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APPLICATION OF FRACTIONAL CALCULUS TO A THIRD ORDER LINEAR ORDINARY DIFFERENTIAL EQUATION

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Fractional Calculus is used to obtain a particular solution of the following non-homogeneous third order linear ordinary differential equation of Fuchs type:

$$\varphi_3 . (z^3-z) + \varphi_2 . \{(3\alpha+\beta+\gamma)\, z^2+\beta\gamma z-\alpha\} + \varphi_1 . \{\alpha(3\alpha+2\beta+2\gamma-3)\, z+\alpha\beta\gamma\}$$
$$+ \varphi . \alpha(\alpha-1)\,(\alpha+\beta+\gamma-2)=f, \quad z\neq 0, \ \pm 1$$

α, β and γ are constants, $z \in C$, $\varphi=\varphi(z)$, $\varphi_1=d\varphi/dz$, $\varphi_2=d^2\varphi/dz^2$, $\varphi_3=d^3\varphi/dz$ and $f=f(z)$ is a known function.

1. Introduction. Fractional calculus deals with the derivatives and integrals of arbitrary orders, called the differintegrals [7, 10, 11, 12]. The concept of differintegral of complex order ν, which is a generalization of the ordinary n-th derivative and n-times integral, can be introduced in several ways. One of the simplest definitions of an integral of fractional order is based on an integral transform, called the Riemann — Liouville operator of fractional integration [12]

$$R^{\alpha}f=I_x^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt, \quad \mathrm{Re}\,(\alpha)\geqq 0 \tag{1}$$

$$=\frac{d^n}{dx^n}R_x^{\alpha+n}f, \quad \text{for } \mathrm{Re}\,(\alpha)<0. \tag{2}$$

Another fundamental definition of differintegral of order ν, due to Grunwald [11] is as follows:

$$\frac{d^{\nu}f}{[d(x-a)]^{\nu}}=\lim_{N\to\infty}\Big\{\frac{\left[\frac{x-a}{N}\right]^{-\nu}}{\Gamma(-\nu)}\sum_{j=0}^{N-1}\frac{\Gamma(j-\nu)}{\Gamma(j+1)}f(x-j\,[\frac{x-a}{N}])\Big\}, \tag{3}$$

where ν is arbitrary. It is interesting to observe that here no explicit use is made of classical definitions of derivatives or integrals of f.

K. Nishimoto [7, 10] defines the differintegral as follows: If $f(z)$ is a regular function and it has no branch points inside and on C (where $C=\{\underset{-}{C}, \underset{+}{C}\}$ $\underset{-}{C}$ being an integral curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}\,(z)$, and $\underset{+}{C}$ being an integral curve along the cut joining two points z and $\infty+i\,\mathrm{Im}\,(z)$),

$$f_{\nu}={}_cf_{\nu}\,(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_c\frac{f(\xi)\,d\xi}{(\xi-z)^{\nu+1}} \tag{4}$$

$$\nu\notin Z^-; \quad \nu\in R \ \text{ and } \ f_{-n}=\lim_{\nu\to -n}f_{\nu}\,(n\in Z^+),$$

where $\xi \neq z$, $-\pi \leqq \arg(\xi - z) \leqq \pi$ for $\underset{-}{C}$ and $0 \leqq \arg(\xi - z) \leqq 2\pi$ for $\underset{+}{C}$, then $f_\nu(\nu > 0)$ is the fractional derivative of order ν and $f_\nu(\nu < 0)$ is the fractional integral of order ν, if f_ν exists (consider the principal value of f for many valued functions).

Various authors [1, 2, 5, 13] have defined and studied differintegral operators and their applications. These operators have applications not only in analysis, but also in many physical problems [6, 11]. S. Kalla, B. Ross [3], S. Kalla, B. Al-Saqabi [4] have applied them in summation of series, while K. Nishimoto [8, 9, 10] has obtained the solution of several differential equations by invoking the fractional calculus. In the present paper, we obtain a particular solution of a differential equation of Fuchs type by means of the fractional calculus. The method presented here can be easily extended to solve other similar differential equations.

2. Differential equation and its solution. Here we shall deal with a third order linear non-homogeneous differential equation of Fuchs-type.

Theorem 1. *If $f_\alpha(\neq 0)$ exists, then the non-homogeneous third order linear ordinary differential equation of Fuchs type*

$$\varphi_3 \cdot (z^3 - z) + \varphi_2 \cdot \{(3\alpha + \beta + \gamma) z^2 + \beta\gamma z - \alpha\} + \varphi_1 \cdot \{\alpha(3\alpha + 2\beta + 2\gamma - 3) z + \alpha\beta\gamma\} + \varphi \cdot \alpha(\alpha - 1)(\alpha + \beta + \gamma - 2) = f \quad (z \neq 0, \pm 1) \tag{5}$$

has a particular solution of the form

$$\varphi = \left(\left(f_{-\alpha} \cdot \frac{(z-1)^A (z+1)^B}{z^3 - z}\right)_{-1} \cdot \frac{1}{(z-1)^A (z+1)^B}\right)_{\alpha - 2}, \tag{6}$$

where $\varphi = \varphi(z)$, $f = f(z)$, $z \in C$, and α, β and γ are constants and $A = (\beta + \gamma + \beta\gamma)/2$, $B = (\beta + \gamma - \beta\gamma)/2$.

Proof. Putting

$$\varphi = w_\alpha = w_\alpha(z) \tag{7}$$

yields

$$\varphi_1 = w_{1+\alpha} \tag{8}$$

$$\varphi_2 = w_{2+\alpha}, \tag{9}$$

and

$$\varphi_3 = w_{3+\alpha}, \tag{10}$$

where $w = w(z)$.

Substituting (8), (9) and (10) into (5), we obtain

$$(w_3 \cdot z^3)_\alpha - (w_3 \cdot z)_\alpha + (\beta + \gamma)(w_2 \cdot z^2)_\alpha + \beta\gamma (w_2 \cdot z)_\alpha = f, \tag{11}$$

that is,

$$w_3 + w_2 \cdot \frac{(\beta + \gamma) z + \beta\gamma}{z^2 - 1} = f_{-\alpha} \cdot \frac{1}{z^3 - z}, \tag{12}$$

since [7]

$$(w_3 \cdot z^3)_\alpha = w_{3+\alpha} \cdot z^3 + 3\alpha w_{2+\alpha} \cdot z^2 + 3\alpha(\alpha - 1) w_{1+\alpha} \cdot z + \alpha(\alpha - 1)(\alpha - 2) w_\alpha,$$

$$(w_3 \cdot z)_\alpha = w_{3+\alpha} \cdot z + \alpha w_{2+\alpha}, \quad (w_2 \cdot z^2)_\alpha = w_{2+\alpha} \cdot z^2 + 2\alpha w_{1+\alpha} \cdot z + \alpha(\alpha - 1) w_\alpha,$$

and $(w_2 \cdot z)_\alpha = w_{2+\alpha} \cdot z + \alpha w_{1+\alpha}$.

Putting $w_2 = u = u(z)$,

we have then

$$u_1 + u \cdot \frac{(\beta+\gamma)z+\beta\gamma}{z^2-1} = f_{-\alpha} \cdot \frac{1}{z^3-z} \tag{13}$$

from (12).

A particular solution to this first order linear differential equation is given by

$$u = w_2 = \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \frac{1}{(z-1)^A(z+1)^B}. \tag{14}$$

Therefore we have

$$\varphi = w_\alpha = u_{\alpha-2} = \Big(\Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \frac{1}{(z-1)^A(z+1)^B}\Big)_{\alpha-2}$$

as a particular solution to the equation (5).

Inversely, substituting (6) into the left hand side of (5), we have then that the left hand side of (5) equals to

$$(w_3 . z^3 - w_3 . z + (\beta+\gamma)\, w_2 . z^2 + \beta\gamma w_2 . z)_\alpha = (w_3 . (z^3-z) + w_2 . \{(\beta+\gamma)z^2 + \beta\gamma z\})_\alpha$$

$$= \Big(\Big(\Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \frac{1}{(z-1)^A(z+1)^B}\Big)_1 \cdot (z^3-z)$$

$$+ \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \frac{(\beta+\gamma)z^2+\beta\gamma z}{(z-1)^A(z+1)^B}\Big)_\alpha$$

$$= \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z} \cdot \frac{z^3-z}{(z-1)^A(z+1)^B} \tag{15}$$

$$+ \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \Big(\frac{1}{(z-1)^A(z+1)^B}\Big)_1 \cdot (z^3-z)$$

$$+ \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \cdot \frac{(\beta+\gamma)z^2+\beta\gamma z}{(z-1)^A(z+1)^B}\Big)_\alpha = (f_{-\alpha})_\alpha = f.$$

Changing the order

$$\Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1} \quad \text{and} \quad \frac{1}{(z-1)^A(z+1)^B},$$

we have another solution:

$$\varphi = \Big(\frac{1}{(z-1)^A(z+1)^B} \cdot \Big(f_{-\alpha} \cdot \frac{(z-1)^A(z+1)^B}{z^3-z}\Big)_{-1}\Big)_{\alpha-2} \tag{16}$$

for $\alpha \notin z$ [10].

3. Solution of homogeneous differential equation. In this section we shal obtain a particular solution of the homogeneous differential equation of Fuchs type

Theorem 2. *Homogeneous third order differential equation*

$$\varphi_3 . (z^3-z) + \varphi_2 . \{(3\alpha+\beta+\gamma)z^2 + \beta\gamma z - \alpha\} + \varphi_1 . \{\alpha(3\alpha+2\beta+2\gamma-3)z + \alpha\beta\gamma\}$$

$$+ \varphi . \alpha(\alpha-1)(\alpha+\beta+\gamma-2) = 0 \qquad (z \neq 0, \ \pm 1) \tag{17}$$

has a solution of the form

$$\varphi = \Big(\frac{1}{(z-1)^A(z+1)^B}\Big)_{\alpha-2}, \tag{18}$$

where $\varphi = \varphi(z)$ *and* $z \in C$.

Proof. Putting $\varphi = w_\alpha$ we have that

$$w_3 + w_2 \cdot \frac{(\beta+\gamma)z+\beta\gamma}{z^2-1} = 0 \tag{19}$$

from (17). Hence

$$u_1+u\cdot\frac{(\beta+\gamma)z+\beta\gamma}{z^2-1}=0 \tag{20}$$

form (19), where $w_2=u=u(z)$.

And a solution to the equation (20) is given by $u=\frac{1}{(z-1)^A(z+1)^B}$. Therefore we obtain

$$\varphi=w_\alpha=u_{\alpha-2}=\left(\frac{1}{(z-1)^A(z+1)^B}\right)_{\alpha-2} \tag{21}$$

as a solution to the equation (17).

Theorem 3. *If $f_\alpha(\neq 0)$ exists, then the fractional differintegrated function*

$$\varphi=\left(\left(f_{-\alpha}\cdot\frac{(z-1)^A(z+1)^B}{z^3-z}\right)_{-1}\cdot\frac{1}{(z-1)^A(z+1)^B}\right)_{\alpha-2}+\left(\frac{1}{(z-1)^A(z+1)^B}\right)_{\alpha-2} \tag{22}$$

satisfies the non-homogeneous third order linear ordinary differential equation of Fuchs type (5).

Proof. It is clear by the Theorems 1 and 2.

4. Examples. Here we consider some examples of the theorems of the previous section.

(I) Examples of Theorem 1

(i) Let $\alpha=1$ and $\beta=\gamma=0$, we have then

$$\varphi_3.(z^3-z)+\varphi_2.(3z^2-1)=f \qquad (z\neq 0,\ \pm 1) \tag{23}$$

and

$$\varphi=\left(\left(f_{-1}\cdot\frac{1}{z^3-z}\right)_{-1}\right)_{-1}=\left(f_{-1}\cdot\frac{1}{z^3-z}\right)_{-2} \tag{24}$$

from (5) and (6), respectively.

The function shown by (24) satisfies (23) clearly.

(ii) Let $\alpha=-1/2$, $\beta=0$ and $\gamma=2$, we have that

$$\varphi_3.(z^3-z)\alpha+\varphi_2\cdot\frac{1}{2}(z^2+1)\beta+\varphi_1.\gamma\frac{z}{4}-\varphi\cdot\frac{3}{8}=f \quad (z\neq 0,\ \pm 1) \tag{25}$$

and

$$\varphi=\left(\left(f_{1/2}\cdot\frac{1}{z}\right)_{-1}\cdot(z^2-1)^{-1}\right)_{-5/2} \tag{26}$$

from (5) and (6), respectively.

Moreover, if we put $f=z^{-3/2}$, the equations (25) and (26) are reduced to

$$\varphi_3.(z^3-z)+\varphi_2.1/2(z^2+1)+\varphi_1\cdot\frac{z}{4}-\varphi\cdot\frac{3}{8}=z^{-3/2}\ (z\neq 0,\ \pm 1) \tag{27}$$

and

$$\varphi=\left(\frac{i}{\sqrt{\pi}}\sum_{k=1}^{\infty}z^{-2(k+1)}\right)_{-5/2}\ \ (|z|>1)=\frac{-1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{\Gamma(2k-1/2)}{\Gamma(2k+2)}z^{-2k+1/2} \tag{28}$$

respectively, since [7]

$$(z^{-3/2})_{1/2}=\frac{-2i}{\sqrt{\pi}}z^{-2}$$

and

$$z^{-2}(z^2-1)^{-1}=\sum_{k=1}^{\infty} z^{-2(k+1)} \qquad (|z|>1).$$

That is, (28) is a particular solution to the differential equation (27).

Inversely, substituting (28) into the left hand side of (27), we have that the left hand side of (27) equals to

$$\frac{i}{\sqrt{\pi}}(-2z^{-2})_{-1/2}=z^{-3/2}.$$

Substituting φ and its derivatives from (28) into the left hand side of (27), we have that the left hand side of (27) equals to

$$\frac{1}{\sqrt{\pi}}\Big[\sum_{k=0}^{\infty}\frac{1}{\Gamma(2k+2+2)}\{\Gamma(2k+2+5/2)-1/2\,\Gamma(2k+2+3/2)+1/4\Gamma(2k+2+1/2)$$
$$+3/8\,\Gamma(2k+2-1/2)\}\,z^{-2k-3/2}$$
$$-\sum_{k=1}^{\infty}\frac{1}{\Gamma(2k+2)}\{\Gamma(2k+5/2)+1/2i\,(2k+3/2)\}z^{-2k-3/2}\Big]=\frac{1}{\sqrt{\pi}}\cdot\frac{12}{\Gamma(4)}\,\Gamma(3/2)\,z^{-3/2}=z^{-3/2}.$$

(iii) Let $\alpha=-1/2$, $\beta=0$, $\gamma=2$ and

$$f=\mathrm{K}e^{-z}(z^3-1/2\,z^2-3/4\,z-1/8),$$

we have then

$$\varphi_3.(z^3-z)+\varphi_2\cdot\frac{1}{2}(z^2+1)+\varphi_1\cdot z/4-\varphi\cdot 3/8$$

(29)
$$=\mathrm{K}e^{-z}(z^3-1/2\,z^2-3/4\,z-1/8) \qquad (z\neq 0,\ \pm 1)$$

and

(30) $$\varphi=((f_{1/2}\,\frac{1}{z})_{-1}\,(z^2-1)^{-1})_{-5/2}=\frac{K}{i}(e^{-z})_{-5/2}=\frac{K}{i}(-ie^{-z})=-\mathrm{K}e^{-z}$$

from (5) and (6), respectively, since

$$f=\mathrm{K}e^{-z}(z^3-1/2\,z^2-3/4\,z-1/8)=\frac{K}{i}((e^{-z}.(z^2-1)_1.z)_{-1/2}.$$

Inversely, we obtain that $\varphi_1=\mathrm{K}e^{-z}$, $\varphi_2=-\mathrm{k}e^{-z}$

(31) and $\varphi_3=\mathrm{K}e^{-z}$.

Substituting (30) and (31) into the left hand side of (29), we have that the left hand side of (29) equals to

(32) $$\mathrm{K}e^{-z}(z^3-z-1/2\,z^2-1/2+1/4\,z+3/8)=\mathrm{K}e^{-z}(z^3-1/2\,z^2-3/4\,z-1/8).$$

As,

(33) $$(e^{-z}\cdot(-z^3+2z^2+z))_{-1/2}=ie^{-z}(z^3-1/2\,z^2-3/4\,z-1/8).$$

(II) Examples of Theorem 2

(i) Let $\alpha=1$, $\beta=0$ and $\gamma=2$, we have then

(34) $$\varphi_3.(z^3-z)+\varphi_2.(5z^2-1)+\varphi_1.4z=0 \qquad (z\neq 0,\ \pm 1)$$

and

(35) $$\varphi = \left(\frac{1}{z^2-1}\right)_{-1}$$

from (17) and (18) respectively since $A=B=1$.
Inversely, substituting φ and its derivatives from (35) into the left hand side of (34), we obtain that the left hand side of (34) equals to

$$\frac{6z^3+2z-10z^3+2z+4z^3-4z}{(z^2-1)^2}=0.$$

(ii) Let $\alpha=-1/2$, $\beta=0$ and $\gamma=2$, we have then

(36) $$\varphi_3\,.(z^3-z)+\varphi_2\,.1/2\,(z^2+1)+\varphi_1\,.z/4-\varphi\cdot\frac{3}{8}=0 \quad (z\neq 0,\ \pm 1)$$

and

(37) $$\varphi=((z^2-1)^{-1})_{-5/2}$$

(38) $$=\Big(\sum_{k=0}^{\infty} z^{-2(k+1)}\Big)_{-5/2} \quad (|z|>1)$$

(39) $$=\sum_{k=0}^{\infty}\frac{\Gamma(2k-1/2)}{\Gamma(2k+2)}\,z^{-2k+1/2},$$

from Theorem 2.

Inversely, substituting (38) into the left hand side of (36), we have that the left hand side of (36) equals to

$$(w_3\,.(z^3-z)+w_2\,.2z^2)_{-1/2}=\Big(\sum_{k=0}^{\infty} z^{-2(k+1)}\Big)_1\,.(z^3-z)=0,$$

where

$$w=\Big(\sum_{k=0}^{\infty} z^{-2(k+1)}\Big)_{-2}.$$

If we substitute (37) into the left hand side of (36), we have that the left hand side of (36) equals to

$$(w_3\,.(z^3-z)+w_2\,.2z^2)_{-1/2}=((z^2-1)^{-1})_1\,.(z^3-z)+(z^2-1)^{-1}\,.2z^2)_{-1/2}=(0)_{-1/2}=0,$$

where $w=((z^2-1)^{-1})_{-2}$.
And if we substitute (39) into the left hand side of (36), we have that the left hand side of (36) equals to

$$i\sum_{k=-1}^{\infty}\frac{1}{8\Gamma(2k+2+2)}\{-8\,(2k+2+5/2)+4\Gamma(2k+2+3/2)-2\Gamma(2k+2+1/2)$$

$$-3\Gamma(2k+2-1/2)\}\,z^{-2k-3/2}+i\sum_{k=0}^{\infty}\frac{1}{2(2k+2)}\{2\Gamma(2k+5/2)+(2k+3/2)\}\,z^{-2k-3/2}=0.$$

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