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A COMPARISON THEOREM FOR TCHEBYCHEFF POLYNOMIALS

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In this paper we study Tchebycheff polynomials satisfying some zero boundary conditions and we show that the L_p-norm of such polynomials decreases when using generalized derivatives of lower order in the boundary conditions.

Let $[a, b]$ be a finite subinterval of the real line and $U_N=\{u_i\}_1^N$ be a set of functions in $C^N[a, b]$.

Definition 1. *U_N is called Extended Complete Tchebycheff system (ECT-system) on $[a, b]$ if any $u\in\mathcal{U}_k:=\operatorname{span}\{u_1,\ldots,u_k\}$ has at most $k-1$ zeros in $[a, b]$ counting the multiplicities ($k=1,\ldots,N$).*

Definition 2. *Let U_N be an ECT-system on $[a, b]$. Then the functions from $\mathcal{U}_N$ are called Extended Tchebycheff polynomials of order N.*

Throughout this paper by polynomials we mean Extended Tchebycheff polynomials.

It is well known (see [2, 5]) that for an ECT-system U_N there exist positive on $[a, b]$ functions $w_i\in C^{N-i}[a, b]$, $i=1,\ldots,N$, such that

$$D_N D_{N-1}\ldots D_1 u(x)=0 \quad \text{for all } u\in\mathcal{U}_N,\ x\in[a, b],$$

where

$$D_i:=\frac{d}{dx}\,\frac{1}{w_i(x)},\quad i=1,\ldots,N.$$

Moreover, if set $D^0u:=u$, $u\in\mathcal{U}_N$ and $D^k:=D_kD_{k-1}\ldots D_1$, $k=1,\ldots,N$ we have $D^k u_i(x)=\delta_{k,i-1}\,w_i(x)$, $i=1,\ldots,k+1$, $k=0,\ldots,N-1$, $\delta_{k,j}$ being the Kronecker symbol.

Given the integers $(\lambda_1,\ldots,\lambda_{m_1};\mu_1,\ldots,\mu_{m_2})=:(\mathcal{J}_1;\mathcal{J}_2)=:\mathcal{J}$ with $0\le\lambda_1<\ldots<\lambda_{m_1}\le N-2$, $0\le\mu_1<\ldots<\mu_{m_2}\le N-2$, $m_1+m_2\le N-1$ we denote by $A(\mathcal{J})$ the set of polynomials of the form:

$$u=u_N+a_1u_{N-1}+\cdots+a_{N-1}u_1,\ a_i\in\mathbb{R},\ i=1,\ldots,N-1,$$

satisfying the boundary conditions

$$D^j u(a)=0,\quad j\in J_1,$$
$$D^j u(b)=0,\quad j\in J_2.$$

Introduce the incidence matrix

$$E(\mathcal{J})=\begin{pmatrix} e_{0,0} & e_{0,1} & \ldots & e_{0,N-2}\\ 1 & 0 & & 0\\ \cdot & \cdot & \cdot & \cdot\\ 1 & 0 & & 0\\ e_{n+1,0} & e_{n+1,1}, & \ldots, & e_{n+1,N-2}\end{pmatrix}$$

with $n=N-1-m_1-m_2$ and

$$e_{0,j}=1 \quad \text{if} \quad j\in J_1, \qquad e_{0,j}=0 \quad \text{otherwise},$$

$$e_{n+1,j}=1 \quad \text{if} \quad J\in J_2, \qquad e_{n+1,j}=0 \quad \text{otherwise}.$$

We write $\widehat{\mathcal{J}}<\mathcal{J}$ if $\widehat{\lambda}_i\leq\lambda_i$, $i=1,\dots m_1$ and $\widehat{\mu}_i\leq\mu_i$, $i=1,\dots,m_2$ with at least one strong inequality.

Theorem. *Let p be fixed, $1\leq p<\infty$. Suppose that $E(\mathcal{J})$ satisfies the Polya condition and $\widehat{\mathcal{J}}<\mathcal{J}$. Then*

$$\|u(\widehat{\mathcal{J}};\cdot)\|_p<\|u(\mathcal{J};\cdot)\|_p, \tag{1}$$

$u(\mathcal{J};\cdot)$ being the polynomial of least $L_p[a,b]$-norm in $A(\mathcal{J})$.

Proof. Let k be a fixed integer satisfying

$$\widehat{\lambda}_k+1<\widehat{\lambda}_{k+1} \quad \text{if} \quad k<m_1, \quad \widehat{\lambda}_k+1\leq N-2 \quad \text{if} \quad k=m_1.$$

The theorem will follow by pair-wise comparisons if we prove it in the case $\mathcal{J}=(\lambda_1,\dots,\lambda_{m_1};\mu_1,\dots,\mu_{m_2})$, where

$$\lambda_i=\begin{cases}\widehat{\lambda}_i, & i\neq k\\ \widehat{\lambda}_i+1, & i=k\end{cases}, \quad \mu_i=\widehat{\mu}_i, \quad i=1,\dots,m_2.$$

We show (1) in simpler case. First we note that $u(\mathcal{J};\cdot)$ has n simple zeros in (a,b). Indeed for arbitrary fixed $\{t_i\}_1^n$, $a<t_1<\cdots<t_n<b$ there exist polynomials $\varphi\in A(\mathcal{J})$ and $\varphi_i\in\mathcal{U}_{N-1}$, $i=1,\dots,n$ satisfying the interpolation conditions

$$\begin{aligned}
&\varphi(t_j)=0, && j=1,\dots,n,\\
&D^j\varphi_i(a)=0, && j\in J_1,\ i=1,\dots,n,\\
&D^j\varphi_i(b)=0, && j\in J_2,\ i=1,\dots,n,\\
&\varphi_i(t_j)=\delta_{i,j}, && i,j=1,\dots,n.
\end{aligned}$$

Then $A(\mathcal{J})=\{\varphi-\sum_{i=1}^{n}\alpha_i\varphi_i:\alpha_i\in\mathbb{R},\ i=1,\dots,n\}$ and hence

$$\int_a^b|u(\mathcal{J};x)|^{p-1}\varphi_i(x)\operatorname{sign}u(\mathcal{J};x)\,dx=0,\ i=1,\dots,n.$$

Now it follows in a standart way that $u(\mathcal{J};\cdot)$ must have n sign changes in (a,b).

Let $\{x_i\}_1^n$ be the zeros of $u:=u(\mathcal{J};\cdot)$ in (a,b). The Atkinson – Sharma theorem for ECT-systems (see [3]) yields that there exists unique polynomial $\widehat{u}\in A(\widehat{\mathcal{J}})$ such that $\widehat{u}(x_i)=0$, $i=1,\dots,n$. As usually $Z(f;(a,b))$ is the number of zeros of the function f in (a,b) counting the multiplicities. By Budan — Fourier theorem (see [5])

$$Z(u;(a,b))\leq S^-(D^0u(a),\dots,D^{N-2}u(a),1)-S^+(D^0u(b),\dots,D^{N-2}u(b),1)$$
$$\leq N-1-m_1-m_2=n.$$

Since $Z(u;(a,b))=n$, we have $S^-(D^0u(a),\dots,D^{N-2}u(a),1)=N-1-m_1$.

Therefore the values $D^j u(a)$, $j \in \{0, \ldots, N-2\} \setminus \mathcal{J}_1$ are distinct from zero and must change sign alternatively. Similarly

$$S^-(D^0 \widehat{u}(a), \ldots, D^{N-2} \widehat{u}(a), 1) = N-1-m_1.$$

Then it is not difficult to see that

$$\text{sign } u(x) = \text{sign } \widehat{u}(x), \quad x \in (a, b). \tag{2}$$

Now observe that $S^-(D^0 u_\alpha(a), \ldots, D^{N-2} u_\alpha(a), 1) \leqq N-1-m_1$ for each $u_\alpha = u - \alpha \widehat{u}$, $0 \leqq \alpha < 1$ and by Budan — Fourier theorem $Z(u_\alpha; (a, b)) \leqq n$. But $u_\alpha(x_i) = 0$, $i = 1, \ldots, n$. That is why $S^-(D^0 u_\alpha(a), \ldots, D^{N-2} u_\alpha(a), 1) = N-1-m_1$

and sign $u(x) = $ sign $\widehat{u}(x) = $ sign $u_\alpha(x)$ for $x \in (a, b)$ and $\alpha \in [0, 1]$.

Therefore $|\widehat{u}(x)| \leqq |u(x)|$, $x \in [a, b]$.
Assuming $\widehat{u}(t_0) = u(t_0)$ for some $t_0 \in (a, b) \setminus \{x_1, \ldots, x_n\}$, we get $\widehat{u}(x) \equiv u(x)$ on (a, b), since the Birkhoff interprolation problem

$$\begin{aligned} D^j v(a) &= 0, & j &\in J_1 \setminus \{\lambda_k\}, \\ D^j v(b) &= 0, & j &\in J_2, \\ v(x_i) &= 0, & i &= 1, \ldots, n, \\ v(t_0) &= 0, \end{aligned}$$

has a unique solution in $\mathcal{U}_{N-1}$ (by Atkinson—Sharma theorem), namely $v(x) \equiv 0$ — contradiction.

This yields $|\widehat{u}(x)| < |u(x)|$, $x \in (a, b) \setminus \{x_1, \ldots, x_n\}$ and consequently $\|\widehat{u}\|_p < \|u\|_p$.

The proof is complete.

The case of usual polynomials was studied by G. Nikolov [4]. The idea of using the Budan -- Fourier theorem is due to B. Bojanov (see [1]).

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