$. If $\varphi: t\to(\varphi^1(t),\ \varphi^2(t),\ \varphi^3(t),\ \varphi^4(t))\in T\L_2$ is a required geodesic, then its projection onto $\L_2$ $\ t\to(\varphi^1(t),\ \varphi^2(t))$ is given by

$$\varphi^1(t)=x^1+x^2\,(-sh\frac{t_0}{k}+ch\frac{t_0}{k}\, th\,\frac{t+t_0}{k}),$$

$$\varphi^2(t)=x^2\, ch\frac{t_0}{k}\,(ch\frac{t+t_0}{k})^{-1}.$$

As we know it is possible to take t_0 and t_1 in this representation such that $\varphi^1(0))=x^1$, $\varphi^2(0)=x^2$, $\varphi^1(t_1)=\bar{x}^1$, $\varphi^2(t_1)=\bar{x}^2$.

Let us return to geodesic (1). We introduce notations

$$y^1(t)=m_1\,\alpha^1(t)+m_2\,\alpha^2(t)+m_3\,\alpha^3(t)+m_4,$$

$$y^2(t)=m_1\,\beta^1(t)+m_2\,\beta^2(t)+m_3\,\beta^3(t)$$

and $\alpha_0^i=\alpha^i(0)$, $\alpha_1^i=\alpha^i(t_1)$, $\beta_0^i=\beta^i(0)$, $\beta_1^i=\beta^i(t_1)$ for $i=1,2,3$.

The relations $\varphi(0)=x$ and $\varphi(t_1)=\bar{x}$ imply that we have to find m_i, $i=1,\ldots,4$ which satisfy the following system of linear equations

$$y^1 = m_1 \alpha_0^1 + m_3 \alpha_0^3 + m_4,$$

$$y^2 = m_1 \beta_0^1 + m_3 \beta_0^3,$$

$$\bar{y}^1 = m_1 \alpha_1^1 + m_2 \alpha_1^2 + m_3 \alpha_1^3 + m_4,$$

$$\bar{y}^2 = m_1 \beta_1^1 + m_2 \beta_1^2 + m_3 \beta_1^3.$$

This system has exactly one solution since the determinant of this sysem is equal to $t_1 \left(th \frac{t_1+t_0}{k} - th \frac{t_0}{k}\right) \left(ch \frac{t_0}{k}\, ch \frac{t_1+t_0}{k}\right)^{-1} \neq 0.$

3^0. If we have $x^1 = \bar{x}^1$ and $x^2 = \bar{x}^2$, then each geodesic

$$\varphi_a : t \to \left(x^1,\ x^2, \frac{\bar{y}^1 - y^1}{a} t + y^1, \frac{\bar{y}^2 - y^2}{a} t + y^2\right),$$

where a is an arbitrary non-zero number, satisfies the conditions $\varphi_a(0) = x$ and $\varphi_a(a) = \bar{x}$. Thus we proved

Theorem 4. *If two points of $T\L_2$ do not belong to the same fibre, then there exists exactly one geodesic which passes through these points. For two points of $T\L_2$ which belong to the same fibre, there exist infinitely many geodesics passing through these points.*

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