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CONFORMAL-HOLOMORPHIC INVARIANTS

EVSTATY PAVLOV

Some conformal-holomorphic invariants of an almost-complex manifold with a B-metric are discussed.

1. Preliminaries. An almost-complex metric manifold M is said to have B-metric with respect to the almost complex structure I if for every point of M the following condition holds true

$$g(x, Iy)=g(Ix, y); \quad x, y \in Mp.$$

In every tangent space M_p acts a subgroup of the group $0(n, n)$ which has representation

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}; \qquad A, B \text{ — matrixes of type } n\times n.$$

Let $g(x, y)=g(Ix, y)$ and $F(x, y, z)=g((\nabla_x I), y)=(\nabla_x \tilde{g})(y, z)$. The tensor field F has the following properties

$$(1) \qquad F(x, y, z)=F(x, z, y), \; F(x, Iy, Iz)=F(x, y, z),$$

which came out of the special agreement between g and I. When $\nabla I=0$, M is called B-manifold.

A classification of the manifolds with an almost-complex structure of this type is given in [1]. First the authors discuss a vector space W of all tensors of the type (0, 3) having the properties (1). The space W is splitted into three mutually orthogonal subspaces W_i $(i=1, 2, 3)$, where $W=W_1 \otimes W_2 \oplus W_3$. The definitions of W_i are as follows:

$$W_1=\{\gamma \in W \mid \gamma(x, y, z)=\frac{1}{2n}(g(x, y)\varphi(z)+g(x, z)\varphi(y)+\tilde{g}(x, y)\varphi(Iz)+\tilde{g}(x, z)\varphi(Iy))\},$$

where $\varphi(z)=g^{ik}\gamma(l_i l_k, z)$ and $\{l_k\}$, $k=1, 2, \ldots, 2n$ is a basis in M_p.

$$W_2=\{\gamma \in W \mid \gamma(x, y, z)+\gamma(y, z, Ix)+\gamma(z, x, Iy)=\varphi=0\},$$

$$W_3=\{\gamma \in W \mid \gamma(x, y, z)+\gamma(y, z, x)+\gamma(z, x, y)=0\}.$$

From here on eight classes of almost-complex manifolds with B-metric are defined. We call them generalized B-manifolds and they are denoted by the same letters W_i. This classification is invariant with respect to the acting group.

Let (M, g) be a B-manifold and a be another metric on M and

$$(2) \qquad a=\lambda g+\mu\tilde{g}.$$

Here λ and μ are arbitrary functions on M which depend on the point only. Now we say that the manifold (M, a) is CH-equivalent to the B-manifold (M, g). The change of the metric $g \to a$ as above is called CH-change. In [2] we have proved the following

Theorem. *The manifold (M, a) is CH-equivalent to a B-manifold if and only if*

$$a((\nabla' xI)y, z) = [\omega(Iz) - \theta(z)]\tilde{a}(x, y) + [\omega(Iy) - \theta(y)]\tilde{a}(x, z)$$
$$+ [\omega(z) - \theta(Iz)]a(x, y) + [\omega(y) - \theta(Iy)]a(x, z),$$

where ω and θ are closed 1-forms on M and ∇' is the proper connection of a.

As a consequence of this theorem we managed to separate some subclasses of W_1 which are still more special generalized B-manifolds. All these new classes CK_0, CH_0, CK and CH of generalized B-manifolds contain the class B of B-manifolds. It is illustrated by the following scheme:

$$B \subset CK_0 \begin{matrix} \subset CH_0 \subset \\ \\ \subset CK \subset \end{matrix} CH.$$

Let further $\tilde{x} = Ix$ and if φ is a 1-form, then $\tilde{\varphi} = \varphi_0 I$. The definitions of the vector subspaces of W_1, corresponding to the manifolds of classes CK_0, CH_0, CK and CH are: if the 1-forms θ and ω are closed, then

$$B = \{\text{ the zero of } W_1\},$$
$$CK_0 = \{\gamma \in W_1 \mid \alpha = \omega + \tilde{\theta},\ \tilde{\alpha} = \tilde{\omega} - \theta \text{ are closed}\},$$
$$CH_0 = \{\gamma \in W_1 \mid \alpha = \omega + \tilde{\theta} \text{ is closed}\},$$
$$CK = \{\gamma \in W_1 \mid \tilde{\alpha} = \tilde{\omega} - \theta \text{ is closed}\},$$
$$CH = \{\gamma \in W_1 \mid \alpha = \omega + \tilde{\theta}\}.$$

2. CH-change of the metric. We suppose that on M a metric as in (2) is given and $\lambda = l^{2\sigma}\cos(2\tau)$, $\mu = l^{2\sigma}\sin(2\tau)$, $\omega = d\tau$, $\theta = d\sigma$. The smooth functions σ and τ are arbitrary. To see which classes of manifolds pointed out in [1] and [2] are CH-equivalent, additional preliminary study of some tensors is necessary. We define the tensors q, q', p and r as follows: the tensor q from [1] with the help of which the space W_1 had been defined

$$q(x, y, z) = \frac{1}{2n}[g(x, y)\varphi(z) + g(x, z)\varphi(y) + \tilde{g}(x, y)\tilde{\varphi}(z) + \tilde{g}(x, z)\tilde{\varphi}(y)],$$

where $\varphi(z) = g^{ik}F(l_i, l_k, z)$;

$$q'(x, y, z) = g(x, y)\alpha(z) + g(x, z)\alpha(y) + \tilde{g}(x, y)\tilde{\alpha}(z) + \tilde{g}(x, z)\tilde{\alpha}(y),$$

where $\alpha = \omega + \tilde{\theta}$;

$$p(x, y, z) = \frac{1}{4}[2F(x, y, z) + F(y, z, x) + F(z, x, y) + F(\tilde{y}, \tilde{z}, x) + F(\tilde{z}, x, \tilde{y})];$$

$$r(x, y, z) = \frac{-1}{2}[2F(\tilde{x}, y, z) + F(y, x, \tilde{z}) - F(\tilde{z}, x, y) + F(z, x, \tilde{y}) - F(\tilde{y}, x, z)].$$

Besides this, to every tensor S of type (0, 3), having the properties of F we associate $\dot{S}$: $\dot{S}(x, y, z) = S(\tilde{x}, y, z)$. Evidently the tensors $\dot{q}$, $\dot{q}'$, $\dot{p}$ and $\dot{r}$ exist.

Note. Similar associated tensors can be determined not only for F but for every other tensor γ from W. If $\gamma \in W$, then the corresponding associated tensors are: q_γ, q'_γ, p_γ, r_γ etc.

Lemma 1. *Let* $\gamma=\gamma_1+\gamma_2+\gamma_3\in W$. *If* $\gamma_i\in W_i$ *then* $\overset{*}{\gamma}_i\in W_i$.

Proof. $i=1$. Now $\gamma_1=q$ and $\overset{*}{q}(x, y, z)=\overset{*}{\gamma}_1(x, y, z)=(2n)^{-1}[-\tilde{\varphi}(z)g(x, y)-\tilde{\varphi}(y)\,.\,g(x, z)+\varphi(z)\tilde{g}(x, y)+\varphi(y)\tilde{g}(x, z)]$. Here $\overset{*}{q}$ is defined using the form $\overset{*}{\varphi}(z)=g^{ik}\overset{*}{q}(l_i, l_k, z)=-\tilde{\varphi}(z)$. Thus $\overset{*}{q}\in W_1$.

$i=2$. From $\gamma_2\in W_2$ follows $\gamma_2(x, y, \tilde{z})+\gamma_2(y, z, \tilde{x})+\gamma_2(z, x, \tilde{y})=0$. In this equation after the change $x\to\tilde{x}$, $y\to\tilde{y}$, $z\to\tilde{z}$ we have $\overset{*}{\gamma}_2(x, y, \tilde{z})+\overset{*}{\gamma}_2(y, z, \tilde{x})+\overset{*}{\gamma}_3(z, x, \tilde{y})=0$, i. e. $\overset{*}{\gamma}_2\in W_2$.

$i=3$. It is evident since $\overset{*}{\gamma}_3=\overset{*}{\gamma}-(\overset{*}{\gamma}_1+\overset{*}{\gamma}_2)$.

Lemma 2. If $q'\in CH$ then $\overset{*}{q}'\in CH$.

Proof. From the above mentioned lemma it follows that the 1-form corresponding to $\overset{*}{q}'$ is $\tilde{\alpha}=\tilde{\omega}-\theta$. Thus $\overset{*}{q}'\in CH$.

Theorem 1. *If* $F=F_1+F_2+F_3$, $F_i\in W_i$, *then* $F_1=q$, $F_2=p-q$, $F_3=F-p$.

The proof follows from theorem 2 of [1] and the definitions of the tensors p and q.

Theorem 2. *The tensor* $F_3=0$ *if and only if* $r=0$.

Proof. From $F_3=0$ it follows that $F=p$. In the expression for r we substitute F with p. So we have

$$r(x, y, z)=\frac{-1}{8}[2p(\tilde{x}, y, z)+p(y, x, \tilde{z})-p(\tilde{z}, x, y)+p(z, x, \tilde{y})-p(\tilde{y}, x, z)].$$

For the inverse, let $r=0$, i. e.

$$0=2F(\tilde{x}, y, z)+F(y, x, \tilde{z})-F(\tilde{z}, x, y)+F(z, x, \tilde{y})-F(\tilde{y}, x, z).$$

In this equation, after the change $x\to\tilde{x}$ we find

$$4F(x, y, z)=2F(x, y, z)+F(y, z, x)+F(z, x, y)+F(\tilde{y}, \tilde{z}, x)+F(\tilde{z}, x, \tilde{y})=4p(x, y, z).$$

From theorem 1 it follows that $F_3=0$.

Corollary. *The following integrability conditions for the complex structure are equivalent*:

a) $F(x, y, \tilde{z})+F(y, z, \tilde{x})+F(z, x, \tilde{y})=0$; b) $r=0$; c) $F=p$.

Proof. Condition a) is proved in [1]. From condition a) we have

$$F\in W_1\oplus W_2\Leftrightarrow F_3=0\Leftrightarrow r=0\Leftrightarrow F=p.$$

Theorem 3. $r\in W_3$.

Proof. Let us find the associated tensors φ_r, q_r and p_r for the tensor $r\in W$ (see the note in this paragraph). From them we have $\varphi_r=0$, $q_r=0$, $p_r=0$. Then for the projections $r_i\in W_i$ we have $r_1=r_2=0$, $r=r_3$ which follows from Theorem 1.

Let ∇' be the proper connection of the metric a and $\bar{F}(x, y, z)=a((\nabla' xI)y, z)$. In [2] we proved that

$$\bar{F}=\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}'+r). \tag{3}$$

Lemma 3. *Let* $\bar{\varphi}$, $\bar{q}$, $\bar{r}$ *be the associated tensors of* $\bar{F}$. *Then*

$$\bar{\varphi}=2n\,\alpha+\varphi,\quad \alpha=\omega+\tilde{\theta},$$

$$\bar{q}=\lambda(q+q')+\mu(\overset{*}{q}+\overset{*}{q}'),\quad \bar{p}=\lambda(p+p')+\mu(\overset{*}{p}+\overset{*}{q}').$$

The proof of the lemma follows from the definition of these tensors and formula (3).

3. *CH*-**invariants.** We define the following tensor fields of type (2.1) on M:

ν from the equation $g(x, \nu(y, z))=F(x, y, z)$,

$\varkappa$ from the equation $g(x, \varkappa(y, z))=q(x, y, z)$,

$\varkappa'$ from the equation $g(x, \varkappa'(y, z))=q'(x, y, z)$,

δ from the equation $g(x, \delta(y, z))=p(x, y, z)$,

ρ from the equation $g(x, \rho(y, z))=r(x, y, z)$.

Lemma 4. *If $\bar{\nu}$, $\bar{\varkappa}$, $\bar{\delta}$ are analogous to the above defined tensors, but corresponding to the metric $a=\lambda g+\mu\tilde{g}$, then*

$$\bar{\varkappa}=\varkappa+\varkappa',\quad \bar{\delta}=\delta+\varkappa',\quad \bar{\nu}=\nu+\varkappa'+\frac{\lambda\mu}{\lambda^2+\mu^2}\rho-\frac{\mu^2}{\lambda^2+\mu^2}I\circ\rho.$$

Proof. We shall use Lemma 3.

$$\bar{a}(x, \bar{\varkappa}(y, z))=\bar{q}(x, y, z)=\lambda[q(x, y, z)+q'(x, y, z)]+\mu[\overset{*}{q}(x, y, z)+\overset{*}{q}'(x, y, z)]$$
$$=\lambda[g(x, \varkappa(y, z))+g(x, \varkappa'(y, z))]+\mu[\tilde{g}(x, \varkappa(y, z))+\tilde{g}(x, \varkappa'(y, z))].$$

Consequently $\bar{\varkappa}=\varkappa+\varkappa'$. The proof for $\bar{\delta}$ is similar. The proof for $\bar{\nu}$ can be given as follows

$$a(x, \bar{\nu}(y, z))=\bar{F}(x, y, z)=\lambda[q'(x, y, z)+F(x, y, z)]+\mu[\overset{*}{q}'(x, y, z)+\overset{*}{F}(x,y,z)+r(x,y,z)$$
$$=\lambda q'(x, y, z)+\mu\overset{*}{q}'(x, y, z)+\lambda F(x, y, z)+\mu\overset{*}{F}(x, y, z)+\mu r(x, y, z)$$
$$=a(x, \varkappa(y, z))+a(x, \nu(y, z))+\mu g(x, \rho(y, z)).$$

On the other hand, from (2) we have $g=\lambda a(\lambda^2-\mu^2)^{-1}-\mu\tilde{a}(\lambda^2+\mu^2)^{-1}$. Then

$$a(x, \bar{\nu}(y, z))=a(x, \varkappa'(y, z))+a(x, \nu(y, z))+(\lambda^2+\mu^2)^{-1}[\lambda\mu a(x, \rho(y, z))-\mu^2\tilde{a}(x, \rho(y, z))],$$

which proves the last equation of the lemma.

Theorem 4. *The tensor field $\delta-\varkappa$ does not depend on any CH-change of the metric.*

The proof follows from Lemma 4.

Theorem 5. *The tensor fields $\delta-\varkappa$, $\nu-\delta$ and $\nu-\varkappa$ do not depend on any conformal change of the metric.*

The proof follows from Lemma 4 when $\mu=0$.

Theorem 6. *The tensor field $\delta-\varkappa$ vanishes on M if and only if M belongs to the class $W_1\oplus W_3$.*

The proof is evident from the scheme

$$\delta=\varkappa\Leftrightarrow p=q\Leftrightarrow F_2=0.$$

Theorem 7. *The tensor fields $\nu-\delta$ vanishes on M if and only if M belongs to the class $W_1\oplus W_2$.*

The proof follows from the scheme $\nu=\delta\Leftrightarrow F=p\Leftrightarrow F_3=0$.

As a consequence of the theorems 6 and 7 we have

Theorem 8. *The manifold M belongs to the class W_1 if and only if $\nu=\delta=\varkappa$, i. e. simultaneous vanishing of the CH-invariant and conformal invariant tensors.*

Table 1

Metric g					Metric $\lambda g+\mu\tilde{g}$				
Class	Sufficient condition	p	q	r	Class	$\overline{F}$	$\overline{p}$	$\overline{q}$	Sufficient condition
B	$F=0$	0	0	0	CH	$\lambda q'+\mu\overset{*}{q}'$	$\lambda q'+\mu\overset{*}{q}'$	$\lambda q'+\mu\dot{q}'$	$\overline{F}=\overline{q}$
$\frac{W_1}{CH}$	$F=q$	F	F	0	$\frac{W_1}{CH}$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}')$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}')$	$\lambda(F+q')+\mu(\overset{*}{F}+\dot{q}')$	$\overline{F}=\overline{q}$
W_2	$F=p-q$	F	0	0	$CH\oplus W_2$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}')$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}')$	$\lambda q'+\mu\overset{*}{q}'$	$\overline{F}=\overline{p}$
W_3	$p=0$	0	0	r	$CH\oplus W_3$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}'+r)$	$\lambda q'+\mu\dot{q}'$	$\lambda q'+\mu\overset{*}{q}'$	$\overline{p}=\overline{q}$
$\frac{W_1\oplus W_2}{CH\oplus W_2}$	$F=p$	F	q	0	$\frac{W_1\oplus W_2}{CH\oplus W_2}$	$\lambda(F+q')+\mu(\dot{F}+\dot{q}')$	$\lambda(F+q')+\mu(\dot{F}+\overset{*}{q}')$	$\lambda(q+q')+\mu(\overset{*}{q}+\dot{q}')$	$\overline{F}=\overline{p}$
$\frac{W_1\oplus W_3}{CH\oplus W_3}$	$p=q$	p	p	r	$\frac{W_1\oplus W_3}{CH\oplus W_3}$	$\lambda(F+q')+\mu(\overset{*}{F}+\dot{q}'+r)$	$\lambda(p+q')+\mu(\overset{*}{p}+\overset{*}{q}')$	$\lambda(p+q')+\mu(\dot{p}+\dot{q}')$	$\overline{p}=\overline{q}$
$\frac{W_2\oplus W_3}{CH\oplus W_2\oplus W_3}$	$q=0$	p	0	r	$CH\oplus W_2\oplus W_3$	$\lambda(F+q')+\mu(\overset{*}{F}+\overset{*}{q}'+r)$	$\lambda(p+q')+\mu(\dot{p}+\overset{*}{q}')$	$\lambda q'+\mu\dot{q}'$	

Table 2'

Metric g					Metric λg				
Class	Sufficient condition	p	q	r	Class	$\bar{F}$	$\bar{p}$	$\bar{q}$	Sufficient condition
B	$F=0$	0	0	0	CK	$\lambda q'$	$\lambda q'$	$\lambda q'$	$\bar{F}=\bar{q}$
$\frac{W_1}{CK}$	$F=q$	F	F	0	$\frac{W_1}{CK}$	$\lambda(F+q')$	$\lambda(F+q')$	$\lambda(F+q')$	$\bar{F}=\bar{q}$
W_2	$F=p-q$	F	0	0	$CK\oplus W_2$	$\lambda(F+q')$	$\lambda(F+q')$	$\lambda q'$	$\bar{F}=\bar{p}$
W_3	$p=0$	0	0	r	$CK\oplus W_3$	$\lambda(F+q')$	$\lambda q'$	$\lambda q'$	$\bar{p}=\bar{q}$
$\frac{W_1\oplus W_2}{CK\oplus W_2}$	$F=p$	F	q	0	$\frac{W_1\oplus W_2}{CK\oplus W_2}$	$\lambda(F+q')$	$\lambda(F+q')$	$\lambda(q+q')$	$\bar{F}=\bar{p}$
$\frac{W_1\oplus W_3}{CK\oplus W_3}$	$p=q$	p	p	r	$\frac{W_1\oplus W_3}{CK\oplus W_3}$	$\lambda(F+q')$	$\lambda(p+q')$	$\lambda(p+q')$	$\bar{p}=\bar{q}$
$\frac{W_2\oplus W_3}{CK\oplus W_2\oplus W_3}$	$q=0$ ($\varphi=0$)	p	0	r	$CK\oplus W_2\oplus W_3$	$\lambda(F+q')$	$\lambda(p+q')$	$\lambda(q+q')$	

Changing of the classes of manifolds after CH-change and conformal change of the metric is given in Tables 1 and 2. The proof for the associated tensors is given in Lemmas 2 and 3 and Theorem 2. In these tables the columns for F and $\bar{F}$ show sufficient conditions for belonging to the same class. To see how each class is mapped let us discuss for example the class $W_2 \oplus W_3$.

After a CH-change of the metric $\bar{F} = \lambda(F + q') + \mu(\overset{*}{F} + \overset{*}{q}' + r)$.

Since $F \in W_2 \oplus W_3$ and $q' \in CH$, then it follows that $\overset{*}{q}' \in CH$ (Lemma 2) and $\overset{*}{F} \in W_2 \otimes W_3$ (Lemma 1). The tensor $r \in W_3$ (Lemma 3). Now it is clear that $\bar{F} \in CH \oplus W_2 \oplus W_3$. The proof for the other classes is similar.

Special case. Let the pair of functions (σ, τ) be conjugate pluricharmonical functions. Then $\alpha = \omega + \tilde{\theta} = 0$, i. e. it means that each class of manifolds is invariant under a CH-change of the metric.

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University of Plovdiv,
Chair of Geometry,
4000 Plovdiv Bulgaria

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