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## JACK'S LEMMA FOR A VECTOR VALUED FUNCTION

V. ANBUHELVI AND R. PARVATHAM

ABSTRACT. We have extended Jack's Lemma (well known in the case of complex functions defined in the complex plane) to holomorphic vector valued functions from  $\mathbb{C}$  to a Banach space  $B$ .

Let  $\mathbb{C}$  be the complex plane and  $E$  be the open unit disc in  $\mathbb{C}$ . The following result was proved by I.S. Jack in [2].

**Theorem A.** Let  $g: E \rightarrow \mathbb{C}$  be a holomorphic function with  $g(0) = 0$  and  $g(\xi) \neq 0$ .

If

$$|g(\xi^0)| = \max_{|\xi| \leq |\xi^0|} |g(\xi)|, \quad \xi \in E$$

then there exists a real number  $m_0 \geq 1$  such that

$$(1) \quad \frac{\xi^0 g'(\xi^0)}{g(\xi^0)} = m_0$$

$$(2) \quad \operatorname{Re} \frac{\xi^0 g''(\xi^0)}{g'(\xi^0)} + 1 \geq m_0$$

Let us observe that the relations (1) and (2) can be presented in the following equivalent form

$$(3) \quad \langle g'(\xi^0) \xi^0, g(\xi^0) \rangle = m_0 |g(\xi^0)|^2$$

$$(4) \quad \operatorname{Re} \langle g''(\xi^0) \xi^{02}, g(\xi^0) \rangle \geq m_0(m_0 - 1) |g(\xi^0)|^2$$

where  $\langle \omega, z \rangle = \omega \bar{z}$  is one-dimensional Euclidean product.

Now let us extend this result for the case of vector valued holomorphic mappings.

Let  $X$  be a finite dimensional Banach space and  $X^*$  be the dual of  $X$ . Given  $x \in X$ ,  $x \neq 0$  define  $T(X) = \{x^* \in X^* / x^*(x) = |x| \text{ and } \|x^*\| = 1\}$ . Note that  $T(X)$  is non-empty.

Let  $D$  be a domain in  $\mathbb{C}$  and  $f: D \rightarrow X$  be a holomorphic Banach (vector) valued function. (For details see [1])

**Theorem.** Let  $f: E \rightarrow X$  with  $f(0) = 0$ . If  $\|f(z_0)\| = \max_{|z| \leq |z_0|} \|f(z)\|$ ,  $z_0 \in E$  then there exists a real number  $m_0 > 1$  such that

$$(5) \quad \langle \langle Df(z_0)z_0, f(z_0) \rangle \rangle = m_0 \langle \langle f(z_0), f(z_0) \rangle \rangle$$

where  $\langle\langle x, y \rangle\rangle = y^* x$ , and

$$(6) \quad \operatorname{Re} \langle\langle D^2 f(z_0)(z_0, z_0), f(z_0) \rangle\rangle \geq m_0(m_0 - 1) \langle\langle f(z_0), f(z_0) \rangle\rangle$$

Proof. Since for  $f(z) = 0$ , (5) and (6) are trivially true, we can assume  $f(z) \not\equiv 0$ . Consider the function

$$g(\xi) = \langle\langle f\left(\frac{\xi z_0}{|z_0|}\right), f(z_0) \rangle\rangle.$$

Now,  $g(\xi)$  is a complex valued holomorphic function in  $E$ ,  $g(0) = 0$ ,  $g(\xi) \not\equiv 0$  in  $E \setminus \{0\}$ .

It is easy to check that from the assumption it follows

$$\begin{aligned} \max_{|\xi| \leq |\xi^0|} |g(\xi)| &= |g(\xi^0)| \quad \text{where } \xi^0 = |z_0| \\ &= \langle\langle f(z_0), f(z_0) \rangle\rangle = |f(z_0)|. \end{aligned}$$

All hypotheses of theorem A being fulfilled, there exists a real number  $m_0$ ,  $m_0 \geq 1$  such that (3) and (4) hold. Since  $g(\xi^0) = \langle\langle f(z_0), f(z_0) \rangle\rangle$ ,  $g'(\xi^0)\xi^0 = \langle\langle z_0 f'(z_0), f(z_0) \rangle\rangle$ , after substituting in (3) we get

$$\begin{aligned} &\langle\langle z_0 f'(z_0), f(z_0) \rangle\rangle, \langle\langle f(z_0), f(z_0) \rangle\rangle > \\ &= m_0 \langle\langle f(z_0), f(z_0) \rangle\rangle, \langle\langle f(z_0), f(z_0) \rangle\rangle > \\ &\langle\langle z_0 f'(z_0), f(z_0) \rangle\rangle = m_0 \langle\langle f(z_0), f(z_0) \rangle\rangle. \end{aligned}$$

This gives (5). As  $g''(\xi_0)\xi_0^2 = \langle\langle D^2 f(z_0)(z_0, z_0), f(z_0) \rangle\rangle$  substituting in (4) we get

$$\operatorname{Re} \langle\langle D^2 f(z_0)(z_0, z_0), f(z_0) \rangle\rangle, |f(z_0)| > \geq m_0(m_0 - 1) |f(z_0)|^2,$$

$$\operatorname{Re} \langle\langle D^2 f(z_0)(z_0, z_0), f(z_0) \rangle\rangle \geq m_0(m_0 - 1) \langle\langle f(z_0), f(z_0) \rangle\rangle.$$

This gives (6).

Thus the proof of the theorem is completed.

It will be interesting to study the applications of this result to geometric function theory.

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*Ramanujan Institute for  
Advanced Study in Mathematics,  
University of Madras,  
Madras - 600 005,  
India.*

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