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COMPLEMENTED BLOCK SUBSPACES OF KÖTHE SPACES

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One of the most important open problems in the theory of nuclear Frèchet spaces is the following question: Is it true that every complemented subspace of a nuclear Fréchet space with a basis has a basis?

It is known that the answer is positive in some special cases (see E. Dubinsky [1], E. Dubinsky, D. Vogt [2], B. Mitjagin [3], B. Mitjagin, G. Henkin [4]).

In this paper we consider only complemented block subspaces and prove that every such subspace has a basis. Since every nuclear Frèchet space with a basis is isomorphic to some Köthe space, we consider only Köthe spaces. It turns out that we do not even need to assume nuclearity.

Let E be a Köthe space and let $\{e_i, i\in I\}$, $I=\{1, 2, \dots\}$, be its natural basis. Suppose $I=\bigcup_{n=1}^{\infty} I_n$ is a decomposition of the set of indices I into disjoint subsets and let E_n be the closed linear hull of the vectors $\{e_i, i\in I_n\}$. Then we have $E=\bigoplus_n E_n$ in the sence that every element $x\in E$ has a unique representation $x=\Sigma_{n=1}^{\infty} x_n$, where $x_n\in E_n$. We will call the subspaces E_n *blocks* of E. We say that the subspace $H\subset E$ is a *block subspace* with respect to the decomposition $E=\bigoplus_n E_n$ if we have $H=\bigoplus_n H\cap E_n$.

Let $Q_n: E\to E$, $n=1, 2, \dots$, be the natural projectors corresponding to the subspaces E_n, i. e.

$$Q_n(x)=\sum_{i\in I_n} e_i^*(x)e_i,$$

where the functionals e_i^* are the adjoint functionals of the basis (e_i). Then, obviously, we have that $x=\Sigma_{n=1}^{\infty} Q_n(x)$ for any $x\in E$. The subspace $H\subset E$ is a block subspace with respect to the decomposition $E=\bigoplus_n E_n$ iff $Q_nH\subset H$, $n=1, 2, \dots$.

Our main result is the following.

Theorem. *Suppose E is a Köthe space, $E=\bigoplus_n E_n$ is a decomposition of E into blocks and H is a complemented block subspace with respect to the given decomposition. If $\sup_n \dim E_n<\infty$, then the subspace H is isomorphic to some coordinate subspace of E (i. e., the subspace generated by some subset of the natural basis of E).*

Lemma 1. *If $H\subset E$ is a complemented block subspace, then there exists a "natural projector" P_0 on H such that $P_0(E_n)=H\cap E_n=Q_n(H)$.*

Proof. If $P: E\to E$ is a projector on H, we put

$$P_0(x)=\sum_{n=1}^{\infty} Q_nPQ_n(x), \quad \forall x\in E.$$

It is easy to check that P_0 is a natural projector on H.

Lemma 2. *Suppose that X is a finite-dimensional linear space, $(e_i)_{i=1}^s$ is a basis in X and $P: X\to X$ is a projector onto the subspace $Y=P(X)$. If $(P_{ij})_{i,j=1}^s$ is*

* Research partially supported by Contract 54/25. 03. 87 with Committee of Science in Bulgaria.

the matrix representing P with respect to the basis $(e_i)_{i=1}^s$ *and* $\dim Y=k$, *then we have*

$$\sum_{1\leq i_1<\dots<i_k\leq s} \det (P_{i_\alpha i_\beta})_{\alpha,\beta=1}^k=1.$$

P r o o f. Choose linearly independent vectors $y_1,\dots,y_k\in Y$. Then, $\forall x\in X$,

$$P(x)=\sum_{j=1}^k h_j(x)\,y_j$$

where the functionals $h_j(x)$ satisfy $h_j(y_i)=\delta_{ij}$. Let $h_j=\Sigma_{\alpha=1}^s h_{j_\alpha} e_\alpha^*$, $y_j=\Sigma_{\beta=1}^s y_{j\beta}e_\beta$; then

$$P(e_\alpha)=\sum_{j=1}^k h_j(e_\alpha)\,y_j=\sum_{j=1}^k h_{j\alpha}\sum_{\beta=1}^s y_{j\beta}e_\beta=\sum_{\beta=1}^s\Big(\sum_{j=1}^k h_{j\alpha}y_{j\beta}\Big)e_\beta.$$

Therefore,

$$P_{\alpha\beta}=\sum_{j=1}^k h_{j\alpha}y_{j\beta}.$$

On the other hand, we have

$$1=\det(h_i(y_j))_{i,j=1}^k=h_1\wedge\cdots\wedge h_k(y_1,\dots,y_k)$$

where $h_1\wedge\cdots\wedge h_k$ is the exterior product of the functionals $h_1,\dots,h_k$. Using the properties of the exterior product and letting Σ denote the set of all permutations of $(1,\dots,k)$, we obtain,

$$h_1\wedge\cdots\wedge h_k(y_1,\dots,y_k)=\sum_{1\leq i_1,\dots,i_k\leq s} h_{1i_1}\cdots h_{ki_k}e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*(y_1,\dots,y_k)$$

$$=\sum_{1\leq i_1<\dots<i_k\leq s}\Big(\sum_{\sigma\in\Sigma} h_{1i_{\sigma(1)}}\dots h_{ki_{\sigma(k)}}e_{i_{\sigma(1)}}^*\wedge\cdots\wedge e_{i_{\sigma(k)}}^*(y_1,\dots,y_k)\Big)$$

$$=\sum_{1\leq i_1<\dots<i_k\leq s}\Big(\sum_{\sigma\in\Sigma}(-1)^{\operatorname{sgn}\sigma}h_{1i_{\sigma(1)}}\cdots h_{ki_{\sigma(k)}}e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*(y_1,\dots,y_k)\Big)$$

$$=\sum_{1\leq i_1<\dots<i_k\leq s}\det(h_{ji_\alpha})_{j,\alpha=1}^k\cdot\det(y_{ji_\beta})_{j,\beta=1}^k=\sum_{1\leq i_1<\dots<i_k\leq s}\det\Big(\sum_{j=1}^k h_{ji_\alpha}y_{ji_\beta}\Big)_{\alpha,\beta=1}^k$$

$$=\sum_{1\leq i_1<\dots<i_k\leq s}\det(P_{i_\alpha i_\beta})_{\alpha,\beta=1}^k.$$

The lemma is proved.

P r o o f o f t h e t h e o r e m. Let $E_n=[\{e_i,\ i\in I_n\}]$. It is more convenient for us to change the indices and write $E_n=[\{e_1^n,\dots,e_{s_n}^n\}]$, where $s_n=\dim E_n=\operatorname{card} I_n$. Let H be a complemented block subspace of E and let $P\colon E\to E$ be a natural projector on H, i. e., $P(E_n)=H\cap E_n$. Then we have

$$P(e_i^n)=P_{i1}^n e_1^n+\cdots+P_{is_n}^n e_{s_n}^n.$$

Further we shall do some computations which depend on the n^{th} block and sometimes we shall omit the index n. We put $r_n=\dim H\cap E_n$. Now we use Lemma 2 and choose for any n indices $j_1,\dots,j_{r_n}$ such that

$$\det(P_{j_\nu j_\mu}^n)_{\nu,\mu=1}^{r_n}\geq d>0$$

where $d=\inf\left(\begin{smallmatrix} s_n \\ r_n \end{smallmatrix}\right)^{-1}$. We have $d>0$ because $\sup s_n<\infty$. Further, we shall write $P_{\nu\mu}$ instead of $P^n_{j_\nu j_\mu}$.

A product of the form $P_{\mu_1\mu_2}P_{\mu_2\mu_3}\cdots P_{\mu_{m-1}\mu_m}$ will be called a *chain* connecting μ_1 and μ_m. If $\mu_1=\mu_m$ the chain will be called a *cycle*. For any n we consider the cycles in the terms of $\det(P^n_{\nu\mu})^{r_n}_{\nu,\mu=1}$ and choose the one with the largest absolute value. Corresponding to this cycle we choose an index $\bar\nu(n)$ from those indices which are used in the cycle. (For example, if the cycle is $P_{\nu_1\nu_2}P_{\nu_2\nu_3}P_{\nu_3\nu_1}$ we choose one of the indices ν_1, ν_2, ν_3).

Now we consider the main inequalities:

$$(1)\qquad \forall k\, \exists\, C_{k,\,m(k)} \ni d|t_{\bar\nu(n)}|\,\|e^n_{\bar\nu n}\|_k \leq C_k \Big\| \sum_{\nu=1}^{r_n} t_\nu P(e^n_\nu)\Big\|_{m(k)}.$$

These inequalities will follow from

$$(2)\qquad \forall k\, \exists\, C_{k,\,m(k)} \ni |A_{\bar\nu(n)\mu}|\,\|e^n_{\bar\nu(n)}\|_k \leq C_k \|e^n_\mu\|_{m(k)}$$

where $A_{\bar\nu(n)\mu}$ is the adjoint subdeterminant of $P_{\bar\nu(n)\mu}$. Indeed we have,

$$C_k\Big\|\sum_{\nu=1}^{r_n} t_\nu P(e^n_\nu)\Big\|_{m(k)} = \sum_{\mu=1}^{s_n}\Big|\sum_{\nu=1}^{r_n} t_\nu P_{\nu\mu}\Big| C_k\|e^n_\mu\|_{m(k)} \geq \sum_{\mu=1}^{r_n}\Big|\sum_{\nu=1}^{r_n} t_\nu P_{\nu\mu}\Big|\,|A_{\bar\nu(n)\mu}|\,\|e^n_{\bar\nu(n)}\|_k$$

$$\geq \Big|\sum_{\nu=1}^{r_n} t_\nu \sum_{\mu=1}^{r_n} P_{\nu\mu}A_{\bar\nu(n)\mu}\Big|\,\|e^n_{\bar\nu(n)}\|_k \geq |t_{\bar\nu(n)}|\,|\det(P_{\nu\mu})|\,\|e^n_{\bar\nu(n)}\|_k \geq d|t_{\bar\nu(n)}|\,\|e^n_{\bar\nu(n)}\|_k.$$

To prove (2) we need the following statement:

Lemma 3. *Every term of* $\det(P_{ij})^m_{i,j=1}$ *is a product of cycles.*

Proof. Every term of the determinant has the form

$$(-1)^{\operatorname{sgn}\sigma}P_{1\sigma(1)}P_{2\sigma(2)}\ldots P_{m\sigma(m)},$$

where σ is a permutation of the indices $1,\ldots,m$. On the other hand, every permutation is a product of "elementary permutations" that is, permutations which interchange two indices. Therefore it is enough to prove that interchanging two indices in a term which is the product of cycles, results again in a term which is the product of cycles. Obviously we have two cases — either to interchange indices in one cycle, or to interchange indices from two different cycles. It is easy to see that in the first case the original cycle becomes a product of two cycles, and in the second case, the two cycles become a single cycle. Since every term of the determinant is a permutation of the indices in the term $P_{11}P_{22}\ldots P_{mm}$, which is a product of cycles, the lemma is proved.

Corollary. *Every term of* $A_{\bar\nu(n)\mu}$ *is a product of some cycles and a chain connecting the indices* μ *and* $\bar\nu(n)$.

Proof. Indeed, if we multiply a term of $A_{\bar\nu(n)\mu}$ by $P_{\bar\nu(n)\mu}$ we get a term of $\det(P^n_{\nu\mu})^{r_n}_{\nu,\mu=1}$ which is a product of cycles by Lemma 3. Removing $P_{\bar\nu(n)\mu}$ from the cycle where it occurs, we get a chain connecting the indices μ and $\bar\nu(n)$.

Now we are ready to prove the inequalities (2). Since the projector P is continuous, we have

$$\forall k \, \exists \, C, k_1 \ni \| Pe_i^n \|_k \leq C \| e_i^n \|_{k_1} \quad \forall n \text{ and } i=1, \ldots, s_n.$$

On the other hand,

$$\| Pe_i^n \|_k = \| \sum_{j=1}^{s_n} P_{ij}^n e_j^n \|_k = \sum_{j=1}^{s_n} | P_{ij}^n | \, \| e_j^n \|_k \geq | P_{ij}^n | \, \| e_j^n \|_k$$

so,

(3) $$\forall k \, \exists \, C, k_1 \ni | P_{ij}^n | \, \| e_j^n \|_k \leq C \| e_i^n \|_{k_1} \quad \forall n \text{ and } i, j=1, \ldots, s_n.$$

Suppose $P_{\mu\mu_1}^n \cdots P_{\mu_t \bar{\nu}(n)}^n$ is a chain connecting the indices $\mu=\mu(n)$ and $\bar{\nu}(n)$. Then using. (3) we get

(4) $$\forall k \, \exists \, \tilde{C}, \tilde{k} \ni | P_{\mu\mu_1}^n \cdots P_{\mu_t \bar{\nu}(n)}^n | \, \| e_{\bar{\nu}(n)}^n \|_k \leq \tilde{C} \| e_\mu^n \|_{\tilde{k}}.$$

Since $\sup_n \dim (E_n \cap H) < \infty$, the constants $\tilde{C}, \tilde{k}$ can be chosen independent of n.

Using again the same argument, we conclude that if $P_{\bar{\nu}(n)\nu_1}^n \cdots P_{\nu_m \bar{\nu}(n)}^n$ is the cycle with the largest absolute value among the terms of $\det (P_{\nu\mu}^n)_{\nu, \mu=1}^{r_n}$ then

(5) $$\forall k \, \exists \, \bar{C}, \bar{k} \ni | P_{\bar{\nu}(n)\nu_1}^n \cdots P_{\nu_m \bar{\nu}(n)}^n | \, \| e_{\bar{\nu}(n)}^n \|_k \leq \bar{C} \| e_{\bar{\nu}(n)}^n \|_{\bar{k}}$$

where the constants $\bar{C}, \bar{k}$ do not depend on n.

By the corollary to Lemma 3, the absolute value of every term in $A_{\bar{\nu}(n)\mu}$ is not bigger than the absolute value of a product of a chain connecting μ and $\bar{\nu}(n)$ with some power of the cycle $P_{\bar{\nu}(n)\nu_1}^n \cdots P_{\nu_m \bar{\nu}(n)}^n$. Hence, using (4) and (5), we get (2).

Let H^1 be the subspace of H generated by the vectors $(Pe_{\bar{\nu}(n)}^n)_{n=1}^\infty$. Then H^1 is a complemented subspace of H and the vectors $(Pe_{\bar{\nu}(n)}^n)_{n=1}^\infty$ form a basis of H^1. Indeed, we have $H^1 = \oplus_n H_n^1$, where H_n^1 is the one-dimensional subspace generated by the vector $Pe_{\bar{\nu}(n)}^n$.

Consider the projectors $\pi_n \colon H \cap E_n \to H_n^1$ given by the formulas

$$\pi_n \Big(\sum_{\nu=1}^{r_n} t_\nu P(e_\nu^n) \Big) = t_{\bar{\nu}(n)} P(e_{\bar{\nu}(n)}^n).$$

Using (1), we obtain

$$\forall k \, \exists \, C_k, m(k) \ni \| \pi_n(x) \|_k \leq C_k \| x \|_{m(k)} \quad \forall x \in H_n.$$

Therefore, the operators π_n define a continuous projector $\pi \colon H \to H$, such that $\pi(H) = H^1$. It is evident that the vectors $P(e_{\bar{\nu}(n)}^n)$ form a basis for H^1. From (1), choosing $t_{\bar{\nu}(n)} = 1$, $t_\nu = 0$ for $\nu \neq \bar{\nu}(n)$, we get

$$\forall k \, \exists \, C_k, m(k) \ni \| e_{\bar{\nu}(n)}^n \|_k \leq \frac{1}{d} C_k \| P(e_{\bar{\nu}(n)}^n) \|_{m(k)}.$$

These inequalities, together with the continuity of P imply that the restriction of P to the subspace E^1, generated by the vectors $\{e_{\bar{\nu}(n)}^n\}_{n=1}^\infty$ is an isomorphism between E^1 and H^1.

Now we can consider ihe subspace $H \ominus H^1$, apply the same argument to it and finish the proof of the theorem by induction on the number $\sup_n \dim H \cap E_n$.

REFERENCES

1. Ed Dubinsky. The structure of nuclear Fréchet spaces. *(Lecture Notes in Math., Vol. 720)*, Berlin, 1979.
2. Ed Dubinsky, D. Vogt. Complemented subspaces in tame power series spaces. *Studia Math.*, (to appear).
3. Б. С. Митягин. Эквивалентность базисов в гильбертовых шкалах. *Studia Math.*, 37, 1971, 111–137.
4. Б. С. Митягин, Г. М. Хенкин. Линейные задачи комплексного анализа. *Успехи матем. наук*, 26, 1972, 93–152.

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Received 6. 8. 1987