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JACK'S LEMMA FOR A VECTOR VALUED FUNCTION

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ABSTRACT. We have extended Jack's Lemma (well known in the case of complex functions defined in the complex plane) to holomorphic vector valued functions from $\mathbb{C}$ to a Banach space B.

Let $\mathbb{C}$ be the complex plane and E be the open unit disc in $\mathbb{C}$. The following result was proved by I.S.Jack in [2].

Theorem A. *Let* $g: E \to \mathbb{C}$ *be a holomorphic function with* $g(0) = 0$ *and* $g(\xi) \not\equiv 0$. *If*

$$|g(\xi^0)| = \max_{|\xi| \leq |\xi^0|} |g(\xi)|, \ \xi \in E$$

then there exists a real number $m_0 \geq 1$ *such that*

$$\frac{\xi^0 g'(\xi^0)}{g(\xi^0)} = m_0 \tag{1}$$

$$\operatorname{Re} \frac{\xi^0 g''(\xi^0)}{g'(\xi^0)} + 1 \geq m_0 \tag{2}$$

Let us observe that the relations (1) and (2) can be presented in the following equivalent form

$$< g'(\xi^0)\xi 0 \, , \, g(\xi^0) > = m_0 |g(\xi^0)|^2 \tag{3}$$

$$\operatorname{Re} < g''(\xi^0)\xi^{0^2} \, , \, g(\xi^0) > \geq m_0(m_0 - 1) \, |g(\xi^0)|^2 \tag{4}$$

where $< \omega, z > = \omega\bar{z}$ is one-dimensional Euclidean product.

Now let us extend this result for the case of vector valued holomorphic mappings.

Let X be a finite dimensional Banach space and X^* be the dual of X. Given $x \in X$, $x \neq 0$ define $T(X) = \{x^* \in X^* \,/\, x^*(x) = |x| \text{ and } \|x^*\| = 1\}$. Note that $T(X)$ is non-empty.

Let D be a domain in $\mathbb{C}$ and $f: D \to X$ be a holomorphic Banach (vector) valued function. (For details see [1])

Theorem. *Let* $f: E \to X$ *with* $f(0) = 0$. *If* $\|f(z_0)\| = \max_{|z| \leq |z_0|} \|f(z)\|$, $z_0 \in E$ *then there exists a real number* $m_0 > 1$ *such that*

$$<< Df(z_0)z_0 \, , \, f(z_0) >> = m_0 << f(z_0) \, , \, f(z_0) >> \tag{5}$$

where $<< x, y >> = y^* x$, *and*

(6) $$\text{Re} << D^2 f(z_0)(z_0, z_0), f(z_0) >> \ \geq \ m_0(m_0 - 1) << f(z_0), f(z_0) >>$$

Proof. Since for $f(z) = 0$, (5) and (6) are trivially true, we can assume $f(z) \not\equiv 0$. Consider the function

$$g(\xi) = << f(\frac{\xi z_0}{|z_0|}), f(z_0) >> .$$

Now, $g(\xi)$ is a complex valued holomorphic function in E, $g(0) = 0$, $g(\xi) \not\equiv 0$ in $E \setminus \{0\}$.

It is easy to check that from the assumption is follows

$$\max_{|\xi| \leq |\xi^0|} |g(\xi)| = |g(\xi^0)| \quad \text{where } \xi^0 = |z_0|$$
$$= << f(z_0), f(z_0) >> = |f(z_0)| .$$

All hypotheses of theorem A being fulfilled, there exists a real number m_0, $m_0 \geq 1$ such that (3) and (4) hold. Since $g(\xi^0) = << f(z_0), f(z_0) >>$, $g'(\xi^0)\xi^0 = << z_0 f'(z_0), f(z_0) >>$, after substituting in (3) we get

$$< << z_0 f'(z_0), f(z_0) >>, << f(z_0), f(z_0) >> >$$
$$= m_0 < << f(z_0), f(z_0) >>, << f(z_0), f(z_0) >> >$$
$$<< z_0 f'(z_0), f(z_0) >> = m_0 << f(z_0), f(z_0) >> .$$

This gives (5). As $g''(\xi_0)\xi^{0^2} = << D^2 f(z_0)(z_0, z_0), f(z_0) >>$ substituting in (4) we get

$$\text{Re} < << D^2 f(z_0)(z_0, z_0) f(z_0) >>, |f(z_0)| > \ \geq \ m_0(m_0 - 1)|f(z_0)|^2 ,$$
$$\text{Re} << D^2 f(z_0)(z_0, z_0), f(z_0) >> \ \geq \ m_0(m_0 - 1) << f(z_0), f(z_0) >> .$$

This gives (6).

Thus the proof of the theorem is completed.

It will be interesting to study the applications of this result to geometric function theory.

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REFERENCES

[1] E. Hille and R. S. Philips. Functional Analysis and Semigroups. *Amer. Math. Soc. Colloq. Publ.* **31** (1957).

[2] I. S. Jack. Function Starlike and Convex of Order a. *J. London Math. Soc.* **3** (1971) 469-474.

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