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ON THE LIAPUNOV AND KELLOGG SURFACES

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This paper describes the difference between the Liapunov surfaces and the Kellogg surfaces. It is interesting to note that the smoothness in Kellogg sense is stronger than of smoothness in Liapunov sense. The difference is illustrated by giving a surface that is everywhere Liapunov but nowhere Kellogg.

1. Intoroduction. A surface σ is smooth in the Liapunov sense, or briefly a Liapunov surface, if it has a tangent plane at each point, and also there exists a local coordinates at any point of the surface, z-axis along normal, x- and y-axis in the tangent plane, such that a portion of the surface in this neighbourhood has the equation $z=f(x, y)$.

If n_p and n_Q are the unit normal vectors at any points P and Q of σ respectively, then the following condition which is called Liapunov condition, must be satisfied

$$\theta<Dr^{\nu}; \quad \theta=\cos^{-1}(n_p, n_Q), \; r=|P-Q| \tag{1}$$

for some $D>0$ and $0<\nu\leqq 1$.[1,2]

We call a surface σ is smooth in the Kellogg sense, or briefly a Kellogg surface if it can be locally represented by $z=f(x, y)$ which has continuous derivatives of the second order in this neighbourhood.[3,4]

A plane and a sphere are Liapunov surfaces, but a cube and the following surface

$$f(x, y)=\begin{cases} x^2\sin(1/x^2)+y^2\sin(1/y^2); & x\neq 0,\ y\neq 0 \\ x^2\sin(1/x^2); & x\neq 0,\ y=0 \\ y^2\sin(1/y^2); & x=0,\ y\neq 0 \\ 0; & x=0,\ y=0 \end{cases}$$

are not Liapunov surfaces, because the cube has no tangent plane on its edges and the second surface does not satisfy the Liapunov condition at origin.

The Liapunov and Kellogg surfaces has been applied by mathematicians in potential theory.[5] The purpose of this paper is to describe a clear distinction between them.

2. Main results

Theorem. *Let σ be a surface with tangent plane at each point, and also there exists a local coordinates at any point of the surface with z-axis along normal, x- and y-axis in the tangent plane, so that a portion of the surface in this neighbourhood has the equation $z=f(x, y)$. This surface is a Liapunov surface if and only if f_x and f_y exist and are Holder continuous.*

Proof: Suppose that f_x and f_y satisfy the Holder condition, that is,

$$|f_x(x, y)-f_x(0, 0)|\leqq D_1(x^2+y^2)^{\nu'/2}, \text{ for some } D_1>0 \text{ and } 0<\nu'\leqq 1 \tag{2}$$

and

(3) $|f_y(x, y)-f_y(0, 0)| \leqq D_2(x^2+y^2)^{\nu''/2}$, for some $D_2>0$ and $0< \nu'' \leqq 1$.

If $\theta = \cos^{-1}(n_p, n_Q)$, then by assuming $|\theta|<\pi/2$, we have

(4) $$\theta^2 \leqq 4\sin^2\theta \leqq 4 \frac{f_x^2+f_y^2}{1+f_x^2+f_y^2} \leqq 4D^2 r^{2\nu};\ f_x=f_x(x, y) \text{ and } f_y=f_y(x, y)$$

where $D^2=2\max(D_1^2, D_2^2)$ and $\nu=\min(\nu', \nu'')$.

Conversely, if $\theta \leqq Dr^\nu$ for some $D>0$ and $0<\nu\leqq 1$, then $\sin\theta<\theta\leqq Dr^\nu$ implies

(5) $$\frac{f_x^2}{1+f_x^2+f_y^2} \leqq \sin^2\theta \leqq D^2(x^2+y^2+z^2)^\nu.$$

Therefore $f_x(x, y) \leqq DMr^\nu$, for any $M \geqq 1+f_x^2+f_y^2$.

Now we shall show that

(6) $$r^2 \leqq K(x^2+y^2), \text{ for some } K>0.$$

By mean value theorem, we have

(7) $$f(x, y)=xf_x(\theta x, \theta y)+yf_y(\theta x, \theta y), \text{ for some } 0<\theta<1.$$

Thus

$$\frac{r^2}{x^2+y^2}=1+\frac{x^2 f_x^2+y^2 f_y^2+2xy\, f_x f_y}{x^2+y^2}$$

(8) $$\leqq 1+2N^2=K, \text{ for any } N \geqq \max(|f_x|, |f_y|).$$

Consequently f_x and similarly f_y are Hölder continuous of class ν.

Corollary 1. Any Kellogg surface is a Liapunov surface.

Corollary 2. If f is a Liapunov surface of class $\nu=1$, it is a Liapunov surface of class $0<\nu<1$.

Corollary 3. If $z=f(x, y)$ is a Liapunov surface, then it is of class $C^{(1)}$.

Let we have two surfaces, that are identical and twice continously differentiable everywhere except that one of them is not twice differentiable in a finite number of points but satisfy Liapunov condition at these points. The single layer potentials generated by continuous source distribution over these surfaces are equal. Therefore in the following section we are going to introduce a surface that is everywhere Liapunov and nowhere Kellogg surface.

3. A surface that is everywhere Liapunov surface and nowhere Kellogg surface. The function $f(x)=\sum_{k=0}^{\infty} a_k(x)$, where $a_0(x)$ is the distance from x to the nearest integer and $a_k(x)=2^{-k}a_0(2^k x)$, is called Waerden function. This function is continuous with no two-sided derivative at any point,[6] nor does it have one-sided derivative anywhere.[7]

Now we consider the following function

(9) $$z=h(x)=\int_0^x f(t)\,dt,$$

where $f(x)$ is the Waerden function. The surface $z=h(x)$ is everywhere Liapunov and nowhere Kellogg. In order to prove this statement it is sufficient to show that the function $h_x=f(x)$ is everywhere Hölder continuous, i. e., for each ν between 0 and 1, there is a constant M_ν such that

(10) $$|f(x+t)-f(x)| \leqq M_\nu |t|^\nu$$

for any two real numbers x and t.

We first consider the case $|t|\leqq 1$. In this case, we can always find an integer n, with $n \geqq 0$, such that

(11) $$2^{-n-1} \leq |t| \leq 2^{-n}.$$

It can easily be shown that

(12) $$|a_k(x+t)-a_k(x)| \leq |t|$$

for any two real numbers x and t. From (11) and (12), we obtain

(13) $$\Sigma_{k=0}^{n-1} |a_k(x+t)-a_k(x)| \leq n|t| \leq n2^{-n(1-\nu)}|t|^{\nu}.$$

Moreover, since

(14) $$0 \leq a_k(x) \leq 2^{-k-1},$$

we have

(15) $$|a_k(x+t)-a_k(x)| \leq 2^{-k-1}.$$

It follows from (11) and (15) that for any $|t|<1$,

(16) $$\Sigma_{k=n}^{\infty} |a_k(x+t)-a_k(x)| \leq 2^{-n} \leq 2|t| \leq 2|t|^{\nu}.$$

Summing inequalities (13) and (16), we obtain

(17) $$\Sigma_{k=0}^{\infty} |a_k(x+t)-a_k(x)| \leq \{2+n2^{-n(1-\nu)}\}|t|^{\nu}.$$

But it is not difficult to inspect that

(18) $$n2^{-n(1-\nu)} \leq [2^{1/\ln 2} . \ln 2^{1-\nu}]^{-1}$$

for any integer $n \geq 0$. By choosing $M_{\nu}=2+[2^{1/\ln 2} . \ln 2^{1-\nu}]^{-1}$, we can readily conclude from (17) the validity of (1) for any $|t| \leq 1$.

Now we consider the case $|t|>1$. It follows from (14) that

(19) $$0 \leq f(x) \leq 1,$$

and for any $|t|>1$, this yields

(20) $$|f(x+t)-f(x)| \leq 1 \leq |t|^{\nu}$$

for any two real numbers x and t. Hence, choosing the same M_{ν} as above ensures the validity of (10) also for the case $|t|>1$, and the proof is complete.

4. Conclusions. It has been shown how the smoothness in Kellogg sense is stronger than of smoothness in Liapunov sense. Our analysis shows that there is always a surface that is everywhere Liapunov but nowhere Kellogg. Therefore, the single layer and double layer potentials generated by a continuous source distributions must be defined over Liapunov surfaces.

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