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THREE FAMILIES OF CHROMATICALLY UNIQUE GRAPHS

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ABSTRACT. Let $P(G)$ denote the chromatic polinomial of a graph G. A graph G is said to be chromatically unique if $P(G) = P(H)$ implies that H is isomorphic to G. In this paper, we prove that a graph (resp., a bipartite graph) obtained from $K_{2,4} \cup P_s$ ($s \geq 3$) (resp., $K_{3,3} \cup P_s$ ($s \geq 7$)) by identifying the end vertices of the part P_s with any two vertices of the complete bipartite graph $K_{2,4}$ (resp., $K_{3,3}$) is chromatically unique. We also show that a bipartite graph obtained from $(K_{3,3} - e) \cup P_s$ ($s \geq 5$) where e is an edge of $K_{3,3}$, by identifying the end vertices of the path P_s with two nonadjacent vertices of $K_{3,3} - e$ is chromatically unique.

1. Introduction. The graphs which we consider here are finite, undirected, simple and loopless. For a graph G, let $V(G)$ denote its vertex set, $E(G)$ denote its edge set and $P(G;\lambda)$, or simply $P(G)$ if there is no likelihood of confusion, denote its chromatic polynomial. Two graphs X and Y are said to be *chromatically equivalent* if $P(X) = P(Y)$. A graph G is said to be *chromatically unique* if $P(G) = P(H)$ implies that H is isomorphic to G, denoted by $H \cong G$.

The search for infinite families of chromatically unique graphs is quite challenging and it is one of the topics on chromatic polinomials that has comme to the fore in recent years. For an expository paper giving a survey on most of the works done on chromatically unique graphs, the reader is refer to Koh and Teo [4].

The complete bipartite graph in which the partite sets consist of the m and n vertices will be denoted by $K_{m,n}$. We denote by $K^1_{m,n}(s)$ (resp $K^2_{m,n}(s)$ and $K^3_{m,n}(s)$), the graph obtained from $K_{m,n} \cup P_s$ by identifying the end vertices of the path P_s on s vertices with two vertices of $K_{m,n}$ which are adjacent (resp., nonadjacent with degree n and nonadjacent with degree m). Note that $K^2_{m,n}(s) \cong K^3_{m,n}(s)$ if and only if $m = n$.

In [2] the author proved the following results.

Theorem 1. *The graphs $K^i_{2,3}(s)$, ($i = 1,2,3$) where $s \geq 3$ are chromatically unique.*

As a natural extension of Theorem 1, we will show that the graph $K^i_{2,4}(s)$ is also chromatically unique for $1 \leq i \leq 3$ and $s \geq 3$. Furthermore, in this paper, we will prove that the graphs $K^1_{3,3}(2s)$ and $K^2_{3,3}(2s-1)$ are chromatically unique for $s \geq 4$. We also show that a bipartite graph obtained from $(K_{3,3} - e) \cup P_s$ $(s \geq 5)$ where e is an edge of $K_{3,3}$, by identifying the end vertices of the path P_s with two nonadjacent vertices of $K_{3,3} - e$ is chromatically unique.

For terms used but not defined here the reader is refer to [1].

2. Preliminaries. This section contains known results which will be used to prove our main theorems in the next section. We need the following notations to state the first known result.

Let H be a nonempty graph with two nonadjacent vertices u and v. The graph denoted by H^* is obtained from H by identifying the vertices u and v. Let G_1 (resp., G_2) be the graph obtained from $H \cup P_s$ by identifying the end vertices of the path P_s with the vertices u and v (resp., any two adjacent vertices) of H. Then by applying Theorem 1 and 3 in [6], and by using the well known fact that $P(C_s; \lambda) = (\lambda-1)^s + (-1)^s(\lambda-1)$, we have

Theorem A (Read [7] and Chia [2]). *Let G_1, G_2 and H^* be the graphs defined as above. Then*

$$P(G_1) = P(G_2) + (-1)^{s-1}P(H^*).$$

Let $K^-_{m,n}$ be the graph obtained by deleting one egde from the complete bipartite graph $K_{m,n}$. Also let C_n be the cycle with n vertices. $N_G(L)$ will denote the number of subgraphs of G isomorphic to L, and $I_G(L)$ will denote the number of induced subgraphs of G isomorphic to L (so that $N_G(L) = I_G(L)$ if L is complete, or if G is bipartite and L is complete bipartite).

The following necessary conditions for two graphs to be chromatically equivalent can be deduced and are well known (see, for example [3] and [9]).

Theorem B. *Let G and H be two chromatically equivalent graphs. Then*

(*i*) $|V(G)| = |V(H)|$;
(*ii*) $|E(G)| = |E(H)|$;
(*iii*) $N_G(C_3) = N_H(C_3)$;
(*iv*) $I_G(C_4) - 2N_G(K_4) = I_H(C_4) - 2N_H(K_4)$;
(*v*) $(q-3)I_G(C_4) + N_G(K_{2,3}) - I_G(C_5) = (q-3)I_H(C_4) + N_H(K_{2,3}) - I_H(C_5)$ *is G has at most one triangle*;
(*vi*) $\chi(G) = \chi(H)$, *where $\chi(G)$ denotes the chromatic number of G*;
(*vii*) *G is connected if and only if H is connected*;
(*viii*) *G is 2-connected if and only if H is 2-connected.*

For bipartite graphs, we have another necessary condition for them to be chromatically equivalent.

Theorem C (Peng [5]). *Let G and H be two chromatically equivalent bipartite graphs. Then*

$$I_G(C_6)+N_G(K_{2,4})-I_G(K_{3,3}^-)-4N_G(K_{3,3}) = I_H(C_6)+N_H(K_{2,4})-I_H(K_{3,3}^-)-4N_H(K_{3,3}).$$

3. Main Results. We shall show the following key result (see also [2]). First the definition:

Definition. *A subgraph F of a graph G is called a chromatically invariance of G if for any graph H with $P(H) = P(G)$, H contains F as a subgraph.*

Theorem 2. *Let G be a 2-connected graph obtained from $F \cup P_s$ by identifying the end vertices of the path P_s with two distinct vertices of a graph F. Suppose that F is chromatic invariance of G. If H is chromatically equivalent to G, then H is also a graph obtained from $F \cup P_s$ by identifying the end vertices of P_s with two distinct vertices of F.*

Proof. It is trivial if $s = 2$. Assume that $S \geq 3$. Since F is chromatic invariance of G, H contains two subgraphs F and $Y = H - F$ with some edges connecting them. Thus by Theorem B (ii),

$$|E(Y)| + e(F,Y) \leq s - 1, \tag{1}$$

where $e(F,Y)$ denotes the number of edges joining vertices of F to vertices of Y. Now let c be the number of componets of Y. Then

$$|E(Y)| \geq s - 2 - c, \tag{2}$$

since $|V(Y)| = s - 2$. Also $e(F,Y) \geq 2c$ because H is 2-connected by Theorem B (viii). Hence from (1) we have

$$|E(Y)| \leq s - 1 - 2c. \tag{3}$$

Therefore (2) and (3) imply that $c = 1$, i.e., Y is connected. Consequently, $|E(Y)| = s - 3$ and $e(F,Y) = 2$. Since H is 2-connected, Y must be the path P_{s-2} whose end vertices are join to two distinct vertices of F. □

Corollary 1. *Let G be a 2-connected graph obtained from $F \cup P_s$ by identifying the end vertices of the path P_s with two adjacent vertices of F. Suppose that F is chromatic invariance of G. If F is edge-transitive, then G is chromatically unique.*

Proof. Let H be a graph with $P(H) = P(G)$. Then by Theorem 2, H is a graph obtained from $F \cup P_s$ by identifying the end vertices of P_s with two distinct vertices x and y of F. By Theorem A, x is adjacent to y. Since F is edge-transitive, we have $H \cong G$. $\square$

Corollary 2. *If the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$ is chromatic invariance of $K^i_{m,n}(s)$ where $s \geq 2$, then $K^i_{m,n}(s)$ is chromatically unique for $1 \leq i \leq 3$.*

Proof. For $i = 1$, the result follows from Corollary 1 because $K_{m,n}$ is edge-transitive. If H is chromatically equivalent to $K^2_{m,n}(s)$ or $K^3_{m,n}(s)$, then by Theorem 2, H is also a $K^i_{m,n}(s)$ $(i = 2, 3)$. If $m = n$, we are done. Otherwise, by Theorem A, $P(K^2_{m,n}(s)) \neq P(K^3_{m,n}(s))$ since $K_{m-1,n}$ is not chromaticallyequivalent to $K_{m,n-1}$ (see Teo and Koh [8]). Hence $K^i_{m,n}(s)$ is chromatically unique for $i = 2, 3$. $\square$

We shall now give three infinite families of chromatically unique graphs. The first one is stated in the following theorem.

Theorem 3. *The graph $K^i_{2,4}(s)$ is chromatically unique for $s \geq 3$ and $1 \leq i \leq 3$.*

Proof. Let $G_1 = K^2_{2,4}(3)$, $G_2 = K^3_{2,4}(3)$, $G_3 = K^1_{2,4}(4)$, $G_4 = K^2_{2,4}(4)$, $G_5 = K^3_{2,4}(4)$ and $G_6 = K^1_{2,4}(5)$. By Corollary 2 to Theorem 2, we need only to show that $K_{2,4}$ is chromatic invariance of $K^i_{2,4}(s)$ for $1 \leq i \leq 3$ and $s \geq 3$. We consider two cases:

G :

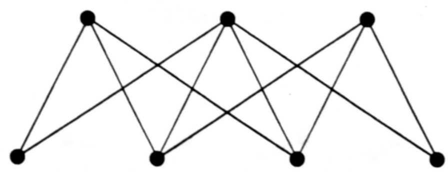


Figure 1. $I_G(K^-_{3,3}) = 2$ and $I_G(C_4) = 7$

Case(a). $K^i_{2,4}(s)(1 \leq i \leq 3) \not\cong G_j(1 \leq j \leq 6)$. Let H be a graph with $P(H) = P(K^i_{2,4}(s))$ for $1 \leq i \leq 3$ and $s \geq 3$. We show that $K_{2,4}$ is chromatic invariance of $K^i_{2,4}(s)$, i.e., $N_H(K_{2,4}) \neq 0$. By Theorem B, we have $I_H(C_4) = 6$ and $N_H(K_{2,3}) \geq 4$. Suppose that $I_H(K^-_{3,3}) = 1$. Note that $I_{K^-_{3,3}}(C_4) = 5$ and $N_{K^-_{3,3}}(K_{2,3}) = 2$. If $N_H(K_{2,3}) \geq 3$ and $I_H(K^-_{3,3}) = 1$, then we must have $I_H(C_4) \geq 7$. Also if $I_H(K^-_{3,3}) \geq 2$, then it is easy to see that $I_H(C_4) \geq 7$ (see Fugure 1 for the minimal example). Since $N_H(K_{2,3} \geq 4$, the suppossition that $I_H(K^-_{3,3}) \geq 1$ leads to contradiction and hence we have $I_H(K^-_{3,3}) = 0$.

We now assume that $N_H(K_{2,4}) = 0$ and we shall get a contradiction. Note that if $N_H(K_{2,3}) = 2$, then these two $K_{2,3}$'s have at most one edge in common, for otherwise, H contains $K_{2,4}$ or $K^-_{3,3}$ as an induced subgraph. So if $N_H(K_{2,3}) \geq 3$, we must have $I_H(C_4) \geq 7$, a contradiction. Therefore $N_H(K_{2,4}) \neq 0$ because $N_H(K_{2,3}) \geq 4$.

Case(b). G_j $(1 \le j \le 6)$. The graph $G_1 \cong K_{2,5}$ is chromatically unique (see [8]). Let H be a graph with $P(H) = P(G_2)$ or $P(G_3)$. Then by Theorem B, H is a bipartite graph satisfying $10 \le |E(H)| \le 11$, $7 \le |V(H)| \le 8$ and $7 \le I_H(C_4) \le 8$. Thus $N_H(K_{3,3}) = 0$ and $I_H(C_6) = 0$, for otherwise, $I_H(C_4) \ge 9$ or $I_H(C_4) \le 5$ which is a contradiction. So by Theorem C, we have $N_H(K_{2,4}) - I_H(K_{3,3}^-) = 1$ which implies $N_H(K_{2,4}) \ne 0$.

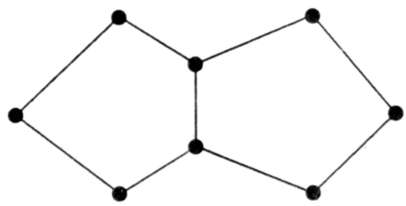
We now consider G_j $(4 \le j \le 6)$. Let H be a graph chromatically equivalent to G_j for any $4 \le j \le 6$. Then by Theorem B, H contains 6 indused subgraphs C_4 but has no triangles, and H is not a bipartite graph.

We first consider G_6. By Theorem B(v), we have $N_H(K_{2,3}) - I_H(C_5) = 3$, i.e., $N_H(K_{2,3}) \ge 3$. By using the arguments similar to Case(a), we can show that $N_H(K_{2,4}) \ne 0$.

We next consider G_5. Theorem B(v) gives us $N_H(K_{2,3}) - I_H(C_5) = 2$. We claim that $I_H(C_5) \ge 1$, for otherwise, $I_H(C_7) = 1$ because $|V(H)| = 8$ and H is not a bypartite graph; but then there is no graph H of order 8 and size 11 such that $N_H(C_3) = I_H(C_5) = 0$ and $I_H(C_7) = 1$. Thus $N_H(K_{2,3}) \ge 3$. By applying the arguments used in Case(a), we can again show that $N_H(K_{2,4}) \ne 0$.

We now consider the last graph G_4. For this graph, we have $N_H(K_{2,3}) = I_H(C_5)$ by Theorem B. As in the case of G_5 above, $I_H(C_5) \ne 0$. If $I_H(C_5) = N_H(K_{2,3}) = 1$, then H must contain a spanning subgraph which is obtained from $K_{2,3} \cup C_5$ by identifying an edge of C_5 with an edge of $K_{2,3}$ because $|V(H)| = 8$. Since H has no trianges, it can easily be confirmed that $|E(H)| = 10$, which is a contradiction. So we must have $I_H(C_5) = N_H(K_{2,3}) \ge 2$. Thus H contains J_1 or J_2 of Figure 2 as a subgraph.

J_1 :



J_2 :

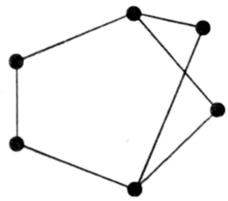


Figure 2.

Since $|V(H)| = 8$, $|E(H)| = 11$ and $I_H(C_4) = 6$, it is not difficult to check that J_1 can not be a subgraph of H. So H contains two vertex disjoint subgraphs J_2 and L, where L is a graph of order 2. Since $|E(H)| = 11$, and $N_H(K_{2,3}) \ge 2$, there are only three possibilities for H. We show these three graphs in Figure 3. But $H \not\cong H'$ or G_5 because $I_{H'}(C_4) = 5$ and $P(G_5) = P(G_4)$. Thus G_4 is chromatically unique.

The proof is now complete. □

Our second and third infinite families of chromatically unique graphs are stated in the following theorems.

Theorem 4. *The graphs* $K^1_{3,3}(2s)$ *and* $K^2_{3,3}(2s-1)$ *are chromatically unique for* $s \geq 4$.

Proof. Let H be a graph with $P(H) = P(K^1_{3,3}(2s))$ or $P(K^2_{3,3}(2s-1))$ where $s \geq 4$. Then by Theorem B, we have $I_H(C_4) = 9$. Since $K^1_{3,3}(2s)$ and $K^2_{3,3}(2s-1)$ are bipartite graphs, and $s \geq 4$, we have $I_H(C_6) + N_H(K_{2,4}) - I_H(K^-_{3,3}) - 4N_H(K_{3,3}) = -4$ by Theorem C. If $N_H(K_{3,3}) = 0$, then $I_H(K^-_{3,3}) \geq 4$. Note that one $K^-_{3,3}$ contains 5 induced subgraphs C_4, and if $I_H(K^-_{3,3}) \geq 4$, then it is not difficult to see that $I_H(C_4) \geq 11$ (see Figure 4 for the minimal case). Therefore $N_H(K_{3,3}) \neq 0$ and by Corollary 2 to Theorem 2, the graphs $K^1_{3,3}(2s)$ and $K^2_{3,3}(2s-1)$ are chromatically unique for $s \geq 4$. □

Remark. The graph $K^1_{3,3}(4)$ is also chromatically unique.

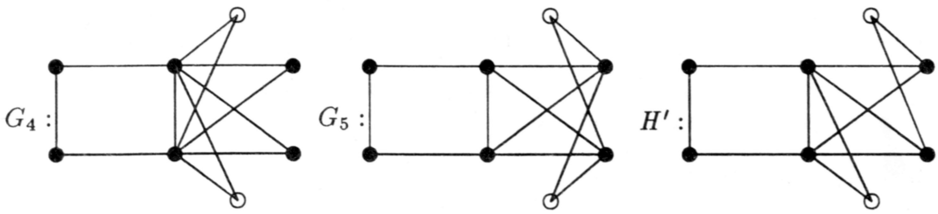


Figure 3.

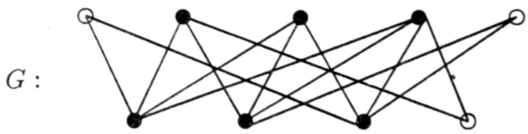


Figure 4. $N_G(K_{3,3}) = 0$, $I_G(K^-_{3,3}) = 4$ and $I_G(C_4) = 11$

Theorem 5. *The bipartite graph obtained from* $K^-_{3,3} \cup P_s$ ($s \geq 5$) *by identifying the end vertices of* P_s *with two nonadjacent vertices of* $K^-_{3,3}$ *is chromatically unique.*

Proof. Let G be a bipartite graph defined as in the theorem and let H be a graph with $P(H) = P(G)$. Then by Theorems B and C, we have $I_H(C_6) - I_H(K^-_{3,3}) = -1$ since $I_H(C_4) = I_G(C_4) = 5$. Thus $I_H(K^-_{3,3}) \neq 0$, i.e., the graph $K^-_{3,3}$ is chromatic invariance of G. Therefore by Theorem 2, H must be a graph obtained from $K^-_{3,3} \cup P_s$ by identifying the end vertices of P_s with two distinct vertices of $K^-_{3,3}$. By Theorem A, it is not difficult to verify that H is isomorphic to G. □

In view of Theorem 3 and 4 it is tempting to state the following conjectures:

Conjectures.

(1) *The graph* $K^i_{2,n}(n)$ *is chromatically unique for* $1 \leq i \leq 3$ *and* $s, n \geq 3$.

(2) *The grapg* $K^1_{m,m}(2s)$ *and* $K^2_{m,m}(2s-1)$ *are chromatically unique for* $m \geq 2$ *and* $s \geq 4$.

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