# Serdica

## Bulgariacae mathematicae publicationes

# Сердика

## Българско математическо списание

*SERDICA* — Bulgaricae mathematicae publicationes
17 (1991) 264-269

# MONOTONE CONVERGENCE OF AN ITERATION PROCESS OF NEWTON'S TYPE

G. L. ILIEV

ABSTRACT. Theorems for monotone and quadratic convergence for an iteration method of Newton's type are proved.

We denote by $R^m$ the set of $m$ -dimensional vectors where $m$ belongs to the set $\mathbb{N}$ of all integers. If $\tau \in R^m$ then $\tau = (\tau_1, \ldots, \tau_m)$ , where $\tau_i \in R^1$ are real numbers, $i = 1, 2, \ldots, m$. We denote by $\mathbf{E}^m$ the open unit cube in $R^m$, i.e.: $\mathbf{E}^m = \{\tau : \tau = (\tau_1, \ldots, \tau_m) \in R^m \ , 0 < \tau_i < 1 \ , i = 1, 2, \ldots, m\}$.

Let $\mathbf{F}$ be a map $R^m \longrightarrow R^m$; $\mathbf{F} = (f_1, \ldots, f_m)$ ; where $f_i$ are functions of $\tau = (\tau_1, \ldots, \tau_m) \in R^m$, $i = 1, \ldots, m$, at least twice continuously differentiable in $\bar{\mathbf{E}}^m$ and where $\bar{\mathbf{E}}^m$ denote the closing of $\mathbf{E}^m$.

We assume the following properties for the set of functions $\mathbf{F} = \{f_i\}_{i=1}^m$:

($i$) The system of equations $f_i(\tau) = 0, \quad i = 1, \ldots, m$ has a unique solution in $\mathbf{E}^m$ which we denote by: $\tau^* := (\tau_1^*, \ldots, \tau_m^*) \in \mathbf{E}^m$

($ii$) If $f_i(\tau) \geq 0$ for $\tau = (\tau_1, \ldots, \tau_m) \in \bar{\mathbf{E}}^m$ then:

$$\frac{\partial^2 f_i(\tau)}{\partial \tau_i^2} > 0 \text{ and } \frac{\partial f_i(\tau)}{\partial \tau_i} > 0.$$

Let $\mathbf{E}_+^m$ denote those points $\tau$ in $\bar{\mathbf{E}}^m$ for which $f_i(\tau) \geq 0$, $i = 1, \ldots, m$. It is obvious that $\mathbf{E}_+^m$ is non-empty because from *(i)* it follows that at least $\tau^* \in \mathbf{E}_+^m$. For every set of functions satisfying *(i)* and *(ii)* we can define the following subset of $\bar{\mathbf{E}}^m$:

$$\mathbf{S}^m = \{\tau = (\tau_1, \ldots, \tau_m) : \ \tau_j \in [\tau_j^*, 1]; \ j = 1, \ldots, m\}.$$

We shall use also the following $m$ sets:

$$\mathbf{S}_i^{m-1} = \{\tau = (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_m) : \tau_j \in [\tau_j^*, 1]; j = 1, \ldots, m; j \neq i\}; i = 1, \ldots, m.$$

The set $\mathbf{E}_{++}^m := \mathbf{E}_+^m \cap \mathbf{S}^m \neq \oslash$ because at least the point $\tau^*$ belongs to it.

($iii$) For any fixed point $\tau \in \mathbf{S}_i^{m-1}$ the function $f_i(\tau_1, \ldots, \tau_{i-1}, *, \tau_{i+1}, \ldots, \tau_m)$ has a zero which is positive and less or equal then 1. This zero must be unique for $\tau \in \mathbf{S}_i^{m-1}$ as it follows from ($ii$) and we denote it by $\otimes_i(\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_m)$.

$(iv)$ The functions $\otimes_i(\tau), \tau \in \mathbf{S}_i^{m-1}$, $i = 1, \ldots, m$, have the properties that $\otimes_i$ is a monotone increasing function in $[\tau_j^*, 1]$ of the $j$-th variable $j = 1, \ldots, m$; $j \neq i$.

**1. Definition.** *We call the set of functions* $\mathbf{F} = \{f_i\}_{i=1}^m$ *a system of strictly convex of the i-th argument monotone tending to* $\tau^*$ *functions if the properties* $(i)$, $(ii)$, $(iii)$ and $(iv)$ *hold for* $\mathbf{F} = \{f_i\}_{i=1}^m$.

For a system of tending functions $\mathbf{F} = \{f_i\}_{i=1}^m$, the problem arises to find a numerical method determining the unique in $\mathbf{E}^m$ zero of the equations:

$$\mathbf{F}(\tau) = 0, \quad \mathbf{F} = (f_1, f_2, \ldots, f_m) \in \mathbf{M}^m. \tag{1}$$

The multivariant Newton's method (N.M.) for solving (1) is described by the iteration formula:

$$\tau^{(k+1)} = \tau^{(k)} - [\mathbf{F}'(\tau^{(k)})]^{-1}\mathbf{F}(\tau^{(k)}); \quad k = 0, 1, 2, \ldots \tag{2}$$

where $\mathbf{F}'$ denotes the matrix representation of the Gataux's derivative (the Jacobian) of the map $\mathbf{F}$ in the point $\tau^{(k)}$ to the solution of (1) (cf. [1]).

We shall consider a strong simplification of (2) which we call - **simple Newton's method (S.N.M.)** and define by the following iteration formula :

$$\tau_i^{(k+1)} = \tau_i^{(k)} - \frac{f_i(\tau^{(k)})}{\frac{\partial f_i(\tau^{(k)})}{\partial \tau_i}}, \quad i = 1, \ldots, m; \ k = 0, 1, 2, \ldots. \tag{3}$$

Here we have replaced the converse matrix of the Jacobian in (2) only by the converse of a matrix which consists of the main diagonal of the Jacobian.

The S.N.M. has been only mentioned in [1] referring to [2]. The method in [1] is called – one -step Jacoby-Newton's method.

**Theorem 1.** *If* $\mathbf{F} = \{f_i\}_{i=1}^m$ *is a system of tending functions then the S.N.M. (3) converges to* $\tau^*$ *for every starting point* $\tau^{(0)} \in \mathbf{E}_{++}^m$ *and the convergence is monotone, i.e.:*

$$0 < \tau_i^* < \tau_i^{(k+1)} < \tau_i^{(k)} < 1.$$

Proof. We shall prove the theorem by induction. For this purpose it is sufficient, as it is clear from below, to prove that if $\tau^{(0)} \in \mathbf{E}_{++}^m$ then $\tau^{(1)} \in \mathbf{E}_{++}^m$.

From $\tau^{(0)} \in \mathbf{E}_{++}^m$ it follows that $f_i(\tau^{(0)}) \geq 0$, $i = 1, \ldots, m$. From this the condition $(ii)$ and the definition of the S.N.M. it follows:

$$0 < \tau_i^{(1)} \leq \tau_i^{(0)} \leq 1 \tag{4}$$

and also:

$$\tau_i^{(1)} \geq \otimes_i(\tau_1^{(0)}, \ldots, \tau_{i-1}^{(0)}, \tau_{i+1}^{(0)}, \ldots, \tau_m^{(0)}). \tag{5}$$

If $\tau^{(0)} \in \mathbf{E}_{++}^m$ then $\tau^{(0)} \in \mathbf{S}^m$ and then $\tau_j^* \leq \tau_j^{(0)} \leq 1$, $j = 1, \ldots, m$; $j \neq i$, holds. From this and the property $(iv)$ it follows:

$$\otimes_i (\tau_1^{(0)}, \ldots, \tau_{i-1}^{(0)}, \tau_{i+1}^{(0)}, \ldots, \tau_m^{(0)}) \geq \otimes_i(\tau_1^{(*)}, \ldots, \tau_{i-1}^{(*)}, \tau_{i+1}^{(*)}, \ldots, \tau_m^{(*)}) = \tau_i^* , \tag{6}$$

From (4), (5) and (6) we have:

$$1 \geq \tau_i^{(1)} \geq \tau_i^* > 0. \tag{7}$$

From the condition $f_i(\tau^{(0)}) \geq 0,\ i = 1, \ldots, m$; the property $(ii)$ and the definition of the S.N.M. we obtain: $f_i(\tau_1^{(0)}, \ldots, \tau_{i-1}^{(0)}, \tau_i^{(1)}, \tau_{i+1}^{(0)}, \ldots, \tau_m^{(0)}) \geq 0$.

Then from $(iv)$, (4) and (7) it follows:

$$f_i(\tau_1^{(1)}, \ldots, \tau_{i-1}^{(1)}, \tau_i^{(1)}, \tau_{i+1}^{(1)}, \ldots, \tau_m^{(1)}) \geq 0. \tag{8}$$

From (7) and (8) it follows that $\tau^{(1)} \in \mathbf{S}^m, \tau^{(1)} \in \mathbf{e}_+^m$, i.e. $\tau^{(1)} \in \mathbf{E}_{++}^m$, which completes the proof.

It is easy to see that, under the conditions and notations in theorem 1

$$\| \tau^{(k+1)} - \tau^* \| < c(\mathbf{F}) \| \tau^{(k)} - \tau^* \|, \quad k = 1, 2, 3, \ldots,$$

where $c(\mathbf{F})$ denotes a positive constant, depending only on the function $\mathbf{F} = \{f_i\}_{i=1}^m$ and $\| \cdot \|$ stands for any norm in the finite-dimensional space $\mathbf{R}^m$. Such a convergence of the vectors $\tau^{(k)}$ to the vector $\tau^*$ is called linear convergence. The above statement follows from the Taylor's formula. The linear convergence in theorem 1 can't be improved except in the case $m = 1$, where from the Taylor's formula we have quadratic convergence, i.e.:

$$| \tau_1^{(k+1)} - \tau_1^* | < c(f_1) | \tau_1^{(k)} - \tau_1^* |^2, \quad k = 0, 1, 2, \ldots,$$

where $c(f_1)$ denotes a positive constant, depending only on the function $f_1$. As we shall see below we can impose a simple condition on the system of tending functions and reach quadratic convergence in the S.N.M.

Let for the system of tending functions $\mathbf{F} = \{f_i\}_{i=1}^m$ the following condition be satisfied:

$(v)$ $\otimes_i$ is at least twice continuously differentiable function in $\mathbf{S}_{m-1}^i$ and

$$\frac{\partial \otimes_i (\tau_1^{(*)}, \tau_2^{(*)}, \ldots, \tau_{i-1}^{(*)}, \tau_{i+1}^{(*)}, \ldots, \tau_m^{(*)})}{\partial \tau_j} = 0, \quad \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, \ldots, m; \quad j \neq i \end{array}.$$

Then we say that $\mathbf{F} = \{f_i\}_{i=1}^m$ is a system of strictly convex of the $i-th$ argument strongly monotone tending to $\tau^*$ functions.

**Theorem 2.** *If under the conditions and notations of theorem 1, $\mathbf{F} = \{f_i\}_{i=1}^m$ is a system of strongly tending functions, then:*

$$\| \tau^{(k+1)} - \tau^* \| < c(\mathbf{F}) \| \tau^{(k)} - \tau^* \|^2, \quad k = 0, 1, 2, \ldots,$$

*where $c(\mathbf{F})$ denotes a positive constant depending only on $\mathbf{F} = \{f_i\}_{i=1}^m$ and $\| * \|$ stands for any norm in $\mathbf{R}^m$.*

Proof. Let $| \tau_i^{(k)} - \tau_i^* | < h,\ \tau^{(k)} = (\tau_1^{(k)}, \ldots, \tau_m^{(k)}) \in \mathbf{E}_{++}^m,\ i = 1, \ldots, m$, where $h > 0$. Then from the Taylor's expansion of $f_i$ as a function of $\tau_i$ alongside the point $\tau_i^*$ it follows:

$$| \tau_i^{(k+1)} - \otimes_i(\tau_1^{(k)}, \ldots, \tau_{i-1}^{(k)}, \tau_{i+1}^{(k)}, \ldots, \tau_m^{(k)}) | < c_1(\mathbf{E}) h^2, \quad i = 1, \ldots, m,$$

On the other side, since the partial derivatives of $\otimes_i$ in the point $(\tau_1^{(*)},\dots,\tau_{i-1}^{(*)},\tau_{i+1}^{(*)},\dots,\tau_m^{(*)})$ are equal to 0, from the Taylor's expansion of $\otimes$ as a function of the variables $\tau_j$, $j=1,\dots,m$, $j\neq i$; alongside the point $(\tau_1^{(*)},\dots,\tau_{i-1}^{(*)},\tau_{i+1}^{(*)},\dots,\tau_m^{(*)})$ it follows:

$$|\otimes_i(\tau_1^{(k)},\dots,\tau_{i-1}^{(k)},\tau_{i+1}^{(k)},\dots,\tau_m^{(k)})-\otimes(\tau_1^{(*)},\dots,\tau_{i-1}^{(*)},\tau_{i+1}^{(*)},\dots,\tau_m^{(*)})|$$

$$=|\otimes_i(\tau_1^{(k)},\dots,\tau_{i-1}^{(k)},\tau_{i+1}^{(k)},\dots,\tau_m^{(k)})-\tau_i^*|<c_2(\mathbf{F})h^2,\quad i=1,\dots,m\,.$$

Theorem 2 follows from the above two inequalities.

**Example.** Let

$$f_1(\tau_1,\tau_2)=(\tau_2-b)\tau_1^2-2a\tau_2\tau_1-a\tau_2^2\,,\quad f_2(\tau_1,\tau_2)=(\tau_1-a)\tau_2^2-2b\tau_1\tau_2-b\tau_1^2\,, \tag{9}$$

where

$$0<a<\frac{1}{4}\,,\quad 0<b<\frac{1}{4}\,. \tag{10}$$

We shall prove (*i*), (*ii*), (*iii*), (*iv*) and (*v*) for $\mathbf{F}=\{f_1 \text{ and } f_2\}$.

It is easy to see that the system of equations

$$f_1(\tau_1,\tau_2)=0\,,\quad f_2(\tau_1,\tau_2)=0 \tag{11}$$

under the assumption (10), has a unique solution in $\mathbf{E}^2$:

$$\tau_1^*=2a^{\frac12}(a^{\frac12}+b^{\frac12})\in(0,1)\,,\quad \tau_2^*=2b^{\frac12}(a^{\frac12}+b^{\frac12})\in(0,1)\,.$$

This proves (*i*) for the set $\mathbf{F}$.

On the other hand, it is easy to see from (9) and (10) ($a>0,\ b>0$) that if $f_1(\tau)\geq 0$ and $f_2(\tau)\geq 0$ for any $\tau=(\tau_1,\tau_2)\in\mathbf{E}^2$ then

$$\frac{\partial^2 f_i(\tau)}{\partial\tau_i^2}>0,\quad \frac{\partial f_i(\tau)}{\partial\tau_i}>0,\quad i=1 \text{ and } 2$$

which proves (*ii*) for $\mathbf{F}$.

For every $\tau_2\in[\tau_2^*,1]$

$$\otimes_1(\tau_2)=\frac{a\tau_2+[a^2\tau_2^2+(\tau_2-b)a\tau_2^2]^{\frac12}}{(\tau_2-b)},\quad 0<\otimes_1(\tau_2)<1,$$

solves $f_1(*,\tau_2)=0$ and for every $\tau_1\in[\tau_1^*,1]$

$$\otimes_2(\tau_1)=\frac{b\tau_1+[b^2\tau_1^2+(\tau_1-a)b\tau_1^2]^{\frac12}}{(\tau_1-a)},\quad 0<\otimes_2(\tau_1)<1\,,$$

solves $f_2(\tau_1,*)=0$. This proves (*iii*).

It is a matter of simple computations to show that:

$$\otimes_1'(\tau_2^*)=0,\quad \otimes_1(\tau_2)\ \text{ is convex for }\ \tau_2\in[\tau_2^*,\infty]\,,$$
$$\otimes_2'(\tau_1^*)=0,\quad \otimes_2(\tau_1)\ \text{ is convex for }\ \tau_1\in[\tau_1^*,\infty]\,.$$

This proves $(iv)$ and $(v)$.

Then $\mathbf{F} = \{f_1$ and $f_2\}$ is a system of strongly tending functions and from theorem 1 and theorem 2 it follows that the S.N.M. for (11) converges monotonely and with quadratic convergence to $\tau^*$ whenever $\tau^{(0)} \in \mathbf{E}^2_{++}$. It is easy to see that in this case $(1,1) \in \mathbf{E}^2_{++}$ because by assuming the opposite we obtain that the system has two zeros in $\mathbf{E}^2_{++}$.

One can prove for this example also that the N.M. for (11) converges monotonely and with quadratic convergence to $\tau^*$ whenever $\tau^{(0)} \in \mathbf{E}^2_{++}$.

The question if the N.M. and S.N.M. always converge (with the same order ?) where in both cases the starting point is in $\mathbf{E}^2_{++}$ and $\mathbf{F}$ is a (strongly) tending system remains open.This is true for our example and we illustrate this by table 1,2 for two cases.

Table 1

| | **a=0.120000** | | **b=0.120000** | |
|---|---|---|---|---|
| | **S.N.M.** | | **N.M.** | |
| k | $\tau_1^{(k)}$ | $\tau_2^{(k)}$ | $\tau_1^{(k)}$ | $\tau_2^{(k)}$ |
| 0 | 1. | 1. | 1. | 1. |
| 1 | 0.657894 | 0.657894 | 0.745098 | 0.745098 |
| 2 | 0.517864 | 0.517864 | 0.590212 | 0.590212 |
| 3 | 0.482579 | 0.482579 | 0.509968 | 0.509968 |
| 4 | 0.480013 | 0.480013 | 0.483151 | 0.483151 |
| 5 | 0.480000 | 0.480000 | 0.480040 | 0.480040 |
| 6 | 0.480000 | 0.480000 | 0.480000 | 0.480000 |

Table 2

| | **a=0.150000** | | **b=0.070000** | |
|---|---|---|---|---|
| | **S.N.M.** | | **N.M.** | |
| k | $\tau_1^{(k)}$ | $\tau_2^{(k)}$ | $\tau_1^{(k)}$ | $\tau_2^{(k)}$ |
| 0 | 1. | 1. | 1. | 1. |
| 1 | 0.692307 | 0.589743 | 0.770320 | 0.695749 |
| 2 | 0.555123 | 0.409350 | 0.626370 | 0.505673 |
| 3 | 0.510754 | 0.352251 | 0.545377 | 0.398768 |
| 4 | 0.505035 | 0.345057 | 0.511666 | 0.353997 |
| 5 | 0.504939 | 0.344939 | 0.505175 | 0.345262 |
| 6 | 0.504939 | 0.344939 | 0.504939 | 0.344939 |

Theorems 1 and 2 can be used for proving the convergence of the Newton's method for obtaining the solution of the so-called "best" convex interpolation of data (see [3,4,5]).This application of theorems 1 and 2 will be treated in another work.

## REFERENCES

[1] J.M.Ortega and W.C.Rheinboldt. Iterative Solution of Non Linear Equations in Several Variables. New York and London, Academic Press, 1970.

[2] L.Wegge. On a Discrete Version of the Newton-Raphson Method. *SIAM J. Numer. Anal.* **3** (1960) 134-142.

[3] G. ILIEV AND W. POLLUL. Convex interpolation by Function with Minimal $\mathbf{L}_p$-Norm ($1 < p < \infty$) of the Second Derivative. *Proceedings of the Thirteenth Spring Conference of the Union of Bulgarian Mathematicians*, 1984, 31-42.

[4] C. MICHELLI, P. SMITH, J. SWETITS, J. WARD. Constrained Approximation. *Constr. Approx.* **1** (1985) 93-102.

[5] L. ANDERSSON, T. ELFVING, An Algorithm for Constrained Interpolation. *SIAM J. Sci. Stat. Comput.* **8** (1987) 1012-1025.

*Institute of Mathematics*
*Bugarian Academy of Sciences*
*P.O.Box 373*
*1090 Sofia, Bulgaria*

*Received 22.07.1991*