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ON THE BAIRE-CATEGORY METHOD IN DIMENSIONAL THEORY

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ABSTRACT. We prove that some sets of mappings of a topological space X into the space $S(\tau)^N$, where $S(\tau)$ being the standard τ-star-space, are residual in the function space of all continuous mappings of X into $S(\tau)^n$. The results on existence of residual sets in spaces of mapping are applied to constructing a sufficient number of compactifications with reminders of given dimension.

Introduction. The present paper is connected with the results of H. J. Kowalski [7], E. Pol [10], W. Olszewski [9], M. M. Choban and Attia [2,3,4]. In section 1 we define the introductory notions. Sections 2 and 3 generalize some results of E. Pol [10]. In section 4 and 5 the class of almost n-dimensional spaces and the class of almost weakly infinite-dimensional spaces are studied. this Sections are applied in Section 6 to constructing a sufficient number of compactifications with remainders of given dimension. The sections 4, 5 and 6 are a continuation of the paper [3].

1. Notations and definitions.

1.1. All spaces are considered to be collectionwise normal and mappings are continuous. By dimension we understand the covering dimension dim. Below $|X|$ is the cardinality of X, $w(X)$ is the weight of a space X, $d(X) = \sup\{|Y| Y$ is a closed discrete subspace of $X\}$.

Our paper uses the terminonology from [5,6,8].

1.2. Let (Y, ρ) be a metric space and $c > 0$. A family $\mathfrak{L}$ of subsets of Y is c-discrete if $\rho(A, B) = \inf\{\rho(x, y): x \in A, y \in B\} \geq c$ for all distinct A, $B \in \mathfrak{L}$.

1.3. By $\beta f: \beta X \to \beta Y$ we denote the continuous extensions of the mapping $f: X \to Y$ onto the Stone-Čech compactifications βX and βY of the spaces X and Y.

1.4. A subset L of a space X is a first category set if L is a union of countably many nowhere dense sets. A subset H is residual in X if the complement $X \setminus H$ is a first category set.

If X is a Čech complete space, then H is residual in X if only if H contains a dense G_δ-subset of X.

Note that a countable intersection of the residual subsets of a space X is residual in X.

1.5. Fix a cardinal number $\tau \geq \aleph_0$. By $S(\tau)$ we denote the τ-star-space, i.e. the space $S(\tau)$ is obtained by identifying all zeros in the set $\cup\{I_\alpha = [0,1]: \alpha \in A\}$, where $|A| = \tau$, equipped with the complete metric

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in I_\alpha \text{ for } \alpha \in A, \\ x + y, & \text{if } x, y \text{ belong to distinct intervals.} \end{cases}$$

In the countable power $S(\tau)^N = \Pi\{S_i(\tau) = S(\tau): i \in N = \{0, 1, \ldots\}\}$ we fix a complete metric

$$\rho(\{x_i\}, \{y_i\}) = (\sum\{2^{-i} d(x_i, y_i)^2 : i \in N\})^{\frac{1}{2}}.$$

Zero 0 of intervals I_α in $S(\tau)$ is the origin of $S(\tau)$.

Let $K_n(\tau) = \{\{x_i : i \in N\} \in S(\tau)^N$: the set $\{i \in N : x_i$ is a positive rational number$\}$ has at most n elements$\}$ $K_\omega(\tau) = \cup\{K_n(\tau): n \in N\}$, $E(\tau)$ be the Stone–Čech compactification of $S(\tau)^N$. The space $K_0(\tau)$ and the generalized Baire space $B(\tau)$ of weight τ are homeomorphic.

1.6. Fix a space X and a metric space (Y, ρ). By $C(X, Y)$ we denote the space of all continuous mapping of X into Y with the topology of uniform convergence. Let

$$\hat{\rho}(f, g) = \min\{2, \sup\{(f(x), g(x)) : x \in X\}\}.$$

If ρ is a complete metric on Y, then $\hat{\rho}$ is a complete metric on a space $C(X, Y)$.

1.7. The perfect preimages of the metric spaces are called paracompact p–spaces (see [1]).

1.8. let $F: X \to Y$ be a mapping. A mapping $g: X \to Z$ is an F–factorization, if g is continuous and $F = fg$ for some continuous mapping $f: Z \to Y$. If F is a perfect mapping, then the mappings g and f are perfect,too.

1.9. A space X is called of countable type every compact subset of X is included in a compact subset of countable character in X. A space X is of countable type if only if $\beta X \backslash X$ is a Lindelőf space.

1.10. The set C is a partition between A and B in a space X if there exist the open disjoint sets V and W satisfying the conditions $A \subseteq V$, $B \subseteq W$ and $X \backslash C = V \cup W$.

1.11. A space X is said to be weakly infinite–dimensional or A–weakly infinite–dimensional (S–weakly infinite–dimensional) if for every sequence $\{(A_i, B_i): i \in N\}$ of pairs of disjoint closed subset of X there exists a sequence $\{C_i: i \in N\}$ (a sequence $\{C_i, : i \in N\}$ and an integer $n \in N\}$ such that $\cap\{C_i: i \in N\} = \emptyset$ ($\cap\{C_i: i < n\} = \emptyset$), where C_i is a partition between A_i and B_i in X for all $i \in N$.

A space X is said to be strongly infinite-dimensional (S-strongly infinite-dimensional) if it is not weakly infinite-dimensional (S-weakly infinite-dimensional).

1.12. $cl_X h$ or $cl\, H$ denotes the closure of a set H in X.

2. Auxiliary assertions.

Lemma 2.1. *Let Y be a closed subspace of a space X. Then the mapping*

$$r: C(X, S(\tau)^N) \to C(X, S(\tau)^N),$$

where $r(f) = f|Y$, is continuous, open and onto.

Proof. The continuity of r is obvious. For every continuous mapping $f: Y \to X$ there exists some continuous extension on X. Therefore r is onto. Fix $\varepsilon > 0$, $f \in C(X, S(\tau)$ and $g \in C(X, S(\tau)$, where $d(f(y), g(y)) < \varepsilon$ for all $y \in Y$. We put $S_\varepsilon(\tau) = \{x \in S(\tau): d(x, 0) \leq \varepsilon l$, $X_1 = g^{-1}(S(\tau))$ and $Y_1 = Y \cap X_1$. For some $f_1 \in C(X_1, S_\varepsilon(\tau))$ we have $f|Y_1 = f_1|Y_1$. The mapping $f_2: X_2 \to S(\tau)$, where $X_2 = X_1 \cup Y$, $f_2|X_1 = f_1$ and $f_2|Y = f$, is continuous and $d(f_2(x), g(x)) < 2\varepsilon$ for each $x \in X_2$. A set $U = g^{-1}(\{x \in S(\tau): d(x, 0) < \varepsilon\})$ is open in X and $U \subseteq X_2$. The family of sets $\{Y_\alpha = g^{-1}(I_\alpha) \backslash U: \alpha \in A\}$ is closed and discrete in X. For every $\alpha \in A$ there exists a continuous mapping $g_\alpha: Y_\alpha \to I_\alpha$ such that $g_\alpha|Y_\alpha \cap X_2 = f_2|Y_\alpha \cap X_2$ and

$d(g_\alpha(x), g(x)) \leq 4\varepsilon$ for all $x \in Y_\alpha$. Then the mapping $h: S(\tau)$, where $h|X_2 = f_2$ and $h|Y_\alpha = g_\alpha$, is continuous and $d(h, g) \leq 4\varepsilon$. Thus r is an open mapping.

Lemma 2.2. *Let $\gamma = \{U_\mu : \mu \in M\}$ be an open discrete family and $\zeta = \{F_\mu : \mu \in M\}$ be a closed family of the space X, $|M| \leq \tau$ and $U_\mu \supseteq F_\mu \neq \emptyset$ for every $\mu \in M$. Then the set $C(X, S(\tau)^N, \gamma, \zeta) = \{f \in C(X, S(\tau)^N)$: for some $\varepsilon > 0$ the family $f(\gamma) = \{f(U_\mu): \mu \in M\}$ is ε-discrete and $\rho(f(F_\mu), f(X \setminus U_\mu \geq \varepsilon\}$ for all $\mu \in M$ is open and dense in $C(X, S(\tau)^N$.*

Proof. Analogous to the proof of Proposition 3.4 in [10].

Proposition 2.3 ([10], Proposition 3.3). *If X is an n-dimensional space, then for any cardinal number τ the set*

$$C_n(X, S(\tau)^N) = \{f \in C(X, S(\tau)^N): cl\, h(X) \subseteq K_n(\tau)\}$$

is residual in $C(X, S(\tau)^N)$.

3. Main results. Let τ be a cardinal number.

Theorem 3.1. *Let X_1 be a closed n-dimensional subspace of a space X and $F: X \to Y$ be a mapping onto a metric space Y of the weight $w(Y) \leq \tau$. Then the set $C_n(X, S(\tau)^N, X_1, F)$ $= \{h \in C(X, S(\tau)^N): cl\, h(X_1) \subseteq K_n(\tau)$ and h is an F-factorization$\}$ is residual in $C(X, S(\tau)^N)$.*

Proof. Take the discrete and open systems $\{\{W_\mu^m : \mu \in M\} : m \in N\}$ and $\{\{H_\mu^m : \mu \in M\} : m \in N\}$ in Y for which: $F_\mu^m = cl\, H_\mu^m \subseteq W_\mu^m$ for all $m \in N$ and $\mu \in M_m$; for every neighbourhood V of any point $y \in Y$ there exist $m \in N$ and $\mu \in M_m$ for which $y \in H_\mu^m \subseteq W_\mu^m \subseteq V$. Then $|M_m| \leq \tau$ for every $m \in N$. From Lemma 2.2 it follows that the set $L = \cap\{C(X, S(\tau)^N), \gamma_m, \zeta_m): m \in N$, where $\gamma_m = \{F^{-1}W_\mu^m : \mu \in M_m\}$ and $\zeta_m = \{F^{-1}(F_\mu^m): \mu \in M_m\}$, is residual in $C(X, S(\tau)^N)$. Let $g \in L$. We put $h(z) = F(g^{-1}(z))$ for every $z \in Z = g(X)$. The mapping $h: Z \to Y$ is single-valued and $F = hg$. If $x_0 \in X$, $z_0 = g(x_0)$, $m \in N$, $\mu \in M_m$, $F(x_0) = h(z_0) \in H_\mu^m$, $\rho(g(F^{-1}(F_\mu^m)), g(X \setminus F^{-1}W_\mu^m)) = \varepsilon > 0$ and $U = \{z \in Z: \rho(z_0, z) < \varepsilon\}$, then $h(U) =\subseteq W_\mu^m$. Thus h is continuous F-factorization. Lemma 2.1 and Proposition 2.3 imply that the set $r^{-1}(C_n(X_1, S(\tau)^N) = \{h \in C(X, S(\tau)^N): cl\, h(X_1) \subseteq K_n(\tau)\}$ is residual in $C(X, S(\tau)^N)$. Then the set $C_n(X, S(\tau)^N, X_1, F) \supseteq L \cap r^{-1}(C_n(X_1, S(\tau)^N))$ is residual in $C(X, S(\tau)^N)$.

Corollary 3.2. *Let X_1 be a closed n-dimensional subspace of a paracompact p-space X and $d(X) \leq \tau$. Then for every continuous mapping $F: X \to Y$ into a metric space Y the set $P_n(X, S(\tau)^N, X_1, F) = \{h \in C(X, S(\tau)^N, X_1, F): h: X \to h(X)$ is a perfect mapping$\}$ is residual in $C(X, S(\tau)^N)$.*

Corollary 3.3. *Let $F: X \to Y$ be a continuous mapping into a metric space Y of weight τ, $\{X_i : i \in N\}$ be a sequence of closed subspaces of X and $\dim X_i \leq n_i$ for every $i \in N$. Then the set $C(X, S(\tau)^N, \{X_i, n_i : i \in N\}, F) = \{f \in C(X, S(\tau)^N): f$ is an F-factorization and $cl\, f(X_i) \subseteq K_{n_i}(\tau)$ for every $i \in N\}$ is residual in $C(X, S(\tau)^N)$. In particular, if X is a paracompact p-space and $d(X) \leq \tau$, then the set $P(X, S(\tau)^N), \{X_i, n_i : i \in N\}, F) = \{h \in C(X, S(\tau)^N, \{X_i, n_i : i \in N\}, F): h: X \to h(X)$ is a perfect mapping$\}$ is residual in $C(X, S(\tau)^N)$.*

Remark 3.4. If $i \in N$ and $F|X_i$ is a homeographic embedding, then $h|X_i$ is a homeographic embedding for every $h \in C(X, S(\tau)^N, \{X_i, n_i : i \in N\}, F)$.

Corollary 3.5. *Let $Y, X_i : i \in N$ be a sequence of closed subspaces of a metric space X, $w(X) \leq \tau$, $\dim Y = n$ and $\dim X_i = n_i$ for every $i \in N$. Then the sets $E_n(X, S(\tau)^N) =$*

$\{f \in C(X, S(\tau)^N)$: *f is a homeographic embedding and* $cl\, f(Y) \subseteq K_n(\tau)\}$ *and* $E(X, S(\tau)^N, \{X_i, n_i: i \in N\}) = \cap\{E_{n_i}(X, S(\tau)^N, X_i): i \in N\}$ *are residual in* $C(X, S(\tau)^N)$.

4. Almost finite–dimensional spaces. A space X is called almost n–dimensional if there exists a compact subset F of X such that $\dim(X\backslash U) \leq n$ for every open subset U in X containing F.

Lemma 4.1. *Let* Φ *be a compact subset of* βX *and* $\dim \Phi \geq n$. *Then the set*

$$\beta C_n(X, S(\tau)^N, \Phi) = \{f \in C(X, S(\tau)^N): \dim(\beta f(\Phi)) \geq n\}$$

is residual in $C(X, S(\tau)^N)$.

Proof. If $\dim \Phi \leq n$ then there exists a finite sequence of pairs $\{(A_i, B_i): i = 1, 2, \ldots, n\}$ of disjoint closed subsets of Φ such that for all closed partitions C_i in Φ between A_i and B_i we have $\cap\{C_i: i = 1, 2, \ldots, n\} \neq$. In βX there exists a family $\{V_i, W_i, U_i, H_i: i = 1, 2, \ldots n\}$ of open subsets such that $A_i \subseteq V_i \subseteq cl_{\beta X} V_i \subseteq U_i$, $B_i \subseteq W_i \subseteq cl_{\beta X} W_i \subseteq H_i$ and $cl_{\beta X} U_i \cap cl_{\beta X} H_i = \emptyset$ for every $i \leq n$. By Lemma 2.2 the set $L = C(X, S(\tau)^N, \{V_i, W_i, U_i, H_i: i \leq n\} = \{f \in C(X, S(\tau)^N): \rho(f(V_i \cap X), f(X\backslash U_i)) > 0$, $\rho(f(W_i \cap X), f(X\backslash H_i)) > 0$ for all $i \leq n$ is open and dense in $C(X, S(\tau)^N)$. Let $f \in L$. By construction $\beta f(A_i) \cap \beta f(B_i) = \emptyset$ for every $i \leq n$. If C_i is a partition in $\beta f(\Phi)$ between $\beta f(A_i)$ and $\beta f(B_i)$, then $\beta f^{-1}(C_i)$ is a partition in Φ between A_i and B_i. Hence $\dim(\beta f(\Phi) \geq n$ and $L \subseteq \beta C_n(X, S(\tau)^N)$.

Corollary 4.2. *Let* Φ *be a compact subset of* βX *and* $\dim \Phi = \infty$. *Then the set* $\beta C_\infty(X, S(\tau)^N, \Phi) = \{f \in C(X, S(\tau)^N): \dim(\beta f(\Phi)) = \infty\} = \cap\{\beta C_n(X, S(\tau)^N, \Phi): n \in N\}$ *is residual in* $C(X, S(\tau)^N)$.

Corollary 4.3. *Let* Φ *be a strongly infinite–dimensional compact subset of* βX. *Then the set* $\beta C_{sid}(X, S(\tau)^N, \Phi) = \{f \in C(X, S(\tau)^N): \beta f(\Phi)$ *strongly infinite–dimensional* $\}$ *is residual in* $C(X, S(\tau)^N)$.

Proof. Fix in Φ a sequence $\{(A_i, B_i): i \in N\}$ of pairs of disjoint closed subset that $\cap\{C_i: i \in N\} \neq \emptyset$, where C_i is a partition between A_i and B_i in Φ. There exists a family of open subsets $\{V_i, W_i, U_i, H_i: i \in N\}$ of βX such that $A_i \subseteq V_i \subseteq cl_{\beta X} V_i \subseteq U_i$, $B_i \subseteq W_i \subseteq cl_{\beta X} W_i \subseteq H_i$ and $cl_{\beta X} U_i \cap cl_{\beta X} H_i = \emptyset$ for every $i \in N$. Then $\cap\{\cap\{C(X, S(\tau)^N, \{V_i, W_i, U_i, H_i: i \leq n\}: n \in N\}$ is a residual subset of $C(X, S(\tau)^N)$ and $\beta C_{sid}(X, S(\tau)^N, \Phi)$.

Theorem 4.4. *Let* X *be an almost* n*–dimensional space of countable type. Then the sets*

$$\begin{aligned} \beta C_n(X, S(\tau)^N) &= \{f \in C(X, S(\tau)^N): \dim(\beta f(\beta X\backslash X)) \leq n\} \\ KC_n(X, S(\tau)^N) &= \{f \in C(X, S(\tau)^N): cl_{E(\tau)}((cl\, f(X))\backslash K_n(\tau)) \subseteq f(X)\} \end{aligned}$$

are residual in $C(X, S(\tau)^N)$ *and* $KC_n(X, S(\tau)^N) \subseteq \beta C_n(X, S(\tau)^N)$.

Proof. There exist a compact subset $\Phi \subseteq X$ and a sequence $\{U_n: n \in N\}$ of open subsets in X such that for every open in X subset $U \supseteq \Phi$ there exists $m \in N$ such that $U_m \subseteq U$; $\Phi \subseteq U_m$ and $\dim X\backslash U_m \leq n$ for every $m \in N$. Let $X_m = X\backslash U_m$ and $n_m = n$. From Lemma 2.2 and Theorem 3.1 the set

$$L = \{f \in C(X, S(\tau)^N): cl\, f(X_m) \subseteq K_n(\tau) \text{ and } \rho(f(\Phi), f(X_m)) > 0\}.$$

is residual in $C(X, S(\tau)^N)$. Suppose that $f \in L$ and $Y = cl\, f(X)$. By construction, $Y \subseteq \Phi \cup K_n(\tau)$, $\dim Y_m \leq n$, where $Y_m = cl_{E(\tau)} f(X_m)$, $\beta f(\beta X\backslash X) \subseteq \cup\{Y_m: m \in N\}$ and $\dim(\beta f(\beta X\backslash X)) \leq n$. Therefore $L \subseteq KC_n(X, S(\tau)^N)$. If $f \in KC_n(X, S(\tau)^N)$, then $\dim Y_m \leq$

n and $\beta f(\beta X\backslash X) \subseteq \cup\{Y_m : m \in N\}$. The space $\beta X\backslash X$ is Lindelőf and the set $Z_m = Y_m \cap \beta f(\beta X\backslash X)$ is closed in $\beta f \beta X\backslash X)$. Hence $\dim Z_m \leq n$, $\dim \beta f(\beta X\backslash X) \leq n$ and $f \in \beta C_n(X, S(\tau)^N)$. The proof is complete.

Theorem 4.5. *For a paracompact space X of countable type the following assertions are equivalent:*

1. X is almost n-dimensional.

2. The set $\beta C_n(X, S(\tau)^N)$ is residual in $C(X, S(\tau)^N)$.

3. The set $KC_n(X, S(\tau)^N)$ is residual in $C(X, S(\tau)^N)$.

4. The set $\beta C_n(X, S(\tau)^N)$ is dense in $C(X, S(\tau)^N)$.

5. The set $KC_n(X, S(\tau)^N)$ is dense in $C(X, S(\tau)^N)$.

Proof. Implications $1 \to 3 \to 2$ and $1 \to 5 \to 4$ follow from Theorem 4.4. Implications $2 \to 4$ and $3 \to 5$ are obvious.

Suppose that $\beta C_n(X, S(\tau)^N)$ is a dense subset. Fix some compact subset Φ of $\beta\backslash X$ and the pairs $\{(A_i, B_i)\}: i = 0, 1, \ldots, n$ of disjoint closed sets in Φ. There exist the closed subsets $\{H_i, F_i : i \leq n\}$ of a space βX such that $A_i \subseteq \text{int}H_i$, $B_i \subseteq \text{int}F_i$ and $H_i \cap F_i = \emptyset$ for every $i \leq n$. Fix the continuous mapping $f_i : X \to I_\alpha \subseteq S(\tau)$ for which: $H_i \cap X \subseteq f_i^{-1}(0)$ and $F_i \cap X \subseteq f_i^{-1}(1)$ for every $i \leq n$ and $f_j(X) = 0$ for every $j > n$. Consider the mapping $f: X \to S(\tau)^N$, where $f(X) = \{f_i(x): i \in N\}$. By construction $\rho(f(H_i \cap X), f(F_i \cap X)) > 2^{-n}$ for every $i \leq n$. There exists a mapping $g \in \beta C_n(X, S(\tau)^N)$ such that $\hat{\rho}(f, g) < 2^{-4n}$. By construction, $\rho(g(H_i \cap X), g(F_i \cap X)) > 2^{-4n}$. Hence $\beta g(H_i) \cap \beta g(F_i) = \emptyset$ and $\beta g(A_i) \cap \beta g(B_i) = \emptyset$ for every $i \leq n$. Since $\dim(\beta g(\Phi)) \leq n$, then there exist the closed subsets $\{C_i : i \leq n\}$ such that $\cap\{C_i : i \leq n\} = \emptyset$ and C_i is a partition between $\beta g(A_i)$ and $\beta g(B_i)$ in $\beta g(\Phi)$. Therefore $P_i = \Phi \cap \beta g^{-1}(C_i)$ is a partition between A_i and B_i in Φ and $\cap\{P_i : i \leq n\}$. Hence $\dim \Phi \leq n$. By Corollary 2.6 [3] the paracompact space X of countable type is almost n-dimensional if only if $\dim \Phi \leq n$ for every compact subset Φ of $\beta X\backslash X$. This proves the implication $4 \to 1$. The proof is complete.

Corollary 4.6. *For a paracompact p-space X the following statements are equivalent:*

1. X is almost n-dimensional and $d(X) \leq \tau$.

2. The set $\beta C_n(X, S(\tau)^N)$ is residual in $C(X, S(\tau)^N)$ and $d(X) \leq \tau$.

3. The set $\beta P_n(X, S(\tau)^N) = \{f \in \beta C_n(X, S(\tau)^N): f: X \to f(X)$ perfect$\}$ is residual in $C(X, S(\tau)^N)$.

4. The set $KP_n(X, S(\tau)^N) = \{f \in KC_n(X, S(\tau)^N): f: X \to f(X)$ perfect$\}$ is residual in $C(X, S(\tau)^N)$.

Corollary 4.7. *For a metrizable space X the following statements are equivalent:*

1. X is almost n-dimensional and $w(X) \leq \tau$.

2. The set $\beta E_n(X, S(\tau)^N) = \{f \in \beta C_n(X, S(\tau)^N): f$ homeomorphic embedding$\}$ is residual in $C(X, S(\tau)^N)$.

3. The set $KE_n(X, S(\tau)^N) = \{f \in KC_n(X, S(\tau)^N): f$ homeomorphic embedding$\}$ is residual in $C(X, S(\tau)^N)$.

4. $KE_n(X, S(\tau)^N) \neq \emptyset$.

Proof. Implications $1 \to 2 \to 3 \to 1$ follow from Corollaries 3.5 and 4.5. Implications $2 \to 4$ is obvious. Let $f \in KE_n(X, S(\tau)^N)$. Then $\Phi = cl(f(X)\backslash K_n(\tau))$ is a compact subset of $f(X)$. Therefore $\dim(f(X)\backslash\Phi) \leq n$ and $X = f(X)$ is almost n-dimensional. The implication $4 \to 1$ is proved.

Theorem 4.8. *For a complete metric space X the following assertions are equivalent:*

1. X is almost n-dimensional space of weight $\leq \tau$.

2. The set $c\beta E_n(X, S(\tau)^N) = \{f \in \beta E_n(X, S(\tau)^N): f(X)$ closed subspace of $S(\tau)^N\}$ is residual in $C(X, S(\tau)^N)$.

3. The set $cKE_n(X, S(\tau)^N) = \{f \in KE_n(X, S(\tau)^N): f(X)$ closed subspace of $S(\tau)^N\}$ is residual in $C(X, S(\tau)^N)$.

4. $c\beta E_n(X, S(\tau)^N) \cup cKE_n(X, S(\tau)^N) \neq \emptyset$.

Proof. Let X be an almost n-dimensional complete metric space of weight $\leq \tau$. H. Toruńczuk [11] has proved that the set $cE(X, S(\tau)^N) = \{f \in C(X, S(\tau)^N): f$ is a closed embedding$\}$ is residual in $C(X, S(\tau)^N)$. Hence in virtue of Corollary 4.7, the implications $1 \to 2$ and $1 \to 3$ are proved. The implications $2 \to 4$ and $3 \to 4$ are obvious. Let $f \in c\beta E_n(X, S(\tau)^N)$. Then $\beta f: \beta X \to E(\tau)$ is an embedding and $\dim(\beta X \backslash X) = \dim \beta f(\beta X \backslash X) \leq n$. By Corollary 2.6 [3] the space X is almost n-dimensional. This proves the implication $4 \to 1$.

5. Almost weakly infinite-dimensional spaces. A space X is almost weakly infinite-dimensional (a.w.i.-d.) if there exists a compact subset Φ of X such that the subspace $X \backslash U$ is finite dimensional for every open set U in X containing Φ.

Theorem 5.1. *Let X be an a.w.i.-d. space of countable type. Then the sets $\beta C_{lfd}(X, S(\tau)^N) = \{f \in S(X, S(\tau)^N): \beta f(\beta X \backslash X)$ locally finite dimensional$\}$ and $H(f) = cl_{E(\tau)}((cl\, f(X)) \backslash K_\omega(\tau)) \subseteq f(X)\}$: $H(f) = cl_{E(\tau)}((cl f(X)) \setminus K_\omega(\tau)) \subseteq f(X)\}$ are residual in $C(X, S(\tau)^N)$ and $KC_{lfd}(X, S(\tau)^N) \subseteq \beta C_{lfd}(X, S(\tau)^N)$.*

Proof. Let $f \in KC_{lfd}(X, S(\tau)^N)$. Then $H = H(f)$ is a compact subset of $f(X)$ and $H \cap \beta f(\beta X \backslash X) \neq \emptyset$. There exists a sequence of open sets $\{U_i: n \in N\}$ in $E(\tau)$ such that $V_n = \{x \in S(\tau)^N: \rho(x, H) < 2^{-n}\} \subseteq U_{n+1} \subseteq cl\, U_{n+1} \subseteq U_n$ for every $n \in N$. Let $Y_n = \beta f(\beta X) \backslash U_n$ and $Z_n = Y_n \cap S(\tau)^N$. If $n \in N$, $x_m \in K_m(\tau) \backslash Z_n$, then $cl_{E(\tau)}\{x_m: m \in N\} \backslash f(X) \neq \emptyset$. Hence $Z_n \subseteq K_m(\tau)$ for some $m \in N$. We consider that $Z_n \subseteq K_n(\tau)$ for every $n \in N$. By construction $Y_n = \beta Z_n$, $\dim Y_n \leq n$, $W_n = E(\tau) \backslash cl\, U_n \subseteq Y_n$ and $\beta f(\beta X \backslash X) \subseteq \cup\{W_n: n \in N\}$. Therefore $\beta f(\beta X \backslash X$ is locally infinite-dimensional. The inclusion $KC_{lfd}(X, S(\tau)^N) \subseteq \beta C_{lfd}(X, S(\tau)^N)$ is proved.

Let X be an a.w.i.-d. space. There exists a nonempty compact subset Φ of X and a sequence $\{U_n: n \in N\}$ of open subsets in X such that: $\Phi = \cap\{U_n: n \in N\}$; $\dim(X \backslash U_i) = n_i$ and $cl\, U_{i+1} \subseteq U_i$ for every $i \in N$; for every open in X subset $U \supseteq \Phi$ there exists $m \in N$ such that $U_m \subseteq U$. We put $X_i = X \backslash U_i$. Then the set $L = \{f \in C(X, S(\tau)^N, \{X_i, n_i: i \in N\}: cl\, f(X_m) \subseteq K_{n_m}(\tau), \rho(f(\Phi), f(X \backslash U_m) > 0 forall m \in N\}$ is residual. Fix $f \in L$. Denote $Y_n = cl_{E(\tau)} f(X_n)$. Then $g = \beta f: \beta X \to E(\tau)$ is a perfect mapping and $Y_n \cap f(\Phi) = 0$ for every $n \in N$. By construction $Y_m \cap S(\tau)^N \subseteq K_{n_m}(\tau)$ and $g(\beta X) = (\cup\{Y_n: n \in N\}) \cup f(\Phi)$. Hence $H(f) \subseteq f(\Phi) \subseteq f(X)$ and the set $KC_\omega(X, S(\tau)^N)$ is residual.The theorem is proved.

Theorem 5.2. *For a paracompact space X of countable type the following assertions are equivalent:*

1. X is almost weakly infinite-dimensional.

2. The set $KC_{lfd}(X, S(\tau)^N)$ is residual in $C(X, S(\tau)^N)$.

3. The set $\beta C_{lfd}(X, S(\tau)^N)$ is residual in $C(X, S(\tau)^N)$.

4. The set $\beta C_{wid}(X, S(\tau)^N) = \{f \in C(X, S(\tau)^N): \beta f(\beta X \backslash X)$ A-weakly infinite dimensional$\}$ is residual in $C(X, S(\tau)^N)$.

Proof. Implications $1 \to 2 \to 3$ follow from Theorem 5.1. Implications $3 \to 4$ is obvious. Let $\beta C_{wid}(X, S(\tau)^N)$ be a residual. If X is not almost weakly infinite-dimensional, then from Theorem 3.6 [3] there exists a strongly infinite-dimensional compact subset Φ of

$\beta X\setminus X$. By Corollary 4.3 the set $L = \beta C_{sid}(X, S(\tau)^N, \Phi)$ is residual. Hence the set $M = L \cap \beta C_{wid}(X, S(\tau)^N)$ is residual. Let $f \in M$. Then $\beta f(\Phi)$ is strongly infinite-dimensional, $\beta f(\beta X\setminus X)$ is weakly infinite-dimensional and $\beta f(\Phi) \subseteq \beta f(\beta X\setminus X)$. This construction complete the proof.

Corollary 5.3. *For a paracompact p-space X the following statements are equivalent:*

1. X is almost weakly infinite-dimensional and $d(X) \leq \tau$.

2. $\beta P_{lfd}(X, S(\tau)^N) = \{f \in \beta C_{lfd}(X, S(\tau)^N): f: X \to f(X)$ perfect$\}$ is residual in $C(X, S(\tau)^N)$.

3. $\beta P_{wid}(X, S(\tau)^N) = \{f \in \beta C_{wid}(X, S(\tau)^N): f: X \to f(X)$ perfect$\}$ is residual in $C(X, S(\tau)^N)$.

4. $KP_{lfd}(X, S(\tau)^N) = \{f \in KC_{lfd}(X, S(\tau)^N): f: X \to f(X)$ perfect$\}$ is residual in $C(X, S(\tau)^N)$.

Corollary 5.4. *For a metric space X the following statements are equivalent:*

1. X is almost weakly infinite-dimensional and $w(X) \leq \tau$.

2. $\beta E_{lfd}(X, S(\tau)^N) = \{f \in \beta C_{lfd}(X, S(\tau)^N): f$ embedding$\}$ is residual in $C(X, S(\tau)^N)$.

3. $\beta E_{wid}(X, S(\tau)^N) = \{f \in \beta C_{wid}(X, S(\tau)^N): f$ embedding$\}$ is residual in $C(X, S(\tau)^N)$.

4. $KE_{lfd}(X, S(\tau)^N) = \{f \in KC_{lfd}(X, S(\tau)^N): f$ embedding$\}$ is residual in $C(X, S(\tau)^N)$.

Theorem 5.5. *For every complete metric space X the following statements are equivalent:*

1. X is almost weakly infinite-dimensional and $w(X) \leq \tau$.

2. $c\beta E_{lfd}(X, S(\tau)^N) = \{f \in \beta E_{lfd}(X, S(\tau)^N): f$ closed embedding$\}$ is residual in $C(X, S(\tau)^N)$.

3. $c\beta E_{wid}(X, S(\tau)^N) = \{f \in \beta E_{wid}(X, S(\tau)^N): f$ closed embedding$\}$ is residual in $C(X, S(\tau)^N)$.

4. $cKE_{lfd}(X, S(\tau)^N) = \{f \in KE_{lfd}(X, S(\tau)^N): f$ closed embedding$\}$ is residual in $C(X, S(\tau)^N)$.

Proof. From Toruńczuk's theorem [11] and Corollary 5.4 follows implication 1 → 4. Implication 4 → 2 follows from Theorem 5.1. Implication 2 → 3 is obvious. Implication 3 → 1 follows from Corollary 5.4. The proof is complete.

Example 5.6. Fix a space X. Denote $S_n(\tau) = \{\{x_i: i \in N\} \in S(\tau)^N: x_i = 0$ for each $i > n\}$ and $E_n(\tau) = cl_{E(\tau)}S_n(\tau)$. Then $\dim E_n(\tau) = \dim S_n(\tau) = n$. The set $C_{bd}(X, S(\tau)^N) = \{f \in C(X, S(\tau)^N): f(X) \subseteq E_n(\tau)$ for some $n\}$ is dense in $C(X, S(\tau)^N)$ and $C_{bd}(X, S(\tau)^N) \subseteq KC_{lfd}(X, S(\tau)^N)$. If $X = R^N$ (or X is not almost weakly infinite-dimensional), then the set $\beta C_{wid}(X, S(\tau)^N)$ is dense and not residual in $C(X, S(\tau)^N)$.

6. On the extensions of metric spaces. Consider the space X and the extensions e_1X and e_2X. The symbol $e_1X > e_2X$ means that there exists a continuous mapping $\pi: e_1X \to e_2X$ such that $\pi(x) = x$ for every $x \in X$.

Let $\mathcal{L} = \{]_\in \mathcal{X}: \alpha \in \mathcal{M}\}$ be a family of extensions of a space X and M be a direct set. The family is directed if: $\beta X = \sup\{e_\mu X: \mu \in M\}$ and for every $\alpha, \mu \in M$, where $\alpha > \mu$, we have $e_\alpha X > e_\mu X$. The family $\mathcal{L}$ is complete if $\mathcal{L}$ is directed, every sequence of bounded continuous functions $\{f_n: X \to R: n \in N\}$ is continuously extendable over some $eX \in \mathcal{L}$ and for every countable subset A of M there exists an element $\mu \in M$ such that $\mu > \alpha$ for every $\alpha \in A$ (see [3]).

Let X be a metric space of weight τ and $f \in E(X, S(\tau)^N) = \{f \in C(X, S(\tau)^N): f$ is an embedding$\}$. Then $b_f X = cl_{E(\tau)} f(X)$ is a compactification of $X, m_f X = S(\tau)^N \cap b_f X$ is a complete metric extension of X.

Proposition 6.1. *Let $f \in E(X, S(\tau)^N)$.*

1. If $f \in \beta C_n(X, S(\tau)^N)$, then $\dim(m_f X \setminus X) \leq n$ and $\dim(b_f X \setminus X \leq n$.

2. If $f \in KC_n(X, S(\tau)^N)$, then $m_f X$ is almost n-dimensional.

3. If $f \in \beta C_{lfd}(X, S(\tau)^N)$, then $m_f X \setminus X$ and $b_f X \setminus X$ are locally finite-dimensional.

4. If $f \in KC_{lfd}(X, S(\tau)^N)$, then $m_f X$ is a.w.i.-d.

5. If $f \in \beta C_{wid}(X, S(\tau)^N)$, then $m_f X \setminus X$ and $b_f X \setminus X$ are A-weakly infinite-dimensional.

6. $b_f X = \beta m_f X$.

7. If f is a closed embedding, then $b_f X = \beta X$.

Proof is obvious.

Proposition 6.2. *Let X be a subspace of a metric space Y, Φ be a compact subset of X. Fix a sequence $\{U_n: n \in N\}$ of open sets in Y such that $\Phi = \cap\{U_n: n \in N\}$ and for every open in Y subset $U \supseteq \Phi$ there exists $n \in N$ such that $U_n \subseteq U$. There exists a G_δ-set Z in Y such that:*

1. $X \subseteq Z$ and $\dim(X \setminus U_n) = \dim(Z \setminus U_n)$ for every $n \in N$.

2. If X is almost n-dimensional, then Z is almost n-dimensional too.

3. If X is almost weakly infinite-dimensional, then Z is almost weakly infinite-dimensional too.

Proof. We consider that X is a dense in Y. In virtue of the enlargement theorem ([5], Theorem 4.1.19), for every $n \in N$ there exists an F_σ-subset H_n in $Y_n = Y \setminus U_n$ such that $X_n = X \setminus U_n \subseteq G_n = Y_n \setminus H_n \subseteq Y_n$ and $\dim X_n = \dim G_n$. Let $Z = Y \setminus \cup \{H_n: n \in N\}$ and $Z_n = Z \setminus U_n$. Then $Z \subseteq Z$ and $X_n \subseteq Z_n \subseteq G_n$. The proof is complete.

Theorem 6.3. *Let X be an almost n-dimensional space of weight $\leq \tau$. Then the families of extensions $M_n(X) = \{m_f X: f \in KE_n(X, S(\tau)^N)\}$ and $B_n(X) = \{b_f X: f \in \beta E_n(X, S(\tau)^N)\}$ are complete.*

Proof. Let $\{m_i X: i \in N\}$ be a sequence of complete metric extensions of X. Consider that $X = \{\{x, x, \ldots, x, \ldots\}: x \in X\} \subseteq Y = \Phi\{m_i X: i \in N\}$. From Proposition 6.2 there exists an almost n-dimensional G_δ-subset Z in Y such that $X \subseteq Z$. By the theorem 4.8 $cKC_n(Z, S(\tau)^N) \neq \emptyset$. Let $g \in cKE_n(Z, S(\tau)^N)$ and $f = g|X$. Then $f \in KE_n(X, S(\tau)^N)$, $m_f X = Z$, $m_f X > m_i X$ and $b_f X > \beta m_i X$ for every $i \in N$. The proof is complete. By Theorem 5.4 in the same way is proved the following fact.

Theorem 6.4. *Let X be an almost weakly infinite-dimensional metric space of weight $\leq \tau$. Then the sets of extensions $M_{lfd}(X) = \{m_f X: f \in KE_{lfd}(X, S(\tau)^N)\}$, $B_{wid}(X) = \{b_f X: f \in \beta E_{wid}(X, S(\tau)^N)\}$ and $B_{lfd}(X) = \{b_f X: f \in \beta E_{lfd}(X, S(\tau)^N)\}$ are complete.*

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