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A NOTE ON MULTIVALENT FUNCTIONS

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The subclass $K_p(p, \alpha)$ consisting of functions which are multivalent in the unit disk U is introduced. The object of the present paper is to derive some interesting results for functions belonging to the class $K_p(p, \alpha)$.

1. Introduction. Let A_p denote the class of functions of the form

$$(1.1)\qquad f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \quad (p\in N=\{1, 2, 3, \dots\})$$

which are analytic in the unit disk $U=\{z: |z|<1\}$. For a function $f(z)$ belonging to A_p, Nunokawa [2] has proved the following result.

Theorem A. *If the function $f(z)$ belonging to A_p satisfies*

$$(1.2)\qquad p+\mathrm{Re}\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\}>0 \quad (z\in U),$$

then $f(z)$ is p-valent in the unit disk U and

$$(1.3)\qquad k+\mathrm{Re}\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\}>0 \quad (z\in U)$$

or $k=0, 1, 2, \dots, p-1$.

In view of (1.2) and (1.3), we note that

$$(1.4)\qquad \mathrm{Re}\{1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\}>1-p \Rightarrow \mathrm{Re}\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\}>1-p.$$

Let $K_p(p, \alpha)$ be the subclass of A_p consisting of functions which satisfy the condition

$$(1.5)\qquad \mathrm{Re}\{1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\}>1-\alpha$$

for some $\alpha(0<\alpha\leq p)$, and for all $z\in U$. Note that (1.4) shows $K_p(p.p)\subseteq K_p(p-1, p-1)$, and that $K_p(p, \alpha)\subseteq K_p(p, p)$.

Let $f(z)$ and $g(z)$ be analytic functions in the unit disk U. Then $f(z)$ is said to be subordinate to $g(z)$ if there exists an analytic function $w(z)$ in U satisfying $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g(w(z))$. We denote by $f(z)\prec g(z)$ this relation. If $g(z)$ is univalent in U, then the subordination $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

2. Some properties of the class $K_p(p, \alpha)$. In order to show our main result, we have to recall here the following result due to Miller and Mocanu [1].

Lemma 1. *Let* $\varphi(u, v)$ *be complex valued function* $\varphi: D\to C$, $D\subset C\times C$ *(C is the complex plane), and let* $u=u_1+iu_2$, $v=v_1+iv_2$. *Suppose that the function* $\varphi(u, v)$ *satisfies the following conditions*:

(i) $\varphi(u, v)$ *is continuous in* D;

(ii) $(1, 0)\in D$ *and* $\operatorname{Re}\{\varphi(1, 0)\}>0$;

(iii) $\operatorname{Re}\{\varphi(iu_2, v_1)\}\leqq 0$ *for all* $(iu_2, v_1)\in D$ *and such that* $v_1\leqq -(1+u_2^2)/2$.

Let $p(z)=1+p_1z+p_2z^2+\dots$ be regular in the unit disk U such that $(p(z), zp'(z))\in D$ for all $z\in U$. If

$$\operatorname{Re}\{\varphi(p(z), zp'(z))\}>0, \quad (z\in U),$$

then $\operatorname{Re}\{p(z)\}>0$ $(z\in U)$.

Applying Lemma 1, we derive

Theorem 1. *If the function* $f(z)$ *is in class* $K_p(p, \alpha)$, *then* $f(z)\in K_p(p-1, \beta-1)$, *where* $0<\alpha\leqq(1+2\sqrt{2})/2$ *and*

$$\beta=\frac{3+2\alpha-\sqrt{4\alpha^2-4\alpha+9}}{4}. \tag{2.1}$$

Proof. We need to show that

$$\operatorname{Re}\{1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\}>1-\alpha \Rightarrow \operatorname{Re}\{1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\}>2-\beta,$$

hat is, that $f(z)\in K_p(p, \alpha)$ satisfies

$$\operatorname{Re}\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\}>1-\beta. \tag{2.2}$$

Define the function $F(z)$ by

$$F(z)=\frac{f^{(p-1)}(z)}{p!}=z+\sum_{n=2}^{\infty} A_nz^n. \tag{2.4}$$

Then, it is sufficient to prove that

$$\operatorname{Re}\{\alpha+\frac{zF''(z)}{F'(z)}\}>0 \Rightarrow \operatorname{Re}\{\frac{zF'(z)}{F(z)}\}>1-\beta.$$

Let

$$\frac{zF'(z)}{F(z)}=(1-\beta)+\beta q(z), \tag{2.5}$$

then $q(z)$ is regular in the unit disk U, and $q(z)=1+q_1z+q_2z^2+\dots$. Taking the logarithmic differentiations of both sides in (2.5), we have

$$\alpha+\frac{zF''(z)}{F'(z)}=\alpha-\beta+\beta q(z)+\frac{\beta zq'(z)}{(1-\beta)+\beta q(z)}, \tag{2.6}$$

or

$$\operatorname{Re}\{\alpha+\frac{zF''(z)}{F'(z)}\}=\operatorname{Re}\{\alpha-\beta+\beta q(z)+\frac{\beta zq'(z)}{(1-\beta)+\beta q(z)}\}>0 \tag{2.7}$$

Setting

$$\varphi(u, v)=\alpha-\beta+\beta u+\frac{\beta v}{(1-\beta)+\beta u}, \tag{2.8}$$

we can see that:

(i) $\varphi(u, v)$ is continuous in $D=(C-\{(\beta-1)/\beta\}\times C$;

(ii) $(1, 0)\in D$ and $\operatorname{Re}\{\varphi(1, 0)\}=\alpha>0$;

(iii) (for all $(iu_2, v_1)\in D$ such that $v_1\leq-(1+u_2^2)/2$,

$$\operatorname{Re}\{\varphi(iu_2, v_1)\}=\alpha-\beta+\frac{\beta(1-\beta)v_1}{(1-\beta)^2+\beta^2u_2^2}$$

$$\leq\alpha-\beta-\frac{\beta(1-\beta)(1+u_2^2)}{2\{(1-\beta)^2+\beta^2u_2^2\}}=-\frac{\beta\{(1-\beta)-2\beta(\alpha-\beta)\}u_2^2}{2\{(1-\beta)^2+\beta^2u_2^2\}}\leq 0,$$

because $0<\alpha\leq(1+2\sqrt{2})/$, β is given by (2.1), and $0<\beta<1$. Therefore, by using Lemma 1, we obtain $\operatorname{Re}\{q(z)\}>0$ which implies that

$$\operatorname{Re}\{\frac{zF'(z)}{F(z)}\}>1-\beta.$$

This completes the proof of Theorem 1.

Corollary 1. *If the function $f(z)$ is in the class $K_p(p, \alpha)$, then $f(z)\in K_p(p-1, \alpha)$, where $0<\alpha\leq(1+2\sqrt{2})/2$.*

Corollary 2. *If the function $f(z)$ is in the class $K_p(p, \alpha)$ with $0<\alpha\leq(1+\sqrt{2})/2$, then $f(z)$ is p-valent convex of order $(1-\alpha)$ for $0<\alpha\leq 1$, and $\operatorname{Re}\{zf'(z)/f(z)\}>-\alpha$.*

Proof. Using Corollary 1, we have

$$\operatorname{Re}\{1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\}>1-\alpha\Rightarrow\operatorname{Re}\{1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\}>1-\alpha$$

$$\Rightarrow\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}>1-\alpha\Rightarrow\operatorname{Re}\{\frac{zf'(z)}{f(z)}\}>-\alpha.$$

Next, we need the following lemma by Obradović and Owa [3].

Lemma 2. *If the function $f(z)$ defined by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ in the unit disk U satisfies*

$$\operatorname{Re}\{\frac{zf'(z)}{f(z)}\}>\alpha \tag{2.9}$$

for some $\alpha(0\leq\alpha<1)$, and for all $z\in U$, then

$$\operatorname{Re}\{(\frac{f(z)}{z})^{\beta}\}>\frac{1}{1+2\beta(1-\alpha)}, \tag{2.10}$$

where $0<\beta\leq 1/2(1-\alpha)$.

Now, we prove

Theorem 2. *If the function $f(z)$ is in the class $K_p(p, \alpha)$ with $0<\alpha\leq 1$, then*

$$\frac{1-(1-2\gamma)|z|}{1+|z|}\leq\left|\frac{f^{(p)}(z)}{p!}\right|^{\beta}\leq\frac{1+(1-2\gamma)|z|}{1-|z|} \tag{2.11}$$

for $z\in U$, where $0<\beta\leq 1/2\alpha$ and $\gamma=1/(1+2\alpha\beta)$.

Proof. Note that Lemma 2 implies that

$$\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}>\alpha\Rightarrow\operatorname{Re}\{(f'(z))^{\beta}\}>\frac{1}{1+2\beta(1-\alpha)},$$

because

$$\operatorname{Re}\{1+\frac{zf''(z)}{f'(z)}\}>\alpha \Leftrightarrow \operatorname{Re}\{\frac{z(zf'(z))'}{zf'(z)}\}>\alpha.$$

By virtue of Theorem 1, we see that

$$f(z)\in K_p(p,\ \alpha) \Rightarrow \operatorname{Re}\{1+\frac{zF''(z)}{F'(z)}\}>1-\alpha$$

for $F(z)$ given by (2.4). It follows from the above that

$$f(z)\in K_p(p,\ \alpha) \Rightarrow \operatorname{Re}\{(F'(z))^\beta\}>\gamma=\frac{1}{1+2\alpha\beta}.$$

Thus, with the help of the subordination, we have

$$(2.12) \qquad (F'(z))^\beta \prec \frac{1+(1-2\gamma)z}{1-z}.$$

so that

$$(2.13) \qquad \left|(F'(z))^\beta-\frac{1+(1-2\gamma)|z|^2}{1-|z|^2}\right| \leqq \frac{2(1-\gamma)|z|}{1-|z|^2}.$$

This gives the assertion of Theorem 2.

Taking $\alpha=1$ and $\beta=1/2$, Theorem 2 gives

Corollary 3. If the function $f(z)$ is in the class $K_p(p,\ 1)$, then

$$(2.14) \qquad \frac{1}{1+|z|} \leqq \left|\frac{f^{(p)}(z)}{p!}\right|^{1/2} \leqq \frac{1}{1-|z|}$$

for $z\in U$.

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