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# ON A CLASS OF ALMOST CONTACT METRIC MANIFOLDS CONFORMALLY RELATED TO THE SASAKIAN MANIFOLDS

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The class of fhe classification scheme of the almost contact metric manifolds containing the basic classes of α-Sasakian and $W_1$ manifolds is considered. This class is characterized by the Nijenhuis tensor, the differential of the fundamental 2-form and the differential of the structure 1-form. The class under consideration is exactly characterized by the maximal subgroup of the contact conformal group, preserving this class. Curvature properties of the considered manifolds are studied.

**Introduction.** A classification scheme of the almost contact metric manifolds consisting of twelve basic classes is given in [1]. The basic classes in this scheme are defined by some conditions for the covariant derivative of the fundamental 2-form. In this scheme the basic class $W_2$ is the class of the α-Sasakian manifolds and for the basic class $W_1$ it is proved in [2] that it is generated by the cosymplectic manifolds by means of a subgroup of the contact conformal group.

In this paper we consider the class $W_1 \oplus W_2$ of almost contact metric manifolds containing the classes $W_1$ and $W_2$. The necessary and sufficient conditions for an almost contact metric manifold to be in $W_1 \oplus W_2$ by making use of the fundamental tensors (the Nijenhuis tensor, the differential of the fundamental 2-form and the differential of the structure 1-form) are given. The class $W_1 \oplus W_2$ is not invariant with respect to the contact conformal group. The maximal subgroup $G_1$ of the contact conformal group preserving this class is found. It has been proved that an almost contact metric manifold will be in $W_1 \oplus W_2$ iff it is contact conformally related by a transformation of $G_1$ to an α-Sasakian manifold. In this way the class $W_1 \oplus W_2$ is one of the contact analogues of the class of locally conformal Kaehler manifolds in the almost Hermitian geometry. Some curvature properties of the manifolds in $W_1 \oplus W_2$ are obtained.

**1. Preliminaries**. Let $M$ be a $(2n+1)$-dimensional almost contact metric manifold with structure $(\varphi, \xi, \eta, g)$, where $\varphi$ is a tensor field of type (1,1), $\xi$ is a vector field, $\eta$ is a 1-form and $g$ a Riemannian metric such that

$$\varphi^2 = -Id + \eta \otimes \xi, \quad g \circ \varphi = g - \eta \otimes \eta, \quad \varphi(\xi) = 0, \quad \eta(\xi) = 1.$$

The tensor field $h$ of type (1,1) is defined by $h = -\varphi^2$. The fundamental 2-form $\Phi$ on $M$ is given by $\Phi(x, y) = g(x, \varphi_y)$ for arbitrary vector fields $x$, $y$ on $M$.

The contact Nijenhuis tensor field $N$ on $M$ is determined by $N = [\varphi, \varphi] + d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the usual Nijenhuis tensor field formed by $\varphi$.

Let $\nabla$ be the Riemannian connection on $M$. We denote $F(x, y, z) = g((\nabla_x \varphi) y, z)$ for all vector fields $x$, $y$ and $z$ on $M$. The tensor field $F$ and the covariant derivative of the fundamental 2-form $\Phi$ are related by the equality $F = -\nabla\Phi$.

The following relations are well known

$$d\Phi(x, y, z) = -\{F(x, y, z) + F(y, z, x) + F(z, x, y\}, \tag{1.1}$$

$$d\eta(x, y) = F(x, \varphi_y, \xi) - F(y, \varphi_x, \xi), \tag{1.2}$$

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(1.3) $$g(N(x, y), z) = -2F(z, x, \varphi_y) - d\Phi(\varphi_x, y, z) - d\Phi(x, \varphi_y, z) + \eta(x)d\eta(y, z) + \eta(y)\{F(x, \xi, \varphi_z) + F(z, \xi, \varphi_x)\}.$$

Let $\{l_i\}$, $i = 1, \dots, 2n+1$ be an orthonormal basis for the tangential space $T_pM$ of $M$ at the point $p \in M$. The 1-forms $f$, $f^*$ and $\omega$ associated with $F$ are given by

$$f(x) = \sum_{i=1}^{2n+1} F(l_i, l_i, x), \quad f^*(x) = \sum_{i=1}^{2n+1} F(l_i, \varphi_{l_i}, x), \quad \omega(x) = F(\xi, \xi, x) = g(\varphi H \circ x),$$

where $H = -\nabla_\xi \xi$.

For arbitrary $x, y, z$ in $T_pM$, $p \in M$ we denote

$$p_1(x, y, z) = \eta(x)\eta(y)\omega(z) - \eta(x)\eta(z)\omega(y),$$

$$p_2(x, y, z) = k\{\eta(z)g(x, y) - \eta(y)g(x, z)\},$$

$$p_3(x, y, z) = k^*\{\eta(z)\Phi(x, y) - \eta(y)\Phi(x, z)\},$$

$$p_9(x, y, z) = \frac{1}{2(n-1)}\{g(hx, hy)hf(z) - g(hx, hz)hf(y) - \Phi(x, y)hf(\varphi_z) + \Phi(x, z)hf(\varphi_y)\},$$

where $k = \frac{f(\xi)}{2n}$, $k^* = -\frac{f^*(\xi)}{2n}$, $hf = f - f(\xi)\eta - \omega$.

We follow the classification of the almost contact metric manifolds given in [1]. Using the decomposition of $F$ into 12 basic components we have defined 12 basic classes of almost contact metric manifolds. The tensors $p_1, p_2, p_3, p_9$ are components of $F$ and the classes $W_1$, $W_2$ and $W_1 \oplus W_2$ are defined by the equalities $W_1: F = p_1$, $W_2: F = p_2$,

(1.4) $$W_1 \oplus W_2: \quad F = p_1 + p_2$$

**2. The class $W_1 \oplus W_2$.** It is known that an almost contact metric manifold is an $\alpha$-Sasakian manifold, i. e. a manifold in the class $W_2$ iff the contact Nijenhuis tensor, the differential of the fundamental 2-form and the differential of the structure 1-form satisfy the following conditions

$$N = 0, \quad d\Phi = 0, \quad d\eta = 2k.\Phi.$$

Now we shall characterize the class $W_1 \oplus W_2$ by conditions for these tensors.

Theorem 2.1. *An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ will be in the class $W_1 \oplus W_2$ iff*

(2.1) $$d\Phi = 0,$$

(2.2) $$d\eta = (\omega \circ \varphi) \wedge \eta + 2k\Phi,$$

(2.3) $$N = [(\omega \circ \varphi) \wedge \eta] \otimes \xi.$$

Proof. Let $M$ be a manifold in the class $W_1 \oplus W_2$. Thus using (1.1), (1.2), (1.3), we obtain from (1.4) the equalities (2.1), (2.2) and (2.3).

Now let (2.1), (2.2) and (2.3) hold. From (1.3) we have

$$\eta(y)\eta(z)\omega(\varphi_x) - \eta(x)\eta(z)\omega(\varphi_y) = -2F(z, x, \varphi_y) + \eta(x)\{\eta(z)\omega(\varphi_y) - \eta(y)\omega(\varphi_z) + 2k\Phi(y, z)\} + \eta(y)\{F(x, \xi, \varphi_z) + F(z, \xi, \varphi_x)\}.$$

Replacing $y$ by $\varphi_y$ in the last equation, we get

$$F(z, x, y) = \eta(x)\eta(z)\omega(y) - \eta(y)F(z, \xi, x) - k\eta(x)g(hy, hz)$$

and hence

$$F(z, \xi, y)=\eta(z)\omega(y)-kg(hy, hz).$$

The last two equations imply (1.4), i. e. $M$ is in the class $W_1 \oplus W_2$.

Proposition 2.1. *The 1-form $\omega \circ \varphi$ and the differential of the function $k$ on a manifold $M(\varphi, \xi, \eta, g)$ in $W_1 \oplus W_2$ are related by the equality*

$$\omega \circ \varphi = \frac{dk}{k} \circ h. \tag{2.4}$$

Proof. After the exterior differentiation (2.2), (2.1) implies (2.4).

3. The maximal subgroup of the contact conformal group preserving the class $W_1 \oplus W_2$. Let $u$, $v$ be differentiable functions on $M$ with an almost contact metric structure $(\varphi, \xi, \eta, g)$. The transformation of the structure into the structure $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ defined by [3]

$$\bar\varphi=\varphi, \quad \bar g=l^{2u}g \circ h+l^{2v}\eta \otimes \eta, \quad \bar\xi=l^{-v}\xi, \quad \bar\eta=l^{v}\eta \tag{3.1}$$

is said to be contact conformal transformation. These transformations form the contact conformal group $G$.

In [3] is proved that the classes $W_1$, $W_2$ and $W_1 \oplus W_2$ are not contact conformally invariant. We have

Theorem ([3]). The maximal subgroup $G_1$ of $G$ preserving the class $W_1$ consists of the transformations (3.1) satisfying the condition

$$du=0. \tag{3.2}$$

Theorem ([3]). The maximal subgroup of $G$ preserving the class $W_2$ consists of the transformations (3.1) satisfying the conditions $du=0$, $dv=0$.

Now we shall obtain the maximal subgroup of the contact conformal group preserving the class $W_1 \oplus W_2$. To that purpose we need the following

Lemma ([3]). Let $M$ be an almost contact metric manifold and let the structures $(\varphi, \xi, \eta, g)$ and $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$ on $M$ be contact conformally related by (3.1). Let $\{F, f, f^*, \omega, p_1, p_2, p_3, p_9\}$ respectively $\{\bar F, \bar f, \bar f^*, \bar\omega, \bar p_1, \bar p_2, \bar p_3, \bar p_9\}$ be the tensors corresponding to the structure $\{\varphi, \xi, \eta, g)$ respectively to $(\bar\varphi, \bar\xi, \bar\eta, \bar g)$. Then

$$\begin{aligned} 2\bar F(x, y, z) = {} & 2l^{2u}\{F(x, y, z)-p_3(x, y, z)-p_9(x, y, z)\}-2l^{2v}p_1(x, y, z) \\ & +2\{\bar p_1(x, y, z)+\bar p_3(x, y, z)+\bar p_9(x, y, z)\}+(l^{2v}-l^{2u})\{\eta(y)F(x, \xi, z) \\ & -\eta(z)F(x, \xi, y)+\eta(y)F(\varphi_z, \xi, \varphi_x)-\eta(z)F(\varphi_y, \xi, \varphi_x) \\ & +\eta(x)[F(y, \xi, z)-F(z, \xi, y)-F(\varphi_y, \xi, \varphi_z)+F(\varphi_z, \xi, \varphi_y]\}, \end{aligned} \tag{3.3}$$

$$du(hz)=\frac{1}{2(n-1)}\{h\bar f(\varphi_z)-hf(\varphi_z)\}, \tag{3.4}$$

$$du(\xi)=k^*-l^v . \bar k^*, \tag{3.5}$$

$$dv(hz)=(\bar\omega \circ \varphi)(z)-(\omega \circ \varphi)(z), \tag{3.6}$$

$$\bar p_2=l^{2v}p_2, \quad \bar k=l^{v-2u}k. \tag{3.7}$$

Theorem 3.1. *The subgroup $G_1$ is the maximal subgroup of the contact conformal group $G$, preserving the class $W_1 \oplus W_2$.*

Proof. Let $M(\varphi, \xi, \eta, g)$ be a manifold in the class $W_1 \oplus W_2$ and let the structures $(\varphi, \xi, \eta, g)$, $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be contact conformally related with functions $u$, $v$ satisfying (3.2). From (1.4) it follows $k^*=0$, $hf=0$. Thus (3.4) and (3.5) imply $\bar{k}^*=0$, $h\bar{f}=0$ and hence $\bar{p}_3=0$, $\bar{p}_9=0$. Moreover, using (1.4), we get

$$\eta(y)F(x, \xi, z)-\eta(z)F(x, \xi, y)+\eta(y)F(\varphi_z, \xi, \varphi_x)-\eta(z)F(\varphi_y, \xi, \varphi_x)$$

$$\eta(x)\,[F(y, \xi, z)-F(z, \xi, y)-F(\varphi_y, \xi, \varphi_z)+F(\varphi_z, \xi, \varphi_y)]=2p_1(x, y, z)+2p_2(x, y, z).$$

Thus from (3.3) and (3.7) we derive $\bar{F}=\bar{p}_1+\bar{p}_2$ that is $M(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is in the class $W_1 \oplus W_2$.

Now, let $M(\varphi, \xi, \eta, g)$ and $M(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be two manifolds in $W_1 \oplus W_2$ which are contact conformally related by (3.1). Since $F=p_1+p_2$ and $\bar{F}=\bar{p}_1+\bar{p}_2$ we have $p_3=\bar{p}_3=0$, $p_9=\bar{p}_9=0$ and hence $k^*=\bar{k}^*=0$, $hf=h\bar{f}=0$. Further (3.4) and (3.5) imply (3.2). So it is proved that $G_1$ is the maximal subgroup of the contact conformal group preserving the class $W_1 \oplus W_2$.

In [2] it is proved the following

Theorem. An almost contact metric manifold is in the class $W_1$ iff the structure of the manifold is locally conformal to a cosymplectic structure by a transformation of $G_1$.

The class $W_1 \oplus W_2$ has also an exact characteristic by the subgroup $G_1$ and the class $W_2$ of the α-Sasakian manifolds. The main result of the present paper is

Theorem 3.2. *An almost contact metric manifold is in the class $W_1 \oplus W_2$ iff the structure of the manifold is locally conformal to an α-Sasakian structure by a transformation of $G_1$.*

Proof. If $M(\varphi, \xi, \eta, g)$ is in the class $W_2$ then by an arbitrary transformation of $G_1$ we will obtain $M(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g}) \in W_1 \oplus W_2$. The proof is the same as for the theorem (3.1).

Conversely, let $M(\varphi, \xi, \eta, g)$ be in the class $W_1 \oplus W_2$. We consider the contact conformal transformation of the structure $(\varphi, \xi, \eta, g)$ with functions $u=\text{const}$ and $v=-\ln|k|$. From theorem 3.1 we have $\bar{F}=\bar{p}_1+\bar{p}_2$. Moreover, (2.4) and (3.6) imply $\bar{\omega}=0$. Thus $\bar{p}_1=0$ and therefore $M(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is in the class $W_2$.

**4. Curvature properties for the manifolds in $W_1 \oplus W_2$.** Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold with Riemannian curvature tensor $R$. The following partial orthogonal decomposition of $R$ is given in [4]:

$$R=hR+vR+wR,$$

where

$$hR(x, y, z, u)=R(hx, hy, hz, hu),$$

$$vR(x, y, z, u)=\eta(x)R(\xi, y, hz, hu)+\eta(y)R(x, \xi, hz, hu)+\eta(z)R(hx, hy, \xi, u)$$
$$+\eta(u)R(hx, hy, z, \xi),$$

$$wR(x, y, z, u)=\eta(x)\eta(z)R(\xi, y, \xi, u)+\eta(x)\eta(u)R(\xi, y, z, \xi)+\eta(y)\eta(z)R(x, \xi, \xi, u)$$
$$+\eta(y)\eta(u)R(x, \xi, z, \xi).$$

When $M$ is in $W_1 \oplus W_2$ it is possible to express the components $vR$ and $wR$ of $R$ by the function $k$ and the 1-form $\omega \circ \varphi$ on $M$. In this case the component $hR$ of $R$ satisfies the same identity as the corresponding component for the curvature tensor of an α-Sasakian manifold.

Proposition 4.1. *Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold in the class $W_1 \oplus W_2$. The components $hR$, $vR$ and $wR$ of the Riemannian curvature tensor $R$ on $M$ satisfy the following identities*

$$(4.1)\quad hR(x, y, z, u) - hR(x, y, \varphi_z, \varphi_u) = k^2\{g(hy, hz)g(hx, hu) - g(hx, hz)g(hy, hu) + \Phi(x, z)\Phi(y, u) - \Phi(y, z)\Phi(x, u)\},$$

$$(4.2)\quad vR(x, y, z, u) = \eta(x)\alpha(y, z, u) - \eta(y)\alpha(x, z, u) - \eta(z)\alpha(u, y, x) + \eta(u)\alpha(z, y, x),$$

$$(4.3)\quad wR(x, y, z, u) = \eta(x)\eta(u)S(y, z) - \eta(x)\eta(z)S(y, u) + \eta(y)\eta(z)S(x, u) - \eta(y)\eta(u)S(x, z)'$$

*where*

$$\alpha(x, y, z) = -k\{2g(y, \varphi_z)\omega(\varphi_x) + g(x, \varphi_z)\omega(\varphi_y) - g(x, \varphi_y)\omega(\varphi_z)\},$$
$$S(x, y) = -\omega(\varphi_x)\omega(\varphi_y) - (\nabla_{hx}\omega \circ \varphi)hy + k^2 g(hx, hy).$$

Proof. It follows from (1.4)

$$(\nabla_x \varphi)_y = \eta(x)\eta(y)\varphi H - \eta(x)g(\varphi H, y)\} + k[g(x, y)\xi - \eta(y)x].$$

The last equation implies immediately

$$(4.4)\quad \nabla_{hx}\varphi_y = \varphi\nabla_{hx}hy + kg(hx, hy)\xi,\ \eta(\nabla_{hx}hy) = -k\Phi(x, y),$$

$$(4.5)\quad \nabla_x\xi = -\eta(x)H - k\,.\,\varphi_x.$$

Thus (4.1) is a consequance of (4.4) and (4.5). Making use of (4.5), we obtain

$$R(x, y)\xi = -d\eta(x, y)H + dk(y)\varphi_x - dk(x)\varphi_y + k^2[\eta(y)x - \eta(x)y] + k[\eta(x)\omega(y) - \eta(y)\omega(x)]\xi + \eta(x)\nabla_y H - \eta(y)\nabla_x H.$$

The last equation implies (4.2) and (4.3).

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