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ON SOME FUNCTIONAL EQUATIONS RELATED TO BEATTY COMPLEMENTARY SEQUENCES

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In this paper we extend some previous results of Ostrowski and Aitken on Beatty complementary sequences of natural numbers and we solve some functional equations motivated by this study.

The sequences (a_n) and (b_n) of natural numbers are called **complementary** if taken together they contain each natural number exactly once, i. e., $\{a_n, b_n \mid n \in \mathrm{N}\} = \mathrm{N}$. In 1926, S. Beatty discovered [3] that for any fixed irrational number $x>0$ the sequences $([n(1+x)])$ and $([n(1+1/x)])$, where $[z]$ denotes the greatest integer not exceeding z, are complementary. A remarkable proof was given by A. Ostrowski and A. C. Aitken [1] and the reader may find an interesting approach to this topic in the well known book of R. Honsberger [4]. In this note we clarify why the functions $1+x$ and $1+1/x$ appear in Beatty's sequences and doing this we derive a functional equation characterizing the function "[ ]". When extending the study to some sequences in [0,1] we find also a characterization of the involution $N(x)=1-x$.

Theorem 1. *Let $u, v>1$ be irrational numbers. The sequences $([nu])$ and $([nv])$ are complementary if and only if* $1/u+1/v=1$.

Proof. If $([nu])$ and $([nv])$ are complementary sequences, then in each interval $(N, N+1)$ between two consecutive natural numbers $(N \geqq 1)$ there exists exactly one number of the set $A=\{nu, nv \mid n \geqq 1\}$. Given $k \geqq 1$ we have

$$\#\{n \mid nu<k\}=[k/u] \quad \text{and} \quad \#\{n \mid nv<k\}=[k/v].$$

Thus in the interval $(1, N)$ there are $[N/u]+[N/v]$ terms of the set A and in $(1, N+1)$ there are $[(N+1)/u]+[(N+1)/v]$. Since in $(N, N+1)$ there exists exactly 1 term of A we must have

$$(1) \qquad [(N+1)/u]+[(N+1)/v]-[N/u]-[N/v]=1.$$

From (1) it follows immediately that

$$(2) \qquad [(N+1)/u]+[(N+1)/v]=[2/u]+[2/v]+N-2.$$

Since u and v are irrationals we have

$$(3) \qquad [(N+1)/u-1<[([N+1)/u]<(N+1)/u$$

and

$$(4) \qquad N+1)/v-1<[(N+1)/v]<(N+1)/v.$$

From (3), (4) and (2) we obtain

$$(N+1)(1/u+1/v)-2<[2/u]+[2/v]+N-2<(N+1)(1/u+1/v)$$

whence dividing by $N+1$,

$$(5) \qquad 1/u+1/v-2/(N+1)<([2/u]+[2/v])/(N+1)+(N-2)/(N+1)<1/u+1/v.$$

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When N increases to infinity (5) implies the fundamental relation

(6) $$1/u+1/v=1.$$

Conversely, if (6) holds, following Ostrowski and Aitken argument, the addition of (3) and (4) imply

$$N-1<[(N+1)/u]+[(N+1)/v]<N+1,$$

and consequently

(7) $$[(N+1)/u]+[(N+1)/v]=N.$$

From (7) it is obvious that the sequences $([nu])$ and $([nv])$ are complementary.

Corollary 1. *Let f, g: $[0, \infty]\to[1, \infty]$ be functions such that $f(R^+\setminus Q^+)$ and $g(R^+\setminus Q^+)$ are subsets of $R^+\setminus Q^+$. Then for any irrational $x>0$ the sequence $([nf(x)]$ and $([ng(x)])$ are complementary if and only if $1/f(x)+1/g(x)=1$.*

Thus if $f(x)=1+x$, then we need to choose necessarily what Beatty considered in ([3]) i. e., $g(x)=1+1/x$; if $f(x)\equiv\sqrt{2}$, then $g(x)\equiv 2/(2-\sqrt{2})$, etc. As we have seen a key point in the above proof was the fact that for any irrational $x>0$ we have for all natural N

(8) $$[N(1+x)]+[N(1+1/x)]=N-1.$$

So it has sense to look if through (8) we can obtain a characterization of the function [ ]. This is what we do in the following:

Theorem 2. *Let f: $R^+\to R^+$ be a nondecreasing function such that $f(0)=0$ and $f(nu)+f(n(1-u))=n-1$, for all naturals $n\geqq 1$ and for irrationals u in $(0, 1)$. Then $f(x)=[x]$ for all positive real $x\geqq 0$.*

Proof. Obviously$[x]$ satisfies the given conditions. Assume that $f(0)=0$, f is non decreasing and

(9) $$f(nu)+f(n(1-u))=n-1,$$

for all irrationals u in $(0, 1)$ and natural $n\geqq 1$. Then the substitution $n=1$ into (9), $f(0)=0$ and f is nondecreasing imply $f(x)=[x]=0$ whenever x is in $[0, 1]$. From this (9) it follows inductively that $f(x)=[x]$ on each interval $[N, N+1]$, i. e., $f\equiv[\ ]$ on R^+.

Now we turn our attention to the case of an interesting analog of Beatty sequences in (0, 1) ([2]). Consider any irrational x in (0, 1) and take $A(x)=\{(1+x)/n, (2-x)/n \mid n\geqq 2\}$ then it is easy to prove the following

Lemma 1. *In each interval $((1/(n_0+1), 1/n_0), n_0=2, 3, \dots$, there are exactly 3 elements of the set $A(x)$.*

Using this fact we will show an intriguing characterization of $N(x)=1-x$.

Theorem 3. *Let N: $[0, 1]\to[0, 1]$ be a function such that $N([0, 1]\setminus Q^+)\subset[0, 1]\setminus Q^+$. If for any irrational x in $(0, 1)$ in each interval of the form $(1/(n_0+1), 1/n_0)$ with $n_0\geqq 2$ we can find 3 elements of the set $A_N(x)=\{(1+x)/n, (1+N(x))/n\}$, then $N(x)=1-x$.*

Proof. Let x be an irrational in (0,1). Then in $(1/(n_0+1), 1/n_0)$ we have 3 elements of $A_N(x)$, so in $(1/(n_0+1), 1)$ there will be $3(n_0-1)$ elements. This yields

$$[(1+x)n_0]+[(1+N(x))n_0]=3n_0-1.$$

From this relation it is easy to prove by induction that whenever $x=\sum_{i=1}^{\infty} x_i/10^i$ is the decimal representation of x and $N(x)=\sum_{i=1}^{\infty} a_i/10^i$ we have $x_k+a_k=9$, for all k. Therefore $x+N(x)=\sum_{i=1}^{\infty}(x_i+a_i)/10^i=\sum_{i=1}^{\infty} 9/10^i=1$, i. e., $N(x)=1-x$.

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