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AN OBLIQUE DERIVATIVE PROBLEM FOR SECOND ORDER QUASILINEAR PARABOLIC OPERATORS II

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ABSTRACT. An oblique derivative boundary value problem for a class of quasilinear strictly parabolic operators on a cylinder in $\mathbb{R}^{n+1}$ is studied. It is assumed that the normal component of the vector field l corresponding to the problem vanishes on some subvariety of codimension one of the boundary and that l is of "emergent" type. Existence and uniqueness results in Hölder spaces are proved.

1. Introduction. The aim of the present paper is to study the quasilinear parabolic equation $\mathcal{P}u = 0$ on a cylinder Q. More precisely, we shall be concerned with the existence and uniqueness questions for the solution of this equation which satisfies a boundary condition in terms of the directional derivative $\partial u/\partial l = 0$ on the lateral boundary S of Q, and initial condition $u = \varphi$.

It is well known [1,2] that the initial-boundary value problem (IBVP) is regular if the vector field l is conormal to S. Moreover, under suitable conditions on the data there exists a unique classical solution of the problem.

Our goal is to study the degenerate problem, i. e. the case when l is tangent to S at point of the set $E \subset \overline{S}$. The linear problem was treated by Egorov and Chiong [3] in Sobolev spaces. It was proved that the solvability of IBVP depends on the way in which the normal component of l changes its sign in the positive direction along the integral curves of the field l. This result is analogous to that in elliptic case [5,6]. In the author's work [7] the case of sign-preserving vector field l was considered for quasilinear operators. We have proved that the above problem is well posed in Hölder spaces in the sense that usual existence and uniqueness theorems analogous to those for the Neumann problem are true. The basic assumption is that the lengths of l-curves on E are finite. Further in [8] the linear problem in another situation was solved: l is of "emergent" type, i. e. the sign of l changes from minus to plus along the l-curves in a neighbourhood of E, where E is a submanifold of S with $\operatorname{codim}_S E = 1$. In that case, however, the IBVP as stated above is ill posed [3,8]. The dimension of the kernel of the problem is infinite. To avoid this difficulty the extra boundary condition $u = \mu$ is prescribed on E.

In the present paper we aim at extending our results in order to to get a solvability theorem for the quasilinear equations in the case of "emergent" field l. As in [7], we use Winzell's technique [5] for a global bound of the solution Hölder norms, as well as the approach of P. Popivanov and N.Kutev [6] in the treatment of elliptic equations. The nonemptiness of E leads to a loss of regularity. Thus, the solution in general has one x-derivative and "1/2"

t-derivative less than in the regular case. That is why, it is natural to consider quasilinear operators with coefficients depending on the unknown function only. Finally, let us note that in our case E is a subvariety of S and $\operatorname{codim}_S E = 1$. The more complicated situation when E is an arbitrary subset of S not containing l-curves of infinite length, can be treated in a similar way.

2. Statement of the problem and main result. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and smooth domain, $0 < T < \infty$, $Q := \Omega \times (0,T]$, $S := \partial\Omega \times (0,T]$. On the lateral boundary $\overline{S}$ of the cylinder Q a flat, smooth and unit vector field $l(x,t) = (l^1(x,t), \ldots, l^n(x,t), 0)$ is defined which can be decomposed into

$$l(x,t) = \tau(x,t) + \gamma(x,t).\nu(x)$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$ and $\tau(x,t)$ is a tangential vector to $\overline{S}$. Let

$$E := \{(x,t) \in \overline{S} : \ \gamma(x,t) = 0\}, \qquad S^+(S^-) := \{(x,t) \in \overline{S} : \ \gamma(x,t) > 0 (< 0)\}.$$

Furthermore we impose the following assumptions

$$\text{the set } E \text{ is a submanifold of } \overline{S}, \ \operatorname{codim}_S E = 1; \tag{1}$$

$$\begin{cases} \text{the vector field } \tau \text{ points from } S^- \text{ into } S^+ \text{ on } E \\ \text{and } \tau \text{ is strictly transversal to } E. \end{cases} \tag{2}$$

Remark 2.1. Let us note that the above conditions imply that the set $E \cap \{t = t_0\}$ is non empty for every $t_0 \in [0,T]$ and that the vector $(0, \ldots, 0, 1)$ is not normal to E at any point $(x,t) \in E$. For convenience we consider the case when E is a connected submanifold of S if $n > 2$ and E consists of two connected manifolds without common points, if $n = 2$. The problem will be studied in the Hölder spaces $C^{q,q/2}(\overline{Q})$, ($q > 0$, q is non-integer) equipped with the seminorms $\langle \cdot \rangle_Q^{(k)}$, $0 \leq k \leq q$, and the norm $|\cdot|_Q^{(q)}$ (for the definitions see [1,2]).

Let us consider the following non-classical IBVP

$$\begin{cases} \mathcal{P}u := a^{ij}(x,t)\partial^2 u/\partial x_i \partial x_j + b^i(x,t,u)\partial u/\partial x_i + c(x,t,u) - u_t = 0 \quad \text{in } Q \\ lu := l^i(x,t)\partial u/\partial x_i = 0 \quad \text{on } S; \\ u(x,t) = \mu(x,t) \quad \text{on } E; \ u(x,0) = \varphi(x), \ x \in \overline{\Omega}. \end{cases} \tag{3}$$

We adopt the standard summation convention that repeated indices indicate summation from 1 to n and assume that

$$\{a^{ij}(x,t)\xi^i\xi^j \geq \lambda|\xi|^2 \qquad (x,t) \in \overline{Q}, \ \xi \in \mathbb{R}^n, \ \lambda = \text{const} > 0, \ a^{ij} = a^{ji}; \tag{4}$$

$$\begin{cases} a^{ij} \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q}); \qquad b^i, c \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q} \times \mathbb{R}); \\ l^i \in C^{2+\alpha,(2+\alpha)/2}(\overline{S}); \qquad E \in C^{2+\alpha,(2+\alpha)/2}, \ \partial\Omega \in C^{3+\alpha}, \ 0 < \alpha < 1; \end{cases} \tag{5}$$

$$z.c(x,t,z) \leq c_0 z^2 + b_0, \qquad (x,t,z) \in \overline{Q} \times \mathbb{R}, \ 0 \leq c_0, \ b_0 = \text{const}. \tag{6}$$

The main result is as follows

Theorem. *Under the above assumptions problem (3) has a unique classical solution in the space $C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ for every $\mu \in C^{2+\alpha,(2+\alpha)/2}(E)$ and $\varphi \in C^{3+\alpha}(\overline{\Omega})$, satisfying compatibility conditions up to order one on $E \cap \{t=0\}$ and $\overline{S} \cap \{t=0\}$ respectively.*

Remark 2.2. 1. $f \in C^{q,q/2}(\overline{Q} \times \mathbb{R})$ if $f(x,t,z) \in C^{q,q/2}(\overline{Q} \times [-N,N])$ for $N > 0$ and we consider z as a $(n+1)$-th spatial variable.

2. The data in (3) satisfy compatibility conditions up to order m on $\overline{S} \cap \{t=0\}$ if

$$\partial^k/\partial t^k \left[l^i(x,t)u_i(x,t)\right]\Big|_{\substack{t=0\\ x\in\partial\Omega}} = 0 \qquad k = 0,1,\dots,m\ ,$$

where the derivatives in the left-hand side come from the equation $\mathcal{P}u = 0$ and the initial condition $u(x,0) = \varphi(x)$ (see [2], p.363). In the same way, the data are compatible up to order m on $E \cup \{t=0\}$ (see [1], p.137) if for every point $(x_0,0) \in E \cap \{t=0\}$

$$\partial^k/\partial t^k \left[\tilde{u}(\tilde{x},t)\right]\Big|_{\substack{t=0\\ \tilde{x}_n=\tilde{x}_{n-1}=0}} = \partial^k/\partial t^k \tilde{\mu}(\tilde{x}'',t)_{|t=0}, \qquad k = 0,1,\dots,m$$

where the functions with " $\sim$ " are the corresponding functions in the new coordinates $(\tilde{x},t)$ in which the set E has the form $\tilde{x}_n = \tilde{x}_{n-1} = 0$ near $(x_0,0)$ and the derivatives in the left are obtained as above from the transformed equation and the initial condition.

3. Some preliminaries. For the sequel a comparison principle for quasilinear parabolic operators will be proved. Consider $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$,

$$\mathcal{R}u := a^{ij}(x,t,u,Du)u_{ij} + b(x,t,u,Du) - u_t,$$

and let

$$a^{ij}(x,t,z,p)\xi^i\xi^j \geq \lambda|\xi|^2 \qquad (x,t,z,p) \in \overline{Q} \times \mathbb{R} \times \mathbb{R}^n,\ \lambda = \text{const} > 0,\ a^{ij} = a^{ji},$$

$$a^{ij}, a^{ij}_z, a^{ij}_{p_k}, b, b_z, b_{p_k} \in C^0(\overline{Q} \times \mathbb{R} \times \mathbb{R}^n)$$

where the lower indices mean a differentiation with respect to the corresponding variables. We denote

$$\tilde{d}\left((x,t),(x',t')\right) := \left(|x-x'|^2 + |t-t'|\right)^{1/2}$$

and

$$C^{2,1}(\overline{Q}) := \{u \in C^0(\overline{Q}) :\ D_t^r D_x^\beta u \in C^0(\overline{Q}),\ 2r + |\beta| \leq 2\}$$

Lemma 3.1. *Assume that $\partial\Omega \in C^2$ and that $u, v \in C^{2,1}(\overline{Q})$ satisfy $\mathcal{P}u \geq \mathcal{P}v$ in Q, $u \leq v$ on E, $lu = lv$ on S, $u(x,0) \leq v(x,0)$ for $x \in \overline{Q}$. Then $u \leq v$ in $\overline{Q}$.*

Proof. Putting $w := u - v$ we obtain

$$\overline{a}^{ij}(x,t)w_{ij} + \overline{b}^i(x,t)w_i + \overline{c}(x,t)w - w_t \geq 0 \qquad \text{in } Q\ ,$$

where the coefficients depend on the corresponding coefficients of $\mathcal{R}$ via the mean-value theorem. Moreover $w(x,t) \leq 0$ on E, $lw = 0$ on S and $w(x,0) \leq 0$, $x \in \overline{\Omega}$. The strong interior maximum principle implies that the function $w(x,t)$ cannot attain a positive maximum in $Q \cup E$ since $w(x,0) \leq 0$ on $\overline{\Omega}$, and $w \leq 0$ on E. If we suppose that $w(x,t)$ attains a positive maximum at a point $(x_0,t_0) \in S^+(S^-)$ then $lw(x_0,t_0) > 0(< 0)$ by virtue of the boundary maximum principle (l is non-tangential to S on $S^+ \cup S^-$), which contradicts to the boundary condition $lw = 0$.

To use Winzell's technique for a global bound of the solution Hölder norms, the integration along l-curves between subdomains of Q will be essential. In fact we need the following result which corresponds to Proposition 3.3 in [5].

Lemma 3.2. *There exist an extension L of l in $\overline{Q}$ and a $C^{2+\alpha,(2+\alpha)/2}$ manifold $\mathcal{N} \subset \overline{Q}$ with a "lateral" boundary $E \cup E'$ such that*

(i) L is strictly transversal to $\mathcal{N}$;

(ii) Every integral curve of L through a point in $\mathcal{N}$ in either direction reaches $\overline{S}$ within a parameter length less than a constant $K > 0$;

(iii) If $Q_p := \{e^{qL}(x,t) \in Q : \ (x,t) \in \mathcal{N}, \ -q < p < q\}$ for $p > 0$, then $\{Q_p\}$ is an increasing family and for each $\delta > 0$ there exists a number $\theta = \theta(\delta) > 0$, not depending on p, such that $\tilde{d}(Q_p, Q_K \backslash Q_{p+\delta}) \geq \theta$ whenever $Q_K \not\equiv Q_{p+\delta}$;

(iv) $\tilde{d}(E, Q\backslash Q_K) \geq \tilde{d}_0 = const\ > 0$.

Proof. The assertions (ii), (iii) and (iv) are clear in view of assumptions (1) and (2) as well as of the fact that for a fixed p the map $(x,t) \to e^{pL}(x,t)$ is a diffeomorphism on $\mathcal{N}$. The point (i) follows from the definitions of L and $\mathcal{N}$. We thus concentrate on the construction of L and $\mathcal{N}$. Since $\partial\Omega \in C^{3+\alpha}$ there exists a neighbourhood Γ of $\partial\Omega$ (see Appendix in [4]) such that for every $x \in \Gamma$ there is a closest point $y(x) \in \partial\Omega$, $y(x) \in C^{2+\alpha}(\Gamma, \mathbb{R}^n)$ and dist$(\Omega\backslash\Gamma, \partial\Omega) \geq d_0$. Putting $L(x,t) := l(y(x),t)$, $\overline{\nu}(x) := \nu(y(x))$ for $x \in \Gamma$, we define $\mathcal{N}$ as the union of curves of length at most d_0, originating in E and with tangent vector $(-\overline{\nu})$. It is evident that $E' = e^{-d_0\overline{\nu}}E$. Finally, we extend $L(x,t)$ in the whole of $\overline{Q}$ in a natural way.

The next local apriori estimate will be very important in our investigations. It is proved in Proposition 2.1 in [7].

Proposition 3.3. *Let $W \subset Q$ be a domain in $\mathbb{R}^{n+1}$, $U \subset \overline{W}$ and $\tilde{d}(U, \overline{Q}\backslash\overline{W}) \geq \theta > 0$. If $u \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ then the following estimate is valid*

$$\langle u\rangle_U^{(2+\alpha)} \leq C\left(\langle u\rangle_{\overline{W}\cap\overline{S}}^{(2+\alpha)} + \langle\mathcal{L}u\rangle_W^{(\alpha)} + [u(\cdot,0)]_{2+\alpha,\ \overline{W}\cap\{t=0\}}\right)$$

$$+C'(\theta)\left(|\mathcal{L}u|_{0,W} + \sum_{j=0}^{2}\langle u\rangle_W^{(j)} + \sum_{j=0}^{1}\langle u\rangle_W^{(j+\alpha)}\right)$$

where $\mathcal{L}$ is a linear uniformly parabolic operator.

We will end with a variant of the interpolation inequality between Hölder norms. Let $u \in C^{q,q/2}(\overline{Q})$ and let Ω satisfy the uniform interior cone condition. Then there is a constant C, independent of u, such that

$$\langle u\rangle_Q^{(\sigma q)} \leq C\left(|u|_Q^{(q)}\right)^{\sigma}\left(|u|_{0,Q}\right)^{1-\sigma} \tag{7}$$

where $\sigma \in [0,1]$ and $|u|_{0,Q} = \langle u\rangle_Q^{(0)}$. The proof is a simple consequence of Lemma 3.2, ch.2 in [2].

4. Proof of the main result. The unicity of the solution of (3) follows from Lemma 3.1.

The proof of the solvability of our problem will be carried out by means of the Leray-Schauder theorem (Th.11.3 in [4]).

First of all, we reduce the problem (3) to a problem with zero initial data (see [2]). Namely, we consider the tangential IBVP

$$
(8)\qquad \begin{cases} a^{ij}u_{ij} + b^i(x,t,u+\varphi)u_i + b^i(x,t,u+\varphi)\varphi_i \\ \qquad\qquad +c(x,t,u,u+\varphi) - u_t = -a^{ij}\varphi_{ij} \text{ in } Q \\ lu = -l\varphi \text{ on } S;\ u = \mu - \varphi \text{ on } E;\ u(x,0) = 0,\ x \in \overline{\Omega} \end{cases}
$$

instead of (3). It is clear that if $u \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ solves (8) then the function $u(x,t)+\varphi(x)$ is the desired solution of the original problem (3).

Let us define the Banach space

$$
C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q}) := \left\{ v \in C^{1+\alpha,(1+\alpha)/2}(\overline{Q}) :\ v(x,0) = 0,\ x \in \overline{Q} \right\}
$$

and the non-linear operator

$$
\mathcal{U} :\ C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q}) \to C_0^{2+\alpha,(2+\alpha)/2}(\overline{Q})
$$

in a standard way: for every $v \in C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q})$ the image $\mathcal{U}v$ is the unique classical solution of the linear IBVP

$$
\begin{cases} a^{ij}(\mathcal{U}v)_{ij} + b^i(x,t,v+\varphi)(\mathcal{U}v)_i - (\mathcal{U}v)_t = -b^i(x,t,v+\varphi)\varphi_i \\ \qquad\qquad -c(x,t,v+\varphi) - a^{ij} \qquad \text{in } Q \\ l(\mathcal{U}v) = -l\varphi \text{ on } S;\ (\mathcal{U}v) = \mu - \varphi \text{ on } E;\ (\mathcal{U}v)(x,0) = 0\ x \in \overline{\Omega}. \end{cases}
$$

The above problem is uniquely solvable because of our assumptions and the Theorem in [8]. Moreover, the estimate for $\mathcal{U}v$ (see the Theorem in [8]) claims that $\mathcal{U}$ maps bounded sets in $C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q})$ into bounded sets in $C_0^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ which means that $\mathcal{U}$ is a compact operator from $C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q})$ into itself. There are no difficulties to obtain that $\mathcal{U} :\ C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q}) \to C_0^{1+\alpha,(1+\alpha)/2}(\overline{Q})$ is a continuous operator. To apply Leray-Schauder's theorem it remains to prove the apriori estimate

$$
(9)\qquad |u|_Q^{(1+\alpha)} \le C
$$

for every solution $u \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ of the problem

$$
(10)\qquad \begin{cases} a^{ij}u_{ij} + b^i(x,t,u+\varphi)u_i - \rho[b^i(x,t,u+\varphi)\varphi_i + c(x,t,u+\varphi)] - u_t \\ \qquad\qquad = \rho a^{ij}\varphi_{ij} \qquad \text{in } Q \\ lu = -\rho l\varphi \text{ on } S; u = \rho(\mu - \varphi) \text{ on } E;\ u(x,0) = 0\ x \in \overline{\Omega} \end{cases}
$$

with a constant C independent of u and $\rho \in [0,1]$.

The desired bound will be derived in several steps. To estimate $|u|_{0,Q}$ we consider in Q the auxiliary function $w(x,t) := e^{-Nt}u(x,t) + \rho e^{-Nt}\varphi(x)$, where $N > 0$ is a constant under control. It is evident that $w(x,t)$ solves the problem

$$\begin{cases} \tilde{P}w := a^{ij}w_{ij} + b^i(x,t,e^{Nt}w + (1-\rho)\varphi)w_i + \rho e^{-Nt}c(x,t,e^{Nt}w + (1-\rho)\varphi) \\ \qquad\qquad -Nw - w_t = 0 \qquad \text{in } Q \\ lw = 0 \text{ on } S; \; w = \rho e^{-Nt} \text{on } E; \; w(x,0) = \rho\varphi(x) \end{cases}$$

Let $m := 2|\varphi|_{0,\overline{\Omega}} + |\mu|_{0,E} + 1$. Since $lw = 0 = lm$ on S, $w \le |\mu|_{0,E} < m$ on E and $w(x,0) \le |\varphi|_{0,\overline{\Omega}} < m$ on $\overline{\Omega}$, if $\tilde{P}w \ge \tilde{P}m$ in Q, then Lemma 3.1 would give $w(x,t) \le m$ in $\overline{Q}$. The inequality $\tilde{P}w \ge \tilde{P}m$ in Q is equivalent to

$$0 \ge m(\rho e^{-Nt}c(x,t,e^{Nt} + (1-\rho)\varphi) - Nm) \qquad \text{in } Q \tag{11}$$

because $m > 0$. Now

$$e^{Nt}m + (1-\rho)\varphi \ge 2|\varphi|_{0,\overline{\Omega}} + |\mu|_{0,E} + 1 - |\varphi|_{0,\overline{\Omega}} > 0,$$

$$0 < e^{Nt}m/(e^{Nt}m + (1-\rho)\varphi) < 2$$

and the structure condition (6) implies

$$m(\rho e^{-Nt}c(x,t,e^{Nt}m + (1-\rho)\varphi) - Nm)$$

$$= -Nm^2 + \rho e^{-2Nt}\left(e^{Nt}m/(e^{Nt}m + (1-\rho)\varphi)\right)(e^{Nt}m + (1-\rho)\varphi).c(x,t,e^{Nt}m + (1-\rho)\varphi)$$

$$\le m^2(5c_0 - N) + 2b_0 = 0 \qquad \text{if } N = 5c_0 + 2b_0/m^2.$$

Therefore

$$w(x,t) \le m = 2|\varphi|_{0,\overline{\Omega}} + |\mu|_{0,E} + 1.$$

In a similar way an estimate from below can be deduced, whence

$$|u|_{0,Q} \le C_1\ . \tag{12}$$

Here after the symbols $C_i, i = 1,2,\ldots$ stand for constants which are independent of u and ρ. We consider now the extension $L(x,t)$ of l in $\overline{Q}$. Let $\Sigma := \Sigma(\tilde{d}_0/2)$ where $\Sigma(d) := \{(x,t) \in \overline{Q} : \tilde{d}(E,(x,t)) \le d\}$, $\tilde{d}_0$ being defined in Lemma 3.2. Because of $|l^i\nu^i| = |\gamma| \ge \gamma_0 > 0$ on $\overline{S} \cap Q\backslash\Sigma(\tilde{d}_0/8)$, and Th.7.1, ch.5 in [2], there is a number $\beta \in (0,1)$, independent of u and ρ such that

$$|u|^{(\beta)}_{Q\backslash\Sigma(\tilde{d}_0/8)} \le C_2(n,\lambda,\alpha,\tilde{d}_0,\Omega,T,C_1,\varphi)\ .$$

It follows that $|b^i(x,t,u+\varphi)|^{(\beta)}_{Q\backslash\Sigma(\tilde{d}_0/8)}$ and $|c(x,t,u+\varphi)|^{(\beta)}_{Q\backslash\Sigma(\tilde{d}_0/8)}$ can be estimated independently of u and ρ. Considering now (10) as a linear problem, the local estimates for parabolic equations, (Th.10.1, ch.4 in [2]) give

$$|u|^{(2+\beta)}_{Q\backslash\Sigma(\tilde{d}_0/4)} \le C_3\ .$$

Applying the same argument for the domains $Q\backslash\Sigma \equiv Q\backslash\Sigma(\tilde{d}_0/2)$ and $Q\backslash\Sigma(\tilde{d}_0/4)$ we obtain

$$|u|^{(2+\alpha)}_{Q\backslash\Sigma} \leq C_4 ,$$

hence

$$|u|^{(2+\alpha')}_{Q\backslash\Sigma} \leq C_5(C_4, \alpha, \alpha') \tag{13}$$

for every $\alpha' \in (0, \alpha)$.

Now we use the family $\{Q_p\}_{p\geq 0}$ from Lemma 3.2 in order to estimate $|u|^{(2+\alpha')}_{Q_K}$. Let $s \to \psi(s, x, t)$ be the parameterization of the maximal integral curve of L, passing through $(x, t) \in Q_K$. It follows from Lemma 3.2 (ii) that every point $(x, t) \in Q_K$ can be written as $\psi(p(x, t), x', t)$ where $(x', t) \in \mathcal{N}$, $|p(x, t)| \leq K$ and $p(x, t) \in C^{2+\alpha,(2+\alpha)/2}$. Therefore

$$u(x, t) = u \circ \psi(-p(x, t), x, t) + \int_0^{p(x,t)} (Lu) \circ \psi(q - p(x, t), x, t)\, dq \tag{14}$$

for every $(x, t) \in Q_K$. We set $v := u_{|\mathcal{N}}$ and suppose that v is extended in Q_K as a constant along L-curves. The integral representation yields

$$\langle u\rangle^{(2+\alpha')}_{Q_p} \leq C_6 \left(|v|^{(2+\alpha')}_{\mathcal{N}} + \int_0^p \langle Lu\rangle^{(2+\alpha')}_{Q_q}\, dq + |Lu|^{(2+\alpha'')}_{Q}\right)$$

where $0 < \alpha'' < \alpha' < \alpha < 1$ and $p \in [0, K]$.

The equation in (10) can be written in the form $\mathcal{P}_0 u = \mathcal{P}_1 u$ with

$$\mathcal{P}_0 := a^{ij}(x, t) D_i D_j - D_t$$

and

$$\mathcal{P}_1 u := -\rho[a^{ij}\varphi_{ij} + b^i(x, t, u + \varphi)\varphi_i + c(x, t, u + \varphi)] - b^i(x, t, u + \varphi)u_i.$$

The action of $\mathcal{P}_0$ on the functions, defined in Q_K, which are constants along L-curves, defines a linear strictly parabolic operator $\mathcal{P}'_Q$ on $\mathcal{N}$ (L is strictly transversal to $\mathcal{N}$). Therefore, by means of (14), the function $v \in C^{2+\alpha,(2+\alpha)/2}(\mathcal{N})$ is a solution of the problem

$$\begin{cases} \mathcal{P}'v = \mathcal{P}_1 u - \mathcal{P}_2(Lu) & \text{in } \mathcal{N} \\ v = \rho(\mu - \varphi) \text{ on } E;\ v = u \text{ on } E';\ v(x, 0) = 0\ x \in \mathcal{N} \cap \{t = 0\} \end{cases}$$

where $\mathcal{P}_2$ is a linear operator with $C^{\alpha,\alpha/2}$ coefficients, $\operatorname{ord}\mathcal{P}_2 = 1$ and $\mathcal{P}_2$ does not contain D_t. Because of the parabolic estimate (Th.6.1 in [1]) for v, we have

$$\begin{aligned} |v|^{(2+\alpha')}_{\mathcal{N}} &\leq C_7 \left(|\mathcal{P}_1 u - \mathcal{P}_2(Lu)|^{(\alpha)}_{\mathcal{N}} + |\rho(\mu - \varphi)|^{(2+\alpha')}_{E} + |u|^{(2+\alpha')}_{E'}\right) \\ &\leq C_8 \left(1 + |Lu|^{(1+\alpha')}_{Q} + |\mathcal{P}_1 u|^{(\alpha')}_{Q}\right) \end{aligned}$$

according to (13). On the other hand,

$$|\mathcal{P}_1 u|^{(\alpha')}_{Q} \leq C_9 \left(|u|^{(1+\alpha')}_{Q} + \langle u\rangle^{(1)}_{Q} . \langle u\rangle^{(\alpha')}_{Q}\right) \leq C_{10} \left(1 + |u|^{(2+\alpha'')}_{Q}\right)$$

since

$$|b^i(x,t,u+\varphi)|_Q^{(\alpha')},\quad |c(x,t,u+\varphi)u_i|_Q^{(\alpha')} \le C_9'(C_1)\left(|u|_Q^{(1+\alpha')} + \langle u\rangle_Q^{(1)}.\langle u\rangle_Q^{(\alpha')}\right)$$

and (see (7))

$$\langle u\rangle_Q^{(1)}.\langle u\rangle_Q^{(\alpha')} \le C_9''(C_1)\left(1+|u|_Q^{(2+\alpha'')}\right) .$$

Finally,

(15) $$|v|_{\mathcal{N}}^{(2+\alpha')} \le C_{11}\left(1+|Lu|_Q^{(2+\alpha'')} + |u|_Q^{(2+\alpha'')}\right)$$

and

(16) $$\langle u\rangle_{Q_p}^{(2+\alpha')} \le C_{12}\left(1+\int_0^p \langle Lu\rangle_{Q_q}^{(2+\alpha')}\,dq + |Lu|_Q^{(2+\alpha'')} + |u|^{(2+\alpha'')}\right) .$$

Our aim is to derive a bound of the form

$$\langle Lu\rangle_{Q_q}^{(2+\alpha')} \le C\langle u\rangle_{Q_{q+\delta}}^{(2+\alpha')} + C'(\delta)|u|_Q^{(2+\alpha'')}$$

with a constant C, independent of δ.

Let $\delta > 0$ and $q \in [0,K]$ be arbitrary numbers. If $Q_{q+\delta} \ne Q_K$, we define $W := \{(x,t) \in \overline{Q} :\ \tilde{d}((x,t),Q_q) < \theta/3\}$ where $\theta = \theta(\delta)$ is the constant from Lemma 3.2. In the case $Q_{q+\delta} \equiv Q_K$ we set $W := Q$.

The function $Lu \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$ solves the first IBVP

(17) $$\begin{cases} \mathcal{P}_0(Lu) = \tilde{\mathcal{P}}_0 & \text{in } Q \\ Lu = -\rho l\varphi & \text{on } S;\ Lu(x,0) = 0\ x \in \overline{\Omega} \end{cases}$$

where $\tilde{\mathcal{P}}_0 u := -\rho[L(a^{ij}\varphi_{ij}) + L^k c_k + c_z L(u+\varphi) + L^k b^i_k \varphi_i + b^i_z \varphi_i L(u+\varphi) + b^i L\varphi_i] - L^k b^i_k u_i - b^i_z L(u+\varphi) - b^i L u_i - u_{ij} L a^{ij} - L^k_i u_k + a^{ij} L^k_{ij} u_k + 2a^{ij} L^k_i u_{kj} \in C^{\alpha,\alpha/2}(\overline{Q})$ and the arguments of b, c and their derivatives are $(x,t,u+\varphi)$. Consequently, applying Prop.3.1 for the function Lu and $U = Q_q$ we obtain

(18) $$\langle Lu\rangle_{Q_q}^{(2+\alpha')} \le C_{13}\langle \tilde{\mathcal{P}}_0 u\rangle_W^{(\alpha')} + C_{14}(\theta)\left(1+|\tilde{\mathcal{P}}_0 u|_{0,Q} + |Lu|_Q^{(2+\alpha'')}\right) .$$

We note that $C_{13} = C_{14}$ if $W \equiv Q$. Carefull calculations, based on the expression of $\tilde{\mathcal{P}}_0 u$, give

(19) $$\begin{aligned} \langle \tilde{\mathcal{P}}_0 u\rangle_W^{(\alpha')} \le\ & C_{15}\Big(1+|u|_Q^{(2+\alpha'')} + [\langle u\rangle_Q^{(1)}]^2 + \langle u\rangle_W^{(2+\alpha')} \\ & +\langle u\rangle_W^{(\alpha')}\langle u\rangle_W^{(2)} + \langle u\rangle_W^{(1)}\langle u\rangle_W^{(1+\alpha')} + [\langle u\rangle_W^{(1)}]^2[\langle u\rangle_W^{(\alpha'/\alpha)}]^\alpha\Big) \end{aligned}$$

and

$$|\tilde{\mathcal{P}}_0 u|_{0,Q} \le C_{16}\left(1+|u|_Q^{(2+\alpha'')} + [\langle u\rangle_Q^{(1)}]^2\right) .$$

Let us consider the case $Q \equiv W$. Now

$$[\langle u\rangle_W^{(1)}]^2[\langle u\rangle_W^{(\alpha'/\alpha)}]^\alpha = [\langle u\rangle_Q^{(1)}]^2[\langle u\rangle_Q^{(\alpha'/\alpha)}]^\alpha \le C_{17}|u|_Q^{(2+\alpha')}$$

and

$$[\langle u\rangle_Q^{(1)}]^2 \le C'_{17}\left(1+|u|_Q^{(2+\alpha'')}\right)$$

by means of the interpolation inequality (7). In the same way we estimate the other products of Hölder seminorms. On the other hand,

$$\langle u\rangle_Q^{(2+\alpha'')} \le \langle u\rangle_{Q_k}^{(2+\alpha'')} + \langle u\rangle_{Q\setminus\Sigma}^{(2+\alpha'')} + C_{18}(\tilde d_0)\sum_{j=0}^{2}\langle u\rangle_Q^{(j)}$$

and

$$|u|_Q^{(2+\alpha'')} \le \langle u\rangle_{Q_k}^{(2+\alpha'')} + C_{19}\left(1+|u|_Q^{(2+\alpha'')}\right)$$

because of (13). Finally, if $Q_{q+\delta}\equiv Q_K$, then the inequality (18) has the form

$$\langle Lu\rangle_{Q_q}^{(2+\alpha')} \le C_{20}\langle u\rangle_{Q+\delta}^{(2+\alpha')} + C_{21}\left(1+|u|_Q^{(2+\alpha'')}+|Lu|_Q^{(2+\alpha'')}\right) \tag{20}$$

and C_{21} does not depend on δ.

If $\delta>0$ and $q\in[0,K]$ are such that $Q_{q+\delta}\not\equiv Q_K$, then we define $\tilde W := \{(x,t)\in\overline{Q} : \tilde d((x,t),Q_q) < 2\theta/3\}$. Let $\xi(x)\in C^\infty(\overline{Q})$ be a cut-off function with the properties: $0\le\xi\le 1$ in $\overline{Q}$, $\xi\equiv 1$ in W, $\xi\equiv 0$ outside $\tilde W$ and $|D_t^r D_x^\beta\xi|\le C(\beta,r,\theta)$ for $r+|\beta|>0$.

We estimate the products of the Hölder seminorms in (19) in a similar way as above. For example,

$$\begin{aligned}\langle u\rangle_W^{(\alpha')}\langle u\rangle_W^{(2)} &= \langle\xi u\rangle_W^{(\alpha')}\langle\xi u\rangle_W^{(2)} \le \langle\xi u\rangle_Q^{(\alpha')}\langle\xi u\rangle_Q^{(2)}\\ &\le C_{22}|\xi u|_Q^{(2+\alpha')}|\xi u|_{0,Q} \le C_{23}\left(\sum_{j=0}^{2}\langle\xi u\rangle_Q^{(j)} + \langle\xi u\rangle_Q^{(2+\alpha')}\right)\\ &\le C_{24}\langle u\rangle_{\tilde W}^{(2+\alpha')} + C_{25}(\theta)\left(1+|u|_Q^{(2+\alpha'')}\right)\\ &\le C_{26}\langle u\rangle_{Q_{q+\delta}}^{(2+\alpha')} + C_{27}(\theta)\left(1+|u|_Q^{(2+\alpha'')}\right)\end{aligned}$$

We have used (13), $\tilde W\cap Q_K\subset Q_{q+\delta}$ and

$$\langle\xi u\rangle_Q^{(2+\alpha')} \le \langle u\rangle_{\tilde W}^{(2+\alpha')} + C_{28}(\theta)\left(1+|u|_Q^{(2+\alpha'')}\right),$$

$$\langle u\rangle_{\tilde W}^{(2+\alpha')} \le \langle u\rangle_{Q_{q+\delta}}^{(2+\alpha')} + \langle u\rangle_{Q\setminus\Sigma}^{(2+\alpha')} + C_{29}(\tilde d_0)\sum_{j=0}^{2}\langle u\rangle(j)_Q .$$

Therefore, if $Q_{q+\delta}\not\equiv Q_K$, then the inequality (18) has the form (20) with new constants, the second one depending on $\theta(\delta)$. The parabolic estimate for (17) and arguments as above give

$$|Lu|_Q^{(2+\alpha'')} \le C_{30}\left(|\tilde{\mathcal P}_0 u|_Q^{(\alpha'')} + |\varphi|_{3+\alpha,\Omega}\right) \le C_{31}\left(1+|u|_Q^{(2+\alpha'')}\right) .$$

Finally, the bounds (16) and (20) yield

$$\langle u\rangle_{Q_p}^{(2+\alpha')} \le C_{32}\int_0^p \langle u\rangle_{Q_{q+\delta}}^{(2+\alpha')}\,dq + C_{33}(\theta)\left(1+|u|_Q^{(2+\alpha'')}\right)$$

where C_{32} does not depend on δ.

This is a Gronwall-type inequality. Bearing in mind Prop.4.1 in [5], we choose $\delta > 0$ so small that $C_{32}.\delta.\exp < 1$. Then

$$\langle u\rangle_{Q_p}^{(2+\alpha')} \leq C_{34}\left(1 + |u|_Q^{(2+\alpha'')}\right)$$

for every $p \in [0, K]$. Therefore,

$$\begin{aligned} |u|_Q^{(2+\alpha')} &= \sum_{j=0}^{2}\langle u\rangle_Q^{(j)} + \langle u\rangle_Q^{(2+\alpha')} \leq |u|_Q^{(2+\alpha'')} + \langle u\rangle_{Q_K}^{(2+\alpha')} \\ &+ \langle u\rangle_{Q\setminus\Sigma}^{(2+\alpha')} + C_{35}(\tilde{d}_0)\sum_{j=0}^{2}\langle u\rangle_Q^{(j)} \leq C_{36}\left(1 + |u|_Q^{(2+\alpha)}\right) \end{aligned}$$

and $\alpha'' < \alpha'$. The desired estimate (9) follows now from the interpolation inequality (7) and from (12).

Returning to the problem (8), we note that the Leray-Schauder theorem asserts that the operator $\mathcal{U}$ has a fixed point $u = \mathcal{U}u$ which is a classical solution of (8).

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