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ON THE WEAK POLYNOMIAL IDENTITIES

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This paper deals with weak polynomial identities satisfied by pairs "associative algebras — vector spaces" over a field of characteristic zero. Two main results are obtained here: i) Necessary and sufficient conditions for distributivity of the lattice of subvarieties of a given variety of pairs are found; ii) In the case when the vector space has the structure of a Lie algebra, we prove an analog of the Nagata-Higman theorem: if a pair satisfies the weak identity $x^n=0$ then it satisfies $x_1 \dots x_M=0$ for a suitable M as well.

Introduction. Throughout this paper we shall work over a fixed field K of characteristic zero. Let $A=K\langle X\rangle=K\langle x_1, x_2, \dots \rangle$ be the free associative algebra with free generators $x_1, x_2, \dots$ and let A_m be the subalgebra of rank m generated by $x_1, x_2, \dots x_m$. We denote S_n and GL_m to be the symmetric group and the general linear group acting respectively on the set of symbols $\{1, 2, \dots, n\}$ and on the m-dimensional vector space. Let R be an associative algebra and let G be a vector subspace of R such that R is generated as an algebra by G. The polynomial $f(x_1, \dots, x_n)$ from $K\langle X\rangle$ is a weak identity for the pair (R, G) if $f(g_1, \dots, g_n)=0$ for any $g_1, g_2, \dots, g_n \in G$. The set T of all weak identities for (R, G) is an ideal in $K\langle X\rangle$.

The ideal T satisfies the following condition: if $f(x_1, \dots, x_m) \in T$ then $f(\Sigma \alpha_{1i} x_i, \dots, \Sigma \alpha_{mi} x_i) \in T$ for $\alpha_{ij} \in K$. In other words T is GL-invariant and we shall call it a GL-ideal in $K\langle X\rangle$ corresponding to the pair (R, G). It is well known that the free Lie algebra $L(X)$ is embedded into $K\langle X\rangle$ in a natural way. In particular, when the vector space G is a Lie algebra, the ideal T of all weak identities for the pair (R, G) satisfies the condition: if $f(x_1, \dots, x_m) \in T$ and $v_1, v_2, \dots, v_n \in L(X)$ then $f(v_1, \dots, v_m) \in T$ again. The weak identities are introduced in this form by Razmyslov [8] in his study of the 2×2 matrix algebra.

Let I be a subset of $K\langle X\rangle$. The class of all pairs (R, G) satisfying as weak identities the elements of I forms a variety of pairs. A lot of the properties of varieties of algebras can be transferred verbatim to varieties of pairs. For example all subvarieties of a given variety of pairs form a lattice with respect to intersection and union.

Now let $\mathfrak{M}$ be the variety of pairs with an ideal of weak identities I. We denote A/I by $F(\mathfrak{M})$ and $A_m/(A_m \cap I)$ by $F_m(\mathfrak{M})$ and we shall call $F(\mathfrak{M})$ relatively free algebra for the variety $\mathfrak{M}$. Let $P_n(\mathfrak{M})$ be the set of all multilinear polynomials from $F_n(\mathfrak{M})$ of degree n. The space $P_n(\mathfrak{M})$ has the structure of a left S_n-module with the following action of S_n:

$$\sigma(x_{i_1} \dots x_{i_n})=x_{\sigma(i_1)} \dots x_{\sigma(i_n)}, \ \sigma \in S_n, \ x_{i_1} \dots x_{i_n} \in P_n(\mathfrak{M}).$$

The algebra A_m is isomorphic to the tensor algebra of a vector space of dimension m. Thus $F_m(\mathfrak{M})$ is a left GL_m-module with the action

$$g(x_{i_1} \dots x_{i_n})=g(x_{i_1}) \dots g(x_{i_n}), \ g \in GL_m, \ x_{i_1} \dots x_{i_n} \in F_m(\mathfrak{M}).$$

The irreducible S_n- and GL_m-modules are described by Young diagrams. For any partition $\lambda=(\lambda_1, \dots, \lambda_r)$ of the integer n, we shall denote $M(\lambda)$ and $N_m(\lambda)$ to be the S_n-

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and GL_m-modules corresponding to λ. It is known [4], that the homogeneous component $F_m^{(n)}(\mathfrak{M})$ of $F_m(\mathfrak{M})$ and $P_n(\mathfrak{M})$ have the same module structures: if $P_n(\mathfrak{M})=\Sigma k(\lambda)M(\lambda)$ then $F_n^{(n)}(\mathfrak{M})=\Sigma k(\lambda)N_m(\lambda)$. We shall denote the standart identity $S_k(x_1, \ldots, x_k) = \Sigma(-1)^\sigma x_{\sigma(1)} \ldots x_{\sigma(k)}$. We refer to [1, 2, 4, 12] as a background for the symmetric and general linear group theory and its application to the polynomial identities.

Next we state the first result of this paper in the following

Theorem 1. *Let $\mathfrak{M}$ be a variety of pairs over a field of characteristic zero. The lattice of subvarieties of $\mathfrak{M}$ is distributive if and only if $\mathfrak{M}$ satisfies the weak identity*

$$(1) \qquad \alpha[x, y]y+\beta y[x, y]=0$$

for suitable $\alpha, \beta \in K$, *such that* $(\alpha, \beta)\neq(0, 0)$.

Remark. This result is analogous to that of A. Ananin, A. Kemer [1], for varieties of associative algebras. But if one compares the description of $P_n(\mathfrak{M})$ in both cases, one can see that the lattice of subvarieties in the case of pairs is more complicated. We recall that the Engel identity is $y(adx)^n=[y, x, x, \ldots, x]=0$ and the identity $d_k(x_1, \ldots, x_k, y_1, \ldots, y_{k-1})=\Sigma(-1)^\sigma x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \ldots y_{k-1} x_{\sigma(k)}$ is known as the Capelli identity.

A. Kostrikin [6] proved that a Lie algebra satisfying the Engel identity is locally nilpotent. S. Mishchenko [7] proved that a Lie algebra with the Engel condition satisfying a Capelli identity of a special form is nilpotent. E. Zelmanov [5] has generalized the previous results and proved that any Lie algebra satisfying the Engel identity is nilpotent. We make use of Zelmanov's result in order to prove the following analog of the Nagata-Higman theorem [11, 13].

Theorem 2. *Let $\mathfrak{M}$ be the variety of pairs "associative algebras — Lie algebras" defined by $x^n=0$. Then $\mathfrak{M}$ satisfies the weak Lie identity $x_1 \ldots x_N=0$ for a suitable N.*

Proof of Theorem 1. It is known that the distributivity of the lattice of subvarieties of the variety $\mathfrak{M}$ is equivalent to the distributivity of the lattice of S_n-submodules in $P_n(\mathfrak{M})$. Therefore, we just have to find necessary and sufficient conditions for $P_n(\mathfrak{M})$ to be a sum of non-isomorfic irreducible S_n-submodules for every $n\geq 1$. The S_n-module P_n of the multilinear polynomials in the freeassociative algebra is isomorphic to the group algebra KS_n and $P_n=\Sigma(\dim M(\lambda)) . M(\lambda)$. The least n with $\dim M(\lambda)>1$ for a given λ is $n=3$, when $\dim M(2, 1)=2$. Hence, a necessary condition for the distributivity of the lattice is the existence of an identity, which "glues" both isomorphic modules $M(2,1)$. Such an identity is (1). In order to prove the theorem it suffices to establish that the identity (1) implies the condition $P_n(\mathfrak{M})\subset \Sigma M(\lambda)$ for any $n\geq 3$.

Denote the variety of pairs determined by the weak identity (1) by $\mathfrak{M}$. We shall examine three different cases:

a) $\alpha\beta(\alpha-\beta)(\alpha+\beta)\neq 0$ or $\alpha=0$ (respectively $\beta=0$);
b) $\alpha-\beta=0$;
c) $\alpha+\beta=0$.

Proposition 1.1. *Let* $\alpha\beta(\alpha-\beta)(\alpha+\beta)\neq 0$ *or* $\beta=0, \alpha\neq 0$ (*respectively* $\alpha=0, \beta\neq 0$). *Then* $P_n(\mathfrak{M})\subseteq \Sigma_{t=0}^{n-1} M(n-t, 1^t)$.

The proof of the Proposition follows from Lemma 1.2 and Lemma 1.3.

Lemma 1.2. *If the conditions of Proposition 1.1 are satisfied, then the multiplicity of the S_n-submodule $M(n-t, 1^t)$, $0\leq t\leq n-1$ in $P_n(\mathfrak{M})$ does not exceed* 1.

Proof. First we shall prove that in the GL-ideal of $\mathfrak{M}$ there exists an element of the form

$$x^{\alpha_1}d_3(x, y, z; x^{\alpha_2}, x^{\alpha_3})x^{\alpha_4}-A(\alpha_1, \ldots, \alpha_4)S_3(x, y, z)x^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.$$

Let us write the identity (1) in the form

$$x[x, y]=b[x, y]x, \quad b=-\frac{\beta}{\alpha}, \quad (\alpha\neq 0). \tag{2}$$

By linearization we obtain the identity

$$x[z, y]+z[x, y]=b([x, y]z+[z, y]x). \tag{3}$$

If we multiply (3) from the left (respectively from the right) by t, summation over all permutations of t, x, y with an alternative change of the signs gives

$$\left|\begin{array}{l}(2+b)u_1+(1-b)u_2-(1-2b)u_3=0\\ (2+b)u_2+(1-b)u_3-(1+2b)u_4=0\end{array}\right.$$

Here we denote $u_1=xS_3(x, y, z)$, $u_2=d_3(x, y, z; x, 1)$, $u_3=d_3(x, y, z; 1, x)$, $u_4=S_3(x, y, z)x$.

We add to this system the obvious identity for the freeassociative algebra $u_1-u_2+u_3-u_4=0$. Since its rank equals 3, the u_i's can be expressed by one of them.

The proof will be completed when we notice that by the identities (2) and (3) we can express any generator of the irreducible S_n-module

$$M(m, 1^{k-1})(m=\Sigma\alpha_i+1), \quad x_1^{\alpha_0}d_k(x_1, \ldots, x_k; x_1^{\alpha_1}, \ldots, x_1^{\alpha_{k-1}})x^{\alpha_k},$$

as a multiple of $S_k(x_1, \ldots, x_k)x_1^{m-1}$.

Lemma 1.3. *Assume that the conditions of Proposition* 1.1 *hold and let* $M(\lambda)$, $\lambda=(\lambda_1, \ldots, \lambda_k)$, *be an irreducible* S_n*-module with* $\lambda_2\geq 2$. *Then* $M(\lambda)$ *has multiplicity* 0 *in* $P_n(\mathfrak{M})$.

Proof. For $n=4$ the only modules $M(\lambda)$ with $\lambda_2\geq 2$ are $2M(2^2)$. They are generated by the linearizations of the elements $[x, y]^2$ and $x[x, y]y-y[x, y]x=b[x, y]^2$ (from (2)). But if $b\neq\pm 1$ we have the identity $[x, y]^2=0$. Hence the generators of $2M(2^2)$ are in the GL-ideal of $\mathfrak{M}$.

We shall use induction on n. We assume that $P_{n-1}(\mathfrak{M})\subseteq\Sigma_{k=0}^{n-1}M(n-1-k, 1^k)$. By the Littlewood — Richardson rule we obtain that $P_n(\mathfrak{M})$ is a submodule of $(P_{n-1}(\mathfrak{M})\otimes M(1))\uparrow S_n\cong\Sigma_{k=1}^{n-1}2M(n-k, 1^k)\oplus\Sigma\varkappa_k M(n-k-1, 2, 1^{k-1})$.

Therefore, it suffices to prove that the generators of the modules $M(n-k-1, 2, 1^{k-1})$, $0\leq k\leq n-1$, are in the GL-ideal of $\mathfrak{M}$.

In general, the generator of the irreducible GL_k-module $N_k(n-k-1, 2, 1^{k-1})$ have the following form:

$$\sum_{\sigma\in S_2} x_1^{\alpha_0}d_k(x_1, \ldots, x_k, x_1^{\alpha_1}, \ldots, x_1^{\alpha_{i-1}}, x_1^{\alpha_i'}x_{\sigma(1)}x_1^{\alpha_i''}, x_1^{\alpha_{i+1}}, \tag{4}$$

$$\ldots, x_1^{\alpha_{j-1}}, x_1^{\alpha_j'}x_{\sigma(2)}x_1^{\alpha_j''}, x_1^{\alpha_{j+1}}, \ldots, x^{\alpha_{k-1}})x^{\alpha_k}.$$

As an obvious consequence of (2) we have the identity

$$x^2y=(b+1)xyx-byx^2.$$

Hence it suffices to prove the statement for the elements (4) only in the case $\alpha_i\leq 1$, $i=0, \ldots, k$. Now we use the following identity obtained from (3):

$$(2+b)x[z, y]+(1-b)d_2(z, y; x)-(2b+1)[z, y]x=0.$$

That gives us the possibility to provide induction on the total degree of x_1. First, let us denote

$$t_1=[x, y]S_3(z, t, u), \quad t_2=d_3(z, t, u; [x, y], 1)$$
$$v_1=xd_3(z, t, u; y, 1)-yd_3(z, t, u; x, 1)$$
$$v_2=xd_3(z, t, u; 1, y)-yd_3(z, t, u; 1, x)$$
$$v_3=xS_3(z, t, u)y-yS_3(z, t, u)x$$
$$u_1=d_3(z, t, u; x, y)-d_3(z, t, u; y, x)$$
$$u_2=d_3(z, t, u; x, 1)-d_3(z, t, u; y, 1)x$$
$$u_3=d_3(z, t, u; 1, [x, y])$$
$$u_4=d_3(z, t, u; 1, x)y-d_3(z, t, u; 1, y)x$$
$$u_5=s_3(z, t, u)[x, y].$$

From (2) and (3) we get the following system of identities:

$$(2+b)t_2+(1-b)u_1-(1+2b)u_2=0$$
$$(2+b)v_1+(1-b)v_2-(1+2b)v_3=0$$
$$(2+b)u_2+(1-b)u_4-(1+2b)u_5=0$$
$$(2+b)t_1+(1-b)v_1-(1+2b)v_2=0$$
$$(2+b)v_3+(1-b)u_2-(1+2b)u_4=0$$
$$(2+b)v_2+(1-b)u_1-(1+2b)u_3=0$$
$$(2+b)u_5+(1-b)u_4-(1+2b)u_3=0$$
$$(2+b)u_3+(1-b)u_1-(1+2b)t_2=0$$
$$(2+b)t_2+(1-b)v_1-(1+2b)t_1=0$$
$$2u_3+2t_1-t_2+v_1+u_1-v_2=0$$
$$2u_5-2t_2-u_3+u_1+u_4-u_2=0$$

and obtain

$$(5)\quad \left|\begin{aligned} u_1 &= \frac{2b^3+7b^2-14b-13}{3(1-b^2)}u_4=A_1u_4,\\ u_2 &= -\frac{2b^2+13b+9}{3(1+b)}u_4=A_2u_4,\\ u_3 &= \frac{b^2+10b+7}{3(1-b^2)}u_4=A_3u_4 \end{aligned}\right.$$

(the cases $b=\pm 1$ are considered separately).

Let us linearize the third identity from (5). We denote $v_1(x, y, z; t)=d_3(x, y, z; 1, t)$. Then we have the identity

$$|\, v_1(x, y, z; t)+v_1(t, y, z; x)=A_3(S_3(x, y, z)t+S_3(t, y, z)x).$$

Permuting x, y, z, t we write two other identities

$$\left| \begin{array}{l} v_1(y, z, x; t)+v_1(t, z, x; y)=A_3(S_3(y, z, x)t+S_3(t, z, x)y) \\ v_1(z, x, y; t)+v_1(t, x, y; z)=A_3(S_3(z, x, y)t+S_3(t, x, y)z). \end{array} \right.$$

Then we add the identity for $K\langle X\rangle$:

$$S_4(x, y, z, t)=v_1(x, y, z; t)-v_1(y, z, t; x)+v_1(z, t, x; y)-v_1(t, x, y; z)$$
$$=S_3(x, y, z)t-S_3(y, z, t)x+S_3(z, t, x)y-S_3(t, x, y)z.$$

By summation of the obtained identities and after some calculations we establish

$$d_3(x, y, z; 1, t)=\frac{1}{4}[(3A_3+1)S_3(x, y, z)t+(A_3-1)S_3(y, z, t)x$$
$$+(A_3+1)S_3(z, t, x)y+(A_3-1)S_3(t, x, y)z].$$

Similarly we obtain analogous identities for the elements $tS_3(x, y, z)$ and $d_3(x, y, z; t, 1)$. Hence using only multiplications from the left and from the right and permutations of the variables we can establish the following identities:

$$x_1S_k(x_1, \ldots, x_k)=B_0S_k(x_1, \ldots, x_k)x_1$$
$$d_k(x_1, \ldots, x_k; 1, \ldots, 1, \underset{\underset{i}{\uparrow}}{x_1}, 1, \ldots, 1)=B_iS_k(x_1, \ldots, x_k)x_1, \quad i=1, \ldots, k-1.$$

The last two identities are inferred from $[x, y]^2=0$. If $b\neq 1, -1/2, -2$, then the homogeneous system has only the trivial solution. The cases $b=-1/2, -2$ are symmetric, so we shall consider only one of them, $b=-2$. We obtain from the system

$$u_1=-u_2=u_4=-u_3=-u_5=-t_2$$
$$v_1=-v_2=v_3=-t_1=0.$$

We replace $t=x$, $u=y$, and add the identity from $K\langle X\rangle$

$$[x, y]S_3(x, y, z)-[xd_3(x, y, z; y, 1)-yd_3(x, y, z; x, 1)]$$
$$+[xd_3(x, y, z; 1, y)-yd_3(x, y, z; 1, x)]-[xS_3(x, y, z)y-yS_3(x, y, z)x]$$

to the system. We again establish that the generators of the module $M(2^2, 1)$ are in the GL-ideal of $\mathfrak{M}$. Next, by linearization and calculations similar to those in the proof of Lemma 1.2 we see that

$$t_1=t_2=v_1=v_2=v_3=u_1=u_2=u_3=u_4=u_5=0.$$

Using only multiplications from the left and from the right we obtain the identities:

$$\sum_{\sigma\in S_2} d_k(x_1, \ldots, x_k; 1, \ldots, \underset{\underset{i}{\uparrow}}{x_{\sigma(1)}}, \ldots, \underset{\underset{j}{\uparrow}}{x_{2\sigma(1)}}, \ldots, 1)=0, \quad 1\leq i<j\leq k-1.$$

This completes the proof of the Lemma 1.3.

In order to complete the proof of Theorem 1 we make use of two other results.

In the case when $\alpha=\beta$ V. Drensky and P. Koshlukov [9] proved that the following equality holds

$$P_n(\mathfrak{M})=\sum_{\lambda} M(\lambda)$$

for the variety of pairs $\mathfrak{M}$ with GL-ideal generated by the identity $[x^2, y]=0$.

In the case when $\alpha+\beta=0$ the weak identity (1) is equivalent to $[x, y, z]=0$.

Here we use the result of I. Volichenko [3] who proved that for weak Lie identities the corresponding variety has a distributive lattice of subvarieties. We can apply that result to our case because the identity $[x, y, z]=0$ gives the same consequences in both cases.

It is clear that the Engel identity $[y, \underbrace{x, \ldots, x}_{2n-1}]=y(adx)^{2n-1}=0$ follows as a weak identity from $x^n=0$.

By the Zelmanov theorem there exists P such that $[x_1, \ldots, x_P]=0$ is a Lie consequence of $y(adx)^{2n-1}$. Without loss of generality we can assume that $P=2^M$. Then we shall prove that $x_1 \ldots x_N=0$, where $N=n^M$.

Proof of Theorem 2. Obviously the weak Lie identities $[[y_1, \ldots, y_{n-k}], [y_{n-k+1}, \ldots, y_n]]=0$, $1\leq k\leq n-1$ follow from the identity $[y_1, \ldots, y_n]=0$.

Let us denote $M_0=2^N$, $M_i=2^{N-i}$. Then it follows from $[x_1, \ldots, x_{M_0}]=0$ that $[\lambda^{(1)}_{M_1}, \lambda^{(2)}_{M_1}]=0$, where $\lambda^{(j)}_i$ is a commutator of length i. We shall denote the complete linearization of x^n by $h_n=h_n(y_1, \ldots, y_n)=\Sigma\, y_{\sigma(1)} \ldots y_{\sigma(n)}$. Next we establish $0=h_n(\lambda^{(1)}_{M_1}, \ldots, \lambda^{(n)}_{M_1})=n!\,\lambda^{(1)}_{M_1} \ldots \lambda^{(n)}_{M_1}$. Let us assume that we can get the weak identity $\lambda^{(1)}_{M_i} \ldots \lambda^{(n^i)}_{M_i}=0$ from $x^n=0$. We shall prove next that $\lambda^{(1)}_{M_{i+1}} \ldots \lambda^{(n^{i+1})}_{M_{i+1}}=0$. Denote $\Lambda_{n^i-1}=\Pi^{n^i}_{k=2}\lambda^{(k)}_{M_i}$. Now our inductive assumption is written in the form $\lambda^{(1)}_{M_i}\Lambda_{n^i-1}=0$. Here we use again the fact mentioned in the beginning of the proof. Then $[\lambda^{(1)}_{M_{i+1}}, \lambda^{(2)}_{M_{i+1}}]\Lambda_{n^i-1}=0$. Now it follows

$$0=h_n(\lambda^{(1)}_{M_{i+1}}, \ldots, \lambda^{(n)}_{M_{i+1}})\Lambda_{n^i-1}=n!\,\lambda^{(1)}_{M_{i+1}} \ldots \lambda^{(n)}_{M_{i+1}}\Lambda_{n^i-1}.$$

If we proceed in the same way with the other commutators of Λ_{n^i-1} we shall finally get the desired result. Then the proof of the Theorem follows for $i=N$.

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