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ON THE LOCAL SOLVABILITY OF HYPERBOLIC OVERDETERMINED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

JORDAN B. TABOV

1. Introduction. Overdetermined systems of partial differential equations (OS PDE) and the problem when they have a solution have been the subject of attention of many mathematicians. Among well-known classical results in this field we can mention the Frobenius theorem and the Cartan — Kähler theorem.

A classical theorem of I. Petrovskii, proved in [6], ensures the solvability of smooth systems of PDE in the case when they are hyperbolic.

We consider overdetermined systems of PDE of the form

$$
(1)\qquad \begin{cases}
\dfrac{\partial u}{\partial x^1}=F\left(x^1,\ x^2,\ \dfrac{\partial u}{\partial x^2},\ \dfrac{\partial v}{\partial x^2},\ u,\ v,\ w\right)\\[2mm]
\dfrac{\partial v}{\partial x^1}=G\left(x^1,\ x^2,\ \dfrac{\partial u}{\partial x^2},\ \dfrac{\partial v}{\partial x^2},\ u,\ v,\ w\right)\\[2mm]
\dfrac{\partial w}{\partial x^1}=H\left(x^1,\ x^2,\ \dfrac{\partial u}{\partial x^2},\ \dfrac{\partial v}{\partial x^2},\ u,\ v,\ w\right)\\[2mm]
\dfrac{\partial w}{\partial x^2}=K\left(x^1,\ x^2,\ \dfrac{\partial u}{\partial x^2},\ \dfrac{\partial v}{\partial x^2},\ u,\ v,\ w\right)
\end{cases}
$$

in which u, v and w are unknown functions of x^1 and x^2, and $F(x^1, x^2, \dots, x^7) = F(x)$, $G(x)$, $H(x)$ and $K(x)$ are smooth functions of their arguments.

Besides the definition of hyperbolic systems of the form (1), the present work contains a proof of their solvability. We should note, that all considerations and results are local.

1. Hyperbolic Pfaff systems. Each Pfaff system of the form

$$
(2)\qquad \omega^i(dx)=\omega_1^i\,dx^1+\cdots+\omega_n^i\,dx^n=0,\quad i=1,\ 2,\dots,n-4
$$

of rank $n-4$ determines a 4-dimensional plane $\theta(x)$ in each point x. We shall call $\theta=\theta(x)$ a 4-dimensional distribution.

Definition. *We say, that the distribution $\theta'=\theta'(x)$ is resolving for the system (2), if $\theta'(x)$ is involutive and at every point x the condition $\theta'\subset\theta$ is satisfied.*

Until the end of this section we shall assume, that the system (2) has no non-zero characteristic (see [7]) vectors.

We call a point y singular for the system (2), if there exist three linearly independent vectors ξ_1, ξ_2 and ξ_3 from θ, such that at y the equalities

$$
\partial\omega^i(\xi_j,\ \xi_k)=0,\quad i=1,\ 2,\dots,n-4;\quad j,\ k=1,\ 2,\ 3
$$

hold. Until the end of this section we shall assume that the system (2) has no singular points.

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D e f i n i t i o n (E. Cartan). The system (2) is in involution with respect to the 2-dimensional distributions (in other words, has property 12), if for every field $\xi(x) \subset \theta(x)$ the rank of the system

$$\omega^i(\eta)=0, \quad \partial\omega^i(\xi, \eta)=0, \quad i=1, 2, \dots, n-4,$$

where η is vector argument, does not exceed $n-2$.

Denote by $\Omega_1(y)$ the linear operator in $\theta(y)$, determined by the equality

$$(\Omega_1(y)\xi, \eta)=\partial\omega^1(\xi, \eta)$$

for every $\xi, \eta \in \theta(y)$ (the scalar product is the same as in R^n). Similarly, we define $\Omega_i(y)$, $i=1, 2, \dots, n-4$.

Let $a=(a^1, a^2, \dots, a^{n-4})$ be a vector of the arithmetic space A^{n-4}. Consider the operator

$$\Omega^a(y)=\sum_{i=1}^{n-4} a^i \Omega_i(y)$$

and denote its kernel by $E_a(y)$. We call the vector a regular for the system (2) at the point y, if $\det \Omega^a(y)=0$.

D e f i n i t i o n. *The system (2) is called hyperbolic at the point y, if it has $n-4$ linearly independent regular vectors at y.*

The following result holds (see [8]):

T h e o r e m. *If the system (2) has property 12 in a neighbourhood of x_0 and is hyperbolic at x_0, then in some neighbourhood of this point the system (2) has a 2-dimensional resolving distribution.*

3. Hyperbolic systems of the form (1) and their local solvability. The following Pfaff system corresponds to the system (1):

$$\text{(3)} \qquad \begin{cases} \omega^1(dx)=Fdx^1+x^3dx^2-dx^5=0 \\ \omega^2(dx)=Gdx^1+x^4dx^2-dx^6=0 \\ \omega^3(dx)=Hdx^4+Kdx^2-dx^7=0. \end{cases}$$

It is a special case of the system (2). According to E. Cartan, the system (1) is said to be in involution, if (3) has property 12.

D e f i n i t i o n. *The system (1) is hyperbolic, if (3) is hyperbolic and moreover has the property 12.*

According to the definition from Section 2, the condition for the hyperbolicity of (2) is based on the properties of the quadratic form $q(a)$,

$$q^2(a)=\det(a_1^1\Omega_1+a^2\Omega_2+a^3\Omega_3).$$

In the case of the system (3), this becomes

$$q(a)=\det\begin{pmatrix} a^1\partial_3F+a^2\partial_3G+a^3\partial_3H & a^1\partial_4F+a^2\partial_4G+a^3\partial_4H \\ a^1+a^3\partial_3K & a^2+a^3\partial_4K \end{pmatrix}.$$

Hence the hyperbolicity of (1) is equivalent, when $\partial_4E\neq 0$, to the condition that it is involutive w i t h $q(a)$ n o n-s e m i-d e f i n i t e.

T h e o r e m. *Every hyperbolic OS PDE of the form (1) is locally resolvable.*

The idea of the proof is based on the application of the theorem quoted in Section 2. Indeed, the Pfaff system (3) corresponding to (1) is hyperbolic and consequently it has a 2-dimensional resolving distribution $\theta'(x)$ in a neighbourhood of x_0. Due to the definition, $\theta'(x)$ is involutive, and hence — by the Frobenius theorem — it is completely integrable. This is equivalent to the existence of functionally independent functions $S_i(x)$, $i=1, 2, \dots, 5$, such that the equalities

$$S_i(x)=C_i, \quad i=1, 2, \dots, 5, \tag{4}$$

determine the family of the integral surfaces of $\theta'(x)$ in a neighbourhood of x_0 (here C_i are suitable constants). In particular, for $C_i=S_i(x_0)$ they determine an integral surface through x_0. Let $x^j=f_j(x^1, x^2)$ be functions, given by the implicit function theorem applied to the system (4). Then the functions $u=f_5(x^1, x^2)$, $v=f_6(x^1, x^2)$ and $w=f_7(x^1, x^2)$ are solutions of the system (1).

We have now to check the applicability in our case of the implicit function theorem, namely that the determinant

$$\frac{D(S_1, S_2, \dots, S_5)}{D(x^3\ x^4, \dots, x^7)} \tag{5}$$

is different from zero at x_0. To achieve this, we shall show that there exists for the system (3) a suitable resolving distribution $\theta'(x)$, whose corresponding determinant (5) is non-zero. This requires a more precise analysis of a part of the theorem quoted in Section 2.

From the point of view of the contents of Section 2, (3) determines a distribution $\theta=\theta(x)$ with basis consisting of the fields

$$\xi_1=(1, 0, 0, 0, F, G, H) \qquad \xi_3=(0, 0, 1, 0, 0, 0, 0)$$

$$\xi_2=0, 1, 0, 0, x^3, x^4, K) \qquad \xi_4=(0, 0, 0, 1, 0, 0, 0).$$

In this basis the operator Ω_1 has the following matrix

$$\Omega_1=\begin{pmatrix} 0 & \xi_2F & \xi_3F & \xi_4F \\ -\xi_2F & 0 & 1 & 0 \\ -\xi_3F & -1 & 0 & 0 \\ -\xi_4F & 0 & 0 & 0 \end{pmatrix}$$

and similarly Ω_2 and Ω_3. According to lemma [8, (3.4)], there exist two regular vectors $a=a(x)$ and $b=b(x)$ from A^3, such that

$$\dim E_a(x)=\dim E_b(x)=2, \quad E_a(x)\cap E_b(x)=\{0\}.$$

Denote by E_0 the linear hull of ξ_3 and ξ_4 and suppose that $E_0\cap E_a=\{0\}$. But

$$(\Omega^a\xi_3, \xi_4)=\sum_{i=1}^{3} a^i(\Omega_i\xi_3, \xi_4)=0,$$

since the matrices of Ω_i, $i=1, 2, 3$ have zeroes in their right-hand bottom corner. Having in mind also that Ω^a is antisymmetric and in addition that it is trivial on E_a, we can conclude, that $\Omega^a=0$ on the linear hull of E_0 and E_a. By dimension arguments it follows from the condition $E_0\cap E_a=\{0\}$ that this linear hull coincides with θ, which is impossible, since E_a is the kernel of Ω^a.

Therefore $E_0\cap E_a\neq\{0\}$. Similarly $E_0\cap E_b\neq\{0\}$, and we conclude, that $\dim E_0\cap E_a=\dim E_0\cap E_b=1$. Denote by η_a and η_b non-zero fields resp. from $E_0\cap E_a$ and $E_0\cap E_b$, and let ξ_a form with η_a a base of E_a, and ξ_b form with η_b a base of E_b. The fields

$$\zeta_a=\xi_a+u_1\eta_a, \quad \zeta_b=\xi_b+u_2\eta_b,$$

where u_1 and u_2 are smooth functions, determine a 2-dimensional distribution $\theta(u_1, u_2)$. Moreover, it is proved in [8], that if (u_1, u_2) is a solution of a special (determined by

ω^i) hyperbolic in the sense of Petrovski system of PDE, then $\theta(u_1, u_2)$ is involutive and hence resolving for (3). The initial conditions for u_1 and u_2 can be chosen arbitrarily on a hyperplane through x_0; let (u_1^0, u_2^0) be a solution with initial conditions zero. Then $\theta_0=\theta(u_1^0, u_2^0)$ is the linear hull of the fields ζ_a and ζ_b, which coincide at x_0 with ξ_a and ξ_b. Consequently $\theta_0 \cap E_0=\{0\}$ at x_0.

Since $\theta_0 \subset \theta$, then ζ_a and ζ_b are linear combinations of ξ_1, ξ_2, ξ_3 and ξ_4; having in mind that

$$\det\begin{pmatrix}\xi_1^1 & \xi_1^2\\ \xi_2^1 & \xi_2^2\end{pmatrix}=\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=1,$$

and that $\theta_0 \cap E_0=\{0\}$, we get that $\det\begin{pmatrix}\zeta_a^1 & \zeta_a^2\\ \zeta_b^1 & \zeta_b^2\end{pmatrix}\neq 0$ at x_0. Hence the determinant (5) corresponding to θ_0 is not zero at x_0, since S_i are a base in the space of solutions of the following systems of PDE: $\zeta_a S=0$, $\zeta_b S=0$.

The proof of the solvability of hyperbolic systems of the form (1) is now complete.

4. A generalization. To every system of PDE of the form

$$\text{(6)}\qquad \begin{cases}\dfrac{\partial u}{\partial x^1}=F, \ \dfrac{\partial v}{\partial x^1}=G, \\ \dfrac{\partial w^i}{\partial x^1}=H^i, \dfrac{\partial w^i}{\partial x^2}=K^i, \end{cases} \qquad i=1, 2, \dots, m,$$

where F, G, H^i, K^i are functions of x^1, x^2, $\partial u/\partial x^2=x^3$, $\partial v/\partial x^2=x^4$, $u=x^5$, $v=x^6$, $w^i=x^{i+6}$, $i=1, 2, \dots, m$ corresponds the Pfaff system

$$\begin{cases}Fdx^1+x^3dx^2-dx^5=0\\ Gdx^1+x^4dx^2-dx^6=0\\ H^idx^1+K^idx^2-dx^{i+6}=0, \quad i=1, 2, \dots, m\end{cases}$$

of rank $m+2$ in R^{m+6}. For a hyperbolic Pfaff system of this type, the theorem from Section 2 ensures the existence of suitable 2-dimensional resolving distributions. Thus we have the possibility to define hyperbolic smooth OS PDE of the form (6) and prove their local solvability in the same way as above.

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Bulgarian Academy of Sciences,
Institute of Mathematics,
1090 Sofia, P. O. Box 373 Bulgaria

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