Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

OBJECTION AND COUNTER-OBJECTION EQUILIBRIA IN MANY-PLAYER STOCHASTIC DIFFERENTIAL GAMES

SVETOSLAV D. GAIDOV

In this paper N-player nonzero-sum games are considered. The dynamics is described by Ito stochastic differential equations. The cost-functions are conditional expectations of functionals of Bolza type with respect to the initial situation. The concept of objection and counter-objection equilibrium in many-player stochastic differential games is introduced and analyzed. Sufficient conditions are established guaranteeing the objection and counter-objection equilibrium for the strategies of the players.

1. Introduction. In this paper we follow the approach of W. Fleming and R. Rishel [1] to the optimal control of stochastic dynamic systems, but applied in situations of conflicts, i. e. to stochastic differential games. Let $\{1, \dots, N\}$ be the set of players. The dynamics is described by the equation

$$dx(t)=f(t, x(t), u_1, \dots, u_N)dt+g(t, x(t), u_1, \dots, u_N)dw(t), \quad t\in[t_0, T].$$

The control u_i is chosen by the i-th player in the feedback from $u_i=u_i(t, x(t))$ with the purpose to minimize its personal cost-function

$$J_i(u_1, \dots, u_N)=\mathbf{E}_{t_0, x_0}\{\Psi_i(T, x(T))+\int_{t_0}^{T} L_i(t, x(t), u_1, \dots, u_N)\, dt\}.$$

As a solution of the game the notion of objection and counter-objection equilibrium is proposed. In deterministic differential games this concept is introduced by V. Zhukovskii and considered in [8]. In two-player stochastic differential games the same concept is treated by the author in [2]. The objection and counter-objection equilibrium is based on the notion of Pareto-optimality (see [3], [4]) and represents a further development in the game theory in comparison with the Nash-equilibrium (see [3], [5]).

Let us now give the outlines of the present work. In Section 2 we consider accurately the formalization of the game. Some results and definitions from our papers [3—6] are quoted in Section 3. In Section 4 we introduce the notion of objection and counter-objection equilibrium in many-player stochastic differential games and analyze some of its properties. Sufficient conditions for the objection and counter-objection equilibrium strategies of the players are established in Section 5.

2. Formalization of the game. Let us consider the system (game)

$$\Gamma=\langle I, \Sigma, \{\mathcal{U}_i\}_{i\in I}, \{J_i\}_{i\in I}\rangle.$$

Here $I=\{1, \dots, N\}$ is the set of players participating in the game Γ. The evolution of the dynamic system Σ is described by Ito stochastic differential equation of the type

$$(*) \qquad dx(t)=f(t, x(t), u_1, \dots, u_N)\, dt+g(t, x(t), u_1, \dots, u_N)\, dw(t), \quad t\in[t_0, T]$$

with initial condition $x(t_0)=x_0\in \mathrm{R}^n$, where $T>t_0\geq 0$. The process $w(t)$, $t\in[t_0, T]$ is a standard m-dimensional Wiener process defined on some complete probability space (Ω,

$\mathcal{F}$, P) and is adapted to a family $F=\{\mathcal{F}_t, t\in[t_0, T]\}$ of nondecreasing sub-σ-algebras of $\mathcal{F}$. The vector $x(t)\in \mathrm{R}^n$ is the state process and $u_i\in U_i\subset \mathrm{R}^{n_i}$ is the control of the i-th player, $i\in I$. Now let us make the following assumptions about the functions $f(t, x, u_1, \ldots, u_N)$ and $g(t, x, u_1, \ldots, u_N)$. Suppose

$$f\colon [t_0, T]\times \mathrm{R}^n\times U_1\times\cdots\times U_N\to \mathrm{R}^n$$

and

$$g\colon [t_0, T]\times \mathrm{R}^n\times U_1\times\cdots\times U_N\to \mathrm{R}^n\times \mathrm{R}^m$$

have continuous partial derivatives in $x, u_1, \ldots, u_N$ and let $C>0$ be a constant such that

$$|f(t, 0, \ldots, 0)|+|g(t, 0, \ldots, 0)|\leq C,$$

$$|f_x|+|g_x|+\sum_{i\in I}(|f_{u_i}|+|g_{u_i}|)\leq C.$$

Here $|\cdot|$ is a general symbol for the norms in the respective spaces.

We suppose that each player has complete information about the state vector $x(t)$ at every moment $t\in[t_0, T]$ and constructs his strategy in the game Γ as an admissible feedback control, i. e. $u_i=u_i(t, x(t))$ where

$$u_i(\cdot, \cdot)\colon [t_0, T]\times \mathrm{R}^n\to U_i$$

is a Borel function satisfying the conditions:

(i) There exists a constant $M_i>0$ such that

$$|u_i(t, x)|\leq M_i(1+|x|) \text{ for all } t\in[t_0, T], \quad x\in \mathrm{R}^n;$$

(ii) For each bounded set $B\subset \mathrm{R}^n$ and $T^*\in(T_0, T)$, there exists a constant $K_i>0$ such that for arbitrary $x, y\in B$ and $t\in[t_0, T^*]$

$$|u_i(t, x)-u_i(t, y)|\leq K_i|x-y|.$$

Denote by $\mathcal{U}_i$ the set of strategies of the i-th player, $i\in I$ and $\mathcal{U}=\Pi_{i\in I}\mathcal{U}_i$, $U=\Pi_{i\in I}U_i$. Let a vector of strategies $u=(u_1, \ldots, u_N)\in\mathcal{U}$ be called for brevity simply a strategy.

The assumptions given above imply the existence and sample path uniqueness of the solution $X=\{x(t), t\in[t_0, T]\}$ of equation (*) corresponding to the control $u\in\mathcal{U}$ (see [1]). Moreover, X is an a. s. continuous Markov process and its infinitesimal operator $\mathcal{A}(u)$ has the form

$$\mathcal{A}(u)V(t, x)=f'(t, x, u)V_x(t, x)+\frac{1}{2}\operatorname{tr}[a(t, x, u)V_{xx}(t, x)]$$

where $a=gg'$ and prime denotes vector or matrix transpose. Here $V(t, x)$ is a real-valued function with continuous partial derivatives up to second order for all $t\in[t_0, T]$, $x\in \mathrm{R}^n$.

Let L_i, Ψ_i be continuous functions satisfying the polynomial growth conditions

$$|L_i(t, x, u_1, \ldots, u_N)|\leq C_i(1+|x|+\sum_{i\in I}|u_i|)^\alpha,$$

$$|\Psi_i(t, x)|\leq C_i(1+|x|)^\alpha,$$

where α, C_i are positive constants. Now we introduce the cost-function $J_i(u)$ of the i-th player

$$J_i(u)=\mathrm{E}_{t_0, x_0}\{\Psi_i(T, x(T))+\int_{t_0}^{T}L_i(t, x(t), u_1, \ldots, u_N)\,dt\}, \quad i\in I.$$

The object of each player in the game Γ is to minimize his own cost-function.

3. Auxiliary notions and results. For the completeness of the presentation we quote some facts from our previous papers.

Definition 3.1 (see [6]). The strategy $u_i^g \in \mathcal{U}_i$ is a guaranteeing strategy of the i-th player in the game Γ if

$$\min_{u_i} \max_{u_{I \setminus i}} J_i(u_i, u_{I \setminus i}) = \max_{u_{I \setminus i}} J_i(u_i^g, u_{I \setminus i}).$$

Here $I \setminus i = \{1, \ldots, i-1, i+1, \ldots, N\}$ and $u_{I \setminus i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N) \in \Pi_{j \in I \setminus i} \mathcal{U}_j = \mathcal{U}_{I \setminus i}$. Let also $(u_i, u_{I \setminus i}) = u$.

Definition 3.2 (see [3], [5]). The strategy $u^n \in \mathcal{U}$ is a Nash-equilibrium strategy in the game Γ if for each $u_i \in \mathcal{U}_i$

$$J_i(u_1^n, \ldots, u_{i-1}^n, u_i, u_{i+1}^n, \ldots, u_N^n) = J_i(u^n \| u_i) \geqq J_i(u^n), \quad i \in I.$$

Definition 3.3 (see [3], [4]). The strategy $u^p \in \mathcal{U}$ is Pareto-optimal in the game Γ if the relations $J_i(u) \leqq J_i(u^p)$, $i \in I$ for some strategy $u \in \mathcal{U}$ imply the equalities $J_i(u) = J_i(u^p)$, $i \in I$.

Theorem 3.4 (see [3], [4]). The strategy $u^p \in \mathcal{U}$ is Pareto-optimal in the game Γ if there exist a vector $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathrm{R}^N$, $\lambda_i > 0$, $i \in I$, $\lambda_1 + \cdots + \lambda_N = 1$ and a real-valued function $V(t, x)$ such that for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$ the following conditions jointly hold:

(a) V, V_t, V_x, V_{xx} are continuous;

(b) $H_\lambda(t, x, u^p) = 0$;

(c) $H_\lambda(t, x, u) \geqq 0$ for each strategy $u \in \mathcal{U}$;

(d) $V(T, x) = \sum_{i \in I} \lambda_i \Psi_i(T, x)$.

Here for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$, $u \in U$,

$$H_\lambda(t, x, u) = V_t(t, x) + \mathcal{A}(u) V(t, x) + \sum_{i \in I} \lambda_i L_i(t, x, u).$$

4. Objection and counter-objection equilibrium. Basic properties. Now we generalize for many-player games the concept of objection and counter-objection equilibrium, considered in [2] in the case of two players. Let $u \in \mathcal{U}$ be an arbitrary strategy.

Definition 4.1. *The strategy* $u_i^o \in \mathcal{U}$ *is an objection of the i-th player to* $u \in \mathcal{U}$ *if* $J_i(u \| u_i^o) < J_i(u)$.

Definition 4.2. *The strategy* $u_j^{co} \in \mathcal{U}_j$ $(i \neq j)$ *is a counter-objection of the j-th player to the objection* u_i^o *of the i-th one if*

$$J_i(u \| u_i^o, u_j^{co}) \geqq J_i(u) \text{ and } J_j(u \| u_i^o, u_j^{co}) \leqq J_j(u).$$

Here $(u \| u_i^o, u_j^{co}) = (u_1, \ldots, u_{i-1}, u_i^o, u_{i+1}, \ldots, u_{j-1}, u_j^{co}, u_{j+1}, \ldots, u_N)$.

Definition 4.3. *The strategy* $u^* \in \mathcal{U}$ *is an objection and counter-objection equilibrium strategy in* Γ *if it is Pareto-optimal and either there is no objection of any player, or to every objection of any player there exists a counter-objection at least of another one of the rest of the players.*

Now we analyze some properties of the objection and counter-objection strategies and compare them with other optimal strategies.

Property 4.4. Pareto-optimality. By definition we have that objection and counter-objection equilibria are Pareto-optimal.

Property 4.5. Pareto-optimal Nash-equilibria are objection and counter-objection equilibria. Let us recall that Pareto-optimality is required in both cases. Now we prove that Nash-equilibrium implies the non-existence of any objection of any player. Let $u^*\in\mathcal{U}$ be a Pareto-optimal Nash-equilibrium strategy. By Definition 3.2 we have that for each $u_i\in\mathcal{U}_i$

$$J_i(u^*\,\|\,u_i)\geqq J_i(u^*),\quad i\in I.$$

Thus, there is no $i\in I$ and $u_i\in\mathcal{U}_i$ such that

$$J_i(u^*\,\|\,u_i)<J_i(u^*),$$

which means that u^* is an objection and counter-objection strategy in the game Γ.

Property 4.6. Individual rationality. Let $u_i^g\in\mathcal{U}_i$ be a guranteeing (minimax) strategy of the i-th player (see Definition 3.1). Then for each $u_{I\setminus i}\in\mathcal{U}_{I\setminus i}$

$$J_i(u_i^g,\ u_{I\setminus i})\leqq \max_{u_{I\setminus i}} J_i(u_i^g,\ u_{I\setminus i})=\min_{u_i}\max_{u_{I\setminus i}} J_i(u).$$

Suppose u^* is an objection and counter-objection equilibrium strategy and let

$$J_i(u^*)>\min_{u_i}\max_{u_{I\setminus i}} J_i(u).$$

Then $J_i(u^*)>J_i(u_i^g,\ u^*_{I\setminus i})=J_i(u^*\mid u_i^g)$ which means that u_i^g is an objection to u^*. By Definition 4.3 there exists $j\in I\setminus i$ and $u_j^{co}\in\mathcal{U}_j$ such that

$$J_i(u^*\,\|\,u_i^g,\ u_j^{co})\geqq J_i(u^*).$$

Hence $J_i(u^*)\leqq J_i(u^*\,\|\,u_i^g,\ u_j^{co})\leqq\max_{u_{I\setminus i}} J_i(u_i^g,\ u_{I\setminus i})$ and finally we have

$$J_i(u^*)\leqq\min_{u_i}\max_{u_{I\setminus i}} J_i(u)<J_i(u^*)$$

which obviously is wrong. Therefore

$$J_i(u^*)\leqq\min_{u_i}\max_{u_{I\setminus i}} J_i(u).$$

Thus, the values of the cost-functions in an objection and counter-objection equilibrium point are at most equal to the minimax values.

Property 4.7. Saddle-points in two-person zero-sum games are objection and counter-objection equilibria. Let us consider the game $\Gamma_0=\langle\{1,\ 2\},\ \Sigma,\ \{\mathcal{U}_1,\ \mathcal{U}_2\},\ J(u_1,\ u_2)\rangle$ with the objection of minimizing $J(u_1,\ u_2)$ for the first player and maximizing $J(u_1,\ u_2)$ for the second one. Let $(u_1^o,\ u_2^o)$ be a saddle-point of Γ_0, i. e. for each $u_1\in\mathcal{U}_1$ and $u_2\in\mathcal{U}_2$

$$J(u_1^o,\ u_2)\leqq J(u_1^o,\ u_2^o)\leqq J(u_1,\ u_2^o).$$

Consider also the game $\Gamma_2=\langle\{1,\ 2\},\ \Sigma,\ \{\mathcal{U}_1,\ \mathcal{U}_2\},\ \{J_1,\ J_2\}\rangle$, where $J_1=J$ and $J_2=-J$ Here both players choose their strategies with the aim of minimizing their own cost-functions. First we prove that the saddle-point $(u_1^o,\ u_2^o)$ of Γ_0 is Pareto-optimal in Γ_2. Suppose $(u_1^o,\ u_2^o)$ is not Pareto-optimal in Γ_2. Then there exists a pair of strategies, say $(u_1,\ u_2)\in\mathcal{U}_1\times\mathcal{U}_2$, such that

$$J_i(u_1,\ u_2)\leqq J_i(u_1^o,\ u_2^o),\quad i=1,\ 2,$$

where at least one of these two inequalities is strict. Then

$$J_1(u_1,\ u_2)+J_2(u_1,\ u_2)<J_1(u_1^o,\ u_2^o)+J_2(u_1^o,\ u_2^o)$$

and hence

$$0=J(u_1,\ u_2)-J(u_1,\ u_2)<J(u_1^o,\ u_2^o)-J(u_1^o,\ u_2^o)=0$$

which is wrong. Therefore the Pareto-optimality of $(u_1^o,\ u_2^o)$ is established.

Next we show that the saddle-point $(u_1^o,\ u_2^o)$ of Γ_0 is a Nash-equilibrium in Γ_2. Indeed, for each $u_1\in\mathcal{U}_1$ we have

$$J_1(u_1,\ u_2^o)=J(u_1,\ u_2^o)\geqq J(u_1^o,\ u_2^o)=J_1(u_1^o,\ u_2^o)$$

and for each $u_2\in U_2$

$$J_2(u_1^o,\ u_2)=-J(u_1^o,\ u_2)\geqq -J(u_1^o,\ u_2^o)=J_2(u_1^o,\ u_2^o).$$

Finally, we come to the conclusion that the saddle-point of Γ_0 is a Pareto-optimal Nash-equilibrium in Γ_2 and hence, by Property 4.5 it is an objection and counter-objection equilibrium in Γ_2. Therefore the notion of objection and counter-objection equilibrium includes the notion of a saddle-point for zero-sum two-player games as a special case.

5. Sufficient conditions. In this section we establish conditions which are sufficient for the determination of some strategies as objection and counter-objection equlibrium ones. Denote

$$G_i(t,\ x,\ u)=V_t^{(i)}(t,\ x)+\mathcal{A}(u)\,V^{(i)}(t,\ x)+L_i(t,\ x,\ u),\quad i\in I$$

where $t\in[t_0,\ T]$, $x\in\mathsf{R}^n$, $u\in U$.

Now consider the next two assumptions for the strategy $u^*\in\mathcal{U}$.

Assumption 5.1. *There exists a vector* $\lambda=(\lambda_1,\ldots,\lambda_N)\in\mathsf{R}^N$, $\lambda_i>0$, $i\in I$, $\lambda_1+\cdots+\lambda_N=1$ *and a real-valued function* $V(t,\ x)$ *such that for all* $t\in[t_0,\ T]$, $x\in\mathsf{R}^n$ *the following conditions jointly hold:*

(a) $V,\ V_t,\ V_x,\ V_{xx}$ *are continuous;*

(b) $H_\lambda(t,\ x,\ u^*)=0$;

(c) $H_\lambda(t,\ x,\ u)\geqq 0$ *for each strategy* $u\in\mathcal{U}$;

(d) $V(T,\ x)=\sum_{i\in I}\lambda_i\Psi_i(T,\ x)$.

Assumption 5.2. *There exist real-valued functions* $V^{(i)}(t,\ x)$, $i\in I$ *such that for all* $t\in[t_0,\ T]$, $x\in\mathsf{R}^n$ *and* $i\in I$ *the following conditions jointly hold:*

(a) $V^{(i)},\ V_t^{(i)},\ V_x^{(i)},\ V_{xx}^{(i)}$ *are continuous;*

(b) $G_i(t,\ x,\ u^*)=0$;

(c) $V^{(i)}(T,\ x)=\Psi_i(T,\ x)$.

Remark 5.3. The conditions of Assumption 5.1 are equivalent to the conditions of Theorem 3.4, which means that $u^*\in\mathcal{U}$ is Pareto-optimal in the game Γ.

Proposition 5.4. *Let Assumption* 5.2 *hold. Then*

$$V^{(i)}(t_0,\ x_0)=J_i(u^*),\quad i\in I$$

Proof. Suppose $X^*=\{x^*(t),\ t\in[t_0,\ T]\}$ is the solution of Ito equation (*) corresponding to the strategy u^*. Write Ito—Dynkin formula for $V^{(i)}(t,\ x)$, u^* and X^*:

$$V^{(i)}(t,\ x)=\mathbf{E}_{t,x}\{V^{(i)}(T,\ x^*(T))-\int_t^T[V^{(i)}(\tau,\ x^*(\tau))+\mathscr{A}(u^*)V^{(i)}(\tau,\ x^*(\tau))]\,d\tau\},\quad i\in I.$$

This representation in conjunction with conditions (b) and (c) from Assumption 5.2 implies the relation

$$V^{(i)}(t,\ x)=\mathbf{E}_{t,x}\{\Psi_i(T,\ x^*(T))+\int_t^T L_i(\tau,\ x^*(\tau),\ u^*)\,d\tau\},\quad i\in I$$

and hence

$$V^{(i)}(t_0,\ x_0)=\mathbf{E}_{t_0,\,x_0}\{\Psi_i(T,\ x^*(T))+\int_{t_0}^T L_i(t,\ x^*(t),\ u^*)\,dt\},\quad i\in I.$$

Further we consider the following two possibilities called (A) and (B), supposing that Assumption 5.1 and Assumption 5.2 hold:

(A) For all $i\in I$, $t\in[t_0,\ T]$, $x\in R^n$ we have that $G_i(t,\ x,\ u^*\,\|\,u_i)\geq 0$ for each $u_i\in\mathscr{U}_i$. In this case we can formulate such a result.

Proposition 5.6. *Let possibility* (A) *hold. Then there does not exist any objection of any player to* u^*.

Proof. Suppose $X^{(i)}=\{x^{(i)}(t),\ t\in[t_0,\ T]\}$ is the solution of Ito equation (*) corresponding to the strategy $u^*\,\|\,u_i$. Write Ito—Dynkin formula for $V^{(i)}(t,\ x)$, $u^*\,\|\,u_i$ and $X^{(i)}$:

$$V^{(i)}(t,\ x)=\mathbf{E}_{t,x}\{V^{(i)}(T,\ x^{(i)}(T))-\int_t^T[V_t^{(i)}(\tau,\ x^{(i)}(\tau))+\mathscr{A}(u^*\,\|\,u_i)V^{(i)}(\tau,\ x^{(i)}(\tau))]\,d\tau\}.$$

Then possibility (A) and condition (c) of Assumption 5.2 imply that

$$V^{(i)}(t,\ x)\leq\mathbf{E}_{t,x}\{\Psi_i(T,\ x^{(i)}(T))+\int_t^T L_i(\tau,\ x^{(i)}(\tau),\ u^*\,\|\,u_i)\,d\tau\}$$

which leads to

$$V^{(i)}(t_0,\ x_0)\leq\mathbf{E}_{t_0,\,x_0}\{\Psi_i(T,\ x^{(i)}(T))+\int_{t_0}^T L_i(t,\ x^{(i)}(t),\ u^*\,\|\,u_i)\,dt\}.$$

Therefore for each $i\in I$ we have $J_i(u^*)=V^{(i)}(t_0,\ x_0)\leq J_i(u^*\,\|\,u_i)$ for each $u_i\in\mathscr{U}_i$ and hence there is no objection of any player to u^*.

Thus we come to one of our main results.

Theorem 5.7. *Let Assumption* 5.1, *Assumption* 5.2 *and possibility* (A) *hold. Then* $u^*\in\mathscr{U}$ *is an objection and counter-objection equilibrium in the game* Γ.

Now, let us consider the second possibility:

(B) There exist $i\in I$ and $u_i^\circ\in\mathscr{U}_i$ such that for all $t\in[t_0,\ T]$, $x\in R^n$ we have $G_i(t,\ x,\ u^*\,\|\,u_i^\circ)<0$.

Proposition 5.8. *Let possibility (B) hold. Then* u_i° *is an objection of the i-th player to* u^*.

Proof. Ito—Dynkin formula and Proposition 5.4 imply that

$$J_i(u^*)=V^{(i)}(t_0,\ x_0)>J_i(u^*\,\|\,u_i^\circ).$$

Further we shall describe two different approaches to the formulation of sufficient conditions for counter-objection strategies. The first approach is based essentially on the use of the Pareto-optimality of u^*.

Proposition 5.9. *Let* $u_j^{co} \in \mathcal{U}_j$ $(j \neq i)$ *be such that* $G_k(t, x, u^* \| u_i^o, u_j^{co}) \leqq 0$ *for each* $k \in I \setminus i$. *Then* u_j^{co} *is a counter-objection of the j-th player to the objection* u_i^o.

Proof. The application of Ito—Dynkin formula and Proposition 5.4 imply for each $k \in I \setminus i$

$$J_k(u^*) = V^{(k)}(t_0, x_0) \geqq J_k(u^* \| u_i^o, u_j^{co})$$

and in particular for $j \neq i$ $J_j(u^*) \geqq J_j(u^* \| u_i^o, u_j^{co})$. Suppose $J_i(u^*) > J_i(u^* \| u_i^o, u_j^{co})$.

Taking into account that $J_k(u^* \| u_i^o, u_j^{co}) \leqq J_k(u^*)$ for each $k \in I \setminus i$, we get

$$J_k(u^* \| u_i^o, u_j^{co}) \leqq J_k(u^*) \text{ for each } k \in I,$$

where at least one inequality (i-th one) is strict. Then the Pareto-optimality of u_* implies that

$$J_i(u^* \| u_i^o, u_j^{co}) \geqq J_i(u^*)$$

which means that u_j^{co} is a counter-objection.

Thus, we can formulate the following result.

Theorem 5.10. *Let Assumption* 5.1 *and Assumption* 5.2 *hold. Let there exist* j, $j \in I$ $(i \neq j)$ *and* $u_i \in \mathcal{U}_i$, $u_j \in \mathcal{U}_j$ *such that the system*

$$\left| \begin{array}{l} G_i(t, x, u^* \| u_i) < 0 \\ G_k(t, x, u^* \| u_i, u_j) \leqq 0 \text{ for each } k \in I \setminus i \end{array} \right.$$

jholds for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$. *Then* $u^* \in \mathcal{U}$ *is an objection and counter-objection equilibrium in the game* Γ.

The second approach leads directly to the verification of the conditions of the definition for a counter-objection strategy (see Definition 4.2).

Proposition 5.11. *Let* $u_j^{co} \in \mathcal{U}$ $(j \neq i)$ *be such that*

$$\left| \begin{array}{l} G_i(t, x, u^* \| u_i^o, u_j^{co}) \geqq 0 \\ G_j(t, x, u_i^* \| u_i^o, u_j^{co}) \leqq 0. \end{array} \right.$$

Then u_j^{co} *is a counter-objection of the j-th player to the objection* u_i^o.

Proof. Let us apply twice Ito—Dynkin formula and Proposition 5.4. Then we get

$$J_i(u^*) = V^{(i)}(t_0, x_0) \leqq J_i(u^* \| u_i^o, u_j^{co})$$

and

$$J_i(u^*) = V^{(j)}(t_0, x_0) \geqq J_i(u^* \| u_i^o, u_j^{co})$$

which completes the proof of Proposition 5.11.

It is interesting that the above result (Theorem 5.10) can be formulated in a new version.

Theorem 5.12. *Let Assumption* 5.1 *and Asumption* 5.2 *hold. Let there exist* i, $j \in I$ $(i \neq j)$ *and* $u_i \in \mathcal{U}_i$, $u_j \in \mathcal{U}_j$ *such that the three relations*

$$\left| \begin{array}{l} G_i(t, x, u^* \| u_i) < 0 \\ G_i(t, x, u^* \| u_i, u_j) \geqq 0 \\ G_j(t, x, u^* \| u_i, u_j) \leqq 0 \end{array} \right.$$

jointly hold for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$. *Then* $u^* \in \mathcal{U}$ *is an objection and counter-objection equilibrium strategy in the game* Γ.

6. Concluding remarks. In this paper the concept of objection and counter-objection equilibrium is introduced and sufficient conditions for its strategies are established. In spite of the fact that these conditions are heavy, there is an example of a linear-quadratic game where the existence of the objection and counter-objection strategies is proved (see [7]). Moreover, in this case the strategies are found in an explicit form.

The author expresses his gratitude to Prof. V. Zhukovskii for introducing him into the subject of research and for the helpful discussions.

REFERENCES

1. W. H. Fleming, R. W. Rishel. Deterministic and Stochastic Optimal Control. Berlin, 1975.
2. S. D. Gaidov. Optimal strategies in two-player stochastic differential games. *C. R. Acad. Bulgar Sci.*, 39, 1986, 3, 33-36.
3. S. D. Gaidov. Basic optimal strategies in stochastic differential games. *C. R. Acad. Bulgar. Sci.*, 37, 1984, 4, 457-460.
4. S. D. Gaidov. Pareto-optimality in stochastic differential games. *Problems Control Inf. Theory*, 15, 1986, 6, 439-450.
5. S. D. Gaidov. Nash-equilibrium in stochastic differential games. *Computers Math. Appl.*, 12A (Bellman Memorial Issue), 1986, 761-768.
6. S. D. Gaidov. Gauranteeing strategies in many player stochastic differential games. — In: Mathematics and Education in Mathematics, 1986 (Proc. 15th Spring Conf. UBM). Sofia, 1986, 379-383.
7. S. D. Gaidov. Existence of objection and counter-objection equilibria in linear-quadratic many-player stochastic differential games. *Univ. Plovdiv, Travaux scientifiques, Mathematiques*, 25, 1987, (in print).
8. Э. М. Вайсборд, В. Й. Жуковский. Введение в дифференциальные игры нескольких лиц и их приложения. Москва, 1980.

Department of Mathematics
University of Plovdiv
BG-4000 Plovdiv, Bulgaria

Received on 06. 05. 88