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TRAFFIC SIGNAL CONTROL OF A NETWORK SYSTEM

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ABSTRACT. The present paper deals with some methods of traffic signal operation. The arrival process at intersection is assumed to have a Poisson distribution. The flow control is expressed by a formula derived to optimize the nonstationary flow input at intersection.

1. Introduction. Usually traffic signal-control systems are based on procedures aiming at minimizing delays and queues of vehicles at intersections.

In this paper we consider the traffic signal control operation of networks with a Poisson distribution flow input accomplished by optimization techniques.

2. System definitions. The traffic signal of a network comprises intersections (nodes) and links (roads).

Let $E[N,A]$ be a graph of a network, where $N=\{1,2,\dots,n\}$ is a set of intersections (nodes) of E, $A=\{i,j\}=\{i\in N, j\in N, i\neq j\}$ is a set of links formed by pairs of intersections.

Let S_i, S_j be the traffic signals at intersection i and j respectively, where $(i,j)\in A$ as in Fig.1. Let us assume that the normalization state of vehicles is S_j (i.e. they are given a green wave at S_j). Let the speed of vehicles be fixed as well.

Let ϕ_j be the number phases designed for intersection j. We suppose that the time-period of a traffic change is normalized and it is equal to 1.

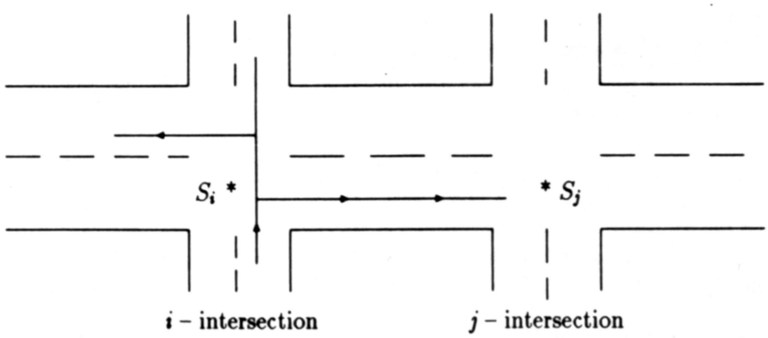

Fig. 1

Let $R^t(i,j)$ and $G^t(i,j)$ be the time probability for S_j to be red and green respectively at t, $t = 0, 1, 2, \ldots, T$.

In the set of equations (1) we assume that vehicles arrival at intersection S_j of link (i,j) may be considered a Poisson process with rate $\lambda^t(i,j)$:

$$(1) \qquad \begin{cases} R^t(i,j) + G^t(i,j) = 1, \\ R^t(i,j) = \begin{cases} 0 \\ 1 \end{cases}, \quad G^t(i,j) = \begin{cases} 0 \\ 1 \end{cases}, \\ \sum_{i=1}^{\phi_j} R^t(i,j) \geq \phi_j - 1 \quad \forall t \text{ and all links } (i,j) \in A. \end{cases}$$

3. Theoretical analysis of the system. In order to consider a network E, with N intersections whose number of phases is ϕ_j, we need $2 \leq \phi_j \leq N$ for the normal state.

Let $Q^t(i,j)$ be a random variable of vehicles waiting at the critical arm of intersections along link $(i,j) \in A$ at time t, and let $E\{Q^t(i,j)\}$ be the expected value of $Q^t(i,j)$.

Let the current time t be discretized, $t = 0, 1, 2, \ldots$.

The problem is:

$$(2) \qquad \min Y_t = \min \sum_{k=0}^{t} \sum_{(i,j)\in A} E\{Q^k(i,j)\} = Y_t^*$$

It is subjected to the restrictions given by (1). Thus we have to solve the total sum of the expected values of all the traffic flows under the following additional stochastic conditions:

Let $X^t(i,j)$, be the number of vehicles arriving at the critical link, $(i,j) \in A$, for t, in Poisson distribution with rate $\lambda^t(i,j)$.

Then, $\quad Pr\{X^t_{(i,j)} = v\} = (\lambda^t_{(i,j)})^v * \exp(-\lambda^t_{(i,j)}/v! \quad$ and

$$(3) \qquad \begin{cases} q^t_r(i,j) = Pr\{Q^t(i,j) = r\}\ , \\ x^t_v(i,j) = Pr\{X^t(i,j) = v\}\ , \end{cases}$$

where $q^t_r(i,j)$ and $x^t_v(i,j)$ are the probability density functions of $Q(i,j)$ and $X(i,j)$ respectively at time t, $t = 0, 1, \ldots, T$.

Let us assume that the light signal is green $G^t(i,j)$ at time t along link (i,j). If there are vehicles waiting for crossing at signal S_j , then a vehicle may depart in period $(t+1)$. Thus we may have the following set of equations recursively:

$$(4) \qquad Q^{t+1}_{(i,j)} = \begin{cases} \max\{Q^t_{(i,j)} + X^t_{(i,j)} - 1\ , 0\} & \text{if the signal is green .} \\ \{Q^t_{(i,j)} + X^t_{(i,j)}\} & \text{if the signal is red .} \end{cases}$$

Lemma. *Consider $Q^t(i,j)$ and $X^t(i,j)$ with probability density functions(p.d.f.) as in (3). Then*

$$\begin{aligned} E\{Q^{t+1}_{(i,j)}\} &= E\{Q^t_{(i,j)}\} - G^t_{(i,j)} + \lambda^t_{(i,j)} \\ &\quad - [q^t_0(i,j)\exp(-\lambda^t_{(i,j)})]G^t_{(i,j)} \quad \text{for } t \geq 0\ , \end{aligned}$$

where $E\{Q^{t+1}_{(i,j)}\}$ is the expected value of the queue length at epoch period $t+1$ along link (i,j).

Proof. From equation (4) we can write Chapman-Kolmogorovs equations as follows:

$$q_0^{t+1}(i,j) = q_0^t(i,j)*x_0^t(i,j) + \{q_1^t(i,j)*x_0^t(i,j)\}G_{(i,j)}^t + \{q_0^t(i,j) * \{x_0^t(i,j)\}G_{(i,j)}^t \tag{5}$$

$$\vdots$$

$$q_r^{t+1}(i,j) = \sum_{v=0}^{r+1}\{q_{r-v+1}^t(i,j)*x_v^t(i,j)\}G_{(i,j)}^t + \sum_{v=0}^{r}\{q_{r-v}^t(i,j) * x_v^t(i,j)\}R_{(i,j)}^t \qquad \text{for } r \geq 1. \tag{6}$$

Let $Q_{(i,j)}^t(z) = \sum_{r=0}^{\infty} q_r^t(i,j)z^r$ and $X_{(i,j)}^t(z) = \sum_{r=0}^{\infty} x_r^t(i,j)z^r = G(z)$, be the probability generating functions of $Q^t(i,j)$ and $X^t(i,j)$ respectively at t for $\|z\| \leq 1$. Since $X^t(i,j)$ is a random variable with the following discrete distribution

$$X_r = Pr\{x^t(i,j) = v_r\}, \qquad (r = 0,1,2,\ldots)\,,$$

then the probability generating function (p.g.f.) of the distribution of X is:

$$G(z) = \sum_{r=0}^{\infty} x_r\, z^r \qquad \text{with } G(1) = 1 \text{ and}$$

$$G'(1) = \left.\frac{dG(z)}{dz}\right|_{z=1} = E\{x^t(i,j)\} = Ex$$

where Ex is the expected value of x, and $G''(1) = E\{x(x-1)\}$. Thus $Ex = G'(1)$, and the variance of x is

$$\mathbf{var}(x) = G''(1) + G'(1) - \{G'(1)\}^2\,.$$

Now, it is convenient to put again $G(z) = G_{(i,j)}^t(z)$. According to the generating function properties and equation (4) we have:

$$Q_{(i,j)}^{t+1}(z) = z^{-1}\{G_{(i,j)}^t(z) - R_{(i,j)}^t\}X_{(i,j)}^t(z) * Q_{(i,j)}^t(z) + \{q_0^t(i,j) * x_0^t(i,j)\}G_{(i,j)}^t * (1 - 1/z)\,. \tag{7}$$

where $G_{(i,j)}^t(z)$ is the generating function and $G_{(i,j)}^t$ is the probability of the traffic signal when S_j is green at time t. Therefore, we can write the queue length as follows:

$$E\{Q_{(i,j)}^{t+1}\} = E\{Q_{(i,j)}^t\} - G_{(i,j)}^t + \lambda_{(i,j)}^t - [1 - q_0^t(i,j) * \exp(-\lambda_{(i,j)}^t)] * G_{(i,j)}^t \quad \text{for } t \geq 0\,. \tag{8}$$

Now the objective is to exercise operational signal-control on the traffic network. Usually, this problem is reduced to the problem of minimizing the total expected delay (the queue length) at all the intersections of the network from epoch $\{0\}$ to epoch $\{T\}$.

Let $y_t = \sum_{(i,j)\in A} E\{Q^t_{(i,j)}\}$ and $Y_t = \sum_{k=0}^{t} y_k$. Then, $Y_{t+1} = y_{t+1} + Y_t$. From equation (8), we have

$$y_{t+1} = y_t + \sum_{(i,j)\in A} [\lambda^t_{(i,j)} - G^t_{(i,j)}\{1 - q^t_0(i,j)\exp(-\lambda^t_{(i,j)}\}] . \tag{9}$$

Let $a_t = \sum_{(i,j)\in A}[\lambda^t_{(i,j)} - G^t_{(i,j)}\{1 - q^t_0(i,j)\exp(-\lambda^t_{(i,j)}\}]$.
From (9) we obtain the following relations:

$$\begin{aligned} Y_{t+1} &= y_{t+1} + Y_t = y_t + a_t + Y_t = 2y_t + Y_{t+1} + a_{t-1} + a_t \\ &= 2y_{t-1} + Y_{t-1} + 2a_{t-1} + a_t \ . \end{aligned}$$

Finaly we have:

$$\begin{aligned} Y_{t+1} &= y_t + Y^*_0 + \sum_{k=1}^{t} (t+1-k)a_k \\ &= (t+2)Y^*_0 + \sum_{k=0}^{t} (t+1-k)a_k \\ Y^*_0 &= y_0 = \sum_{(i,j)\in A} E\{Q^{t=0}_{(i,j)}\} \ , \end{aligned}$$

which is the initial condition. Since, $\lambda^t_{i,j)}$, $q^t_0(i,j)$ are known, and a_k depends only on $\{G^k_{(i,j)}\}$, we obtain:

$$Y^*_{t+1} = \min Y_{t+1} = (t+2)Y^*_0 + \sum_{k=0}^{t}(t+1-k)\min a_k \ .$$

In order to solve the problem through the step by step method, we have to minimize

$$a_t = \sum_{(i,j)\in A} [\lambda^t_{(i,j)} - G^t_{(i,j)}\{1 - q^t_0(i,j)\exp(-\lambda^t_{(i,j)})\}] \ , \quad k = 0, 1, \ldots, t$$

by subjecting it to:

$$\begin{aligned} &R^k(i,j) + G^k(i,j) = 1 \ , \\ &G^k(i,j) = \begin{cases} 0 \\ 1 \end{cases} , \\ &\sum_{i=1}^{\phi_j} R^t(i,j) \geq \phi_j - 1 \qquad \forall (i,j) \in A \text{ and } \forall j \in N \ . \end{aligned}$$

Since the set $\{G^k_{(i,j)}\}_i$ is one phase and equal to 1 for every node j and since $\lambda^k_{(i,j)}$ are constant, for fixed k we have:

$$\min_i\{q^k_0(i,j)\exp[-\lambda^k_{(i,j)}]\}$$

where i belongs to the set of a phasing for the node j. Thus, the following algorithm is obtained:

1. Calculate the initial condition $Y_0^* = y_0 = \sum_{(i,j)\in A} E\{Q_{(i,j)}^{t=0}\}$
2. Let $Y_t^* = \min\{Y_t\}$ and y_t be known.
3. Calculate the following expression for $t = t+1$,

3.1. $q_0^{t-1}(i_j, j)\exp[-\lambda_{(i_j,j)}^{t-1}] = \min_i\{q_0^{t-1}(i,j)\exp[-\lambda_{(i,j)}^{t-1}]\}$
and put $G_{(i_j,j)}^{t-1} = 1$, $R_{(i,j)}^{t-1} = 1$, $i \neq i_j$, $\forall j \in N$.

3.2. $y_t = y_{t-1} + \sum_{(i,j)\in A} \lambda_{(i,j)}^{t-1} - \sum_{j\in N,(i_j,j)\in A} G_{(i_j,j)}^{t-1}\{1 - q_0^{t-1}(i_j, j)\exp(-\lambda_{(i_j,j)}^{t-1})\}$.

3.3. $Y_t^* = y_t + Y_{t-1}^*$ and go to step 2.

Remarks.

1. The above algorithm can be applied to a traffic signal system at normal state of traffic lights or a flow a sufficiently large T.

2. This algorithm can be used to solve the problem in real time which is of great significance for the operative management of traffic lights.

4. Example. Let us consider a network with two intersections whose signal design is as follows:

Vehicles arrive at intersections in the form of a Poisson process with time-dependent parameters. The following table illustrates the arrival rates of vehicles for the critical link (i,j), $(i = 1,\ j = 1,2)$ of each epoch phase in time t.

t	*arrival-rates* $\lambda_{(1,1)}^t$	$\lambda_{(1,2)}^t$	$G_{(1,1)}^t$	$G_{(1,2)}^t$
0	1.1	1.2	0	1
1	1.4	1.3	1	0
2	1.5	1.4	0	1
3	1.3	1.5	1	0
4	1.7	1.6	0	1
5	0.6	1.8	1	0
6	1.2	1.9	0	1
7	0.6	1.4	1	0
8	1.3	1.7	0	1

In order to find out the expected number of vehicles waiting for crossing at the critical link $(i,j) \in A$ at t $(t = 0, 1, \ldots)$ we may use (8), assuming that $q_0^t = 1$ and the initial waiting time $y_0 = 0$. Thus we obtain:

t	$y_{(i,j)}$	$y_{(1,1)}$	$y_{(1,2)}$
0	0.00	0.00	0.00
1	1.60	1.10	0.50
2	3.55	1.75	1.80
3	5.70	3.25	2.45
4	7.77	3.82	3.95
5	10.27	5.52	4.75
6	12.22	5.67	6.55
7	14.47	6.87	7.60
8	16.02	7.02	9.00
9	17.47	7.59	9.80

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