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DIRECT AN INVERSE INEQUALITIES FOR SOME DISCRETE OPERATORS OF BOUNDED MEASURABLE FUNCTIONS.

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ABSTRACT. Convergence properties of Jakson polynomials have been considered by Zygmund [6, ch.X], J.Szabados [5], ($p = \infty$) and V.A.Popov, J.Szabados [4] ($1 \le p \le \infty$) have proved a direct inequality for Jakson polynomials in L_p spaces in terms of some moduli of functions. In this paper we give the direct and also inverse inequalities for Jakson polynomials and Valee-Poussin discrete operators in locally global norms [1] in terms of suitable Peetre K-functionals. Our results aim at generalizing some of the results of V.A.Popov and J.Szabados.

1. Introduction. We denote the set of 2π-periodic bounded measurable functions with usual sup-norm by L_∞ and the L_p norm ($1 \le p \le \infty$) of $f \in L_p$ by $\|f\|_p$

For $f \in L_\infty$ instead of usual sup-norm, we denote by $\|f\|_{\delta,p}$ the locally global norm of f wich is given by ($\delta > 0$)

$$\|f\|_{\delta,p} := \left(\int_{-\pi}^{\pi} |f_\delta(x)|^p \, dx\right)^{1/p} , \quad 1 \le p \le \infty ,$$

where $f_\delta(x) := \sup\{|f(t)| : t \in U(\delta, x)\}$ and $U(\delta, x) := \{y : |x - y| \le \delta\}$, (see [1]).

We shall denote the set of all functions $f \in L_\infty$ with norm $\|f\|_{\delta,p}$ by $L_{\delta,p}$.

Let us denote by c the absolute constants which are in general different. Let $x_k = x_{k,2n} := \frac{2\pi k}{2n+1}$, $k = 0, 1, \dots, 2n$.

For $f \in L_\infty$, we have [1] ($1 \le p \le \infty$)

$$(1) \qquad \|f\|_p \le \|f\|_{\delta,p} ,$$

$$(2) \qquad \|f\|_{m\delta,p} \le c m^{1/p} \|f\|_{\delta,p} , \ m \text{ natural} ,$$

$$(3) \qquad \|f\|_{\delta',p} \le \|f\|_{\delta,p} , \ \delta' \le \delta ,$$

$$(4)\qquad \|f\|_{l^p_{2n+1}} := \left(\frac{1}{2n+1}\sum_{k=0}^{2n}|f(x_k)|^p\right)^{1/p} \le c\|f\|_{\frac{1}{n},p}\,,$$
$$\|f\|_{l^\infty_{2n+1}} := \max\{|f(x_k)| : k = 0,1,\dots,2n\}\,.$$

Let $W_p^1 := \{g : g' \in L_p\}$, $\tilde{W}_p^1 := \{g : \tilde{g}' \in L_p\}$, have semi-norms $\|g'\|_p$ and $\|\tilde{g}'\|_p$ respectively, where $\tilde{g}$ is the conjugate of function g.

We denote by T_n the set of all trigonometric polynomials of degree n. Let $P \in T_n$, then [1] $(1 \le p \le \infty)$

$$\|P\|_p \le \|P\|_{\delta,p} \le c(1+n\delta)^{1/p}\|P\|_p\,.$$

From the last inequalities we have for $P \in T_n$

$$(5)\qquad \|P\|_p \le \|P\|_{\frac{1}{n},p} \le c\|P\|_p, \quad (1 \le p \le \infty).$$

Let

$$K_n(u) := \frac{2}{n+1}\left(\frac{\sin(n+1)u/2}{2\,\sin u/2}\right)^2,\ n = 0,1,\dots$$

be the Fejer kernel. Let $t_k := x_{k,n} := \frac{2\pi k}{n+1}$, $k = 0,1,\dots,n$ then

$$J_n(f,x) := \frac{2}{n+1}\sum_{k=0}^{n} f(t_k)\,K_n(x-t_k)$$

is the so-called Jakson polynomial of function $f \in L_\infty$.

Consider the Fejer means $(f \in L_1)$

$$\sigma_n(f,x) := \frac{1}{\pi}\int_0^{2\pi} f(x+t)\,K_n(t)\,dt\,.$$

For $f \in L_p$ it is well known

$$(6)\qquad \|\sigma_n(f)\|_p \le \|f\|_p\,,\ 1 \le p \le \infty\,,$$

and for $f \in \tilde{W}_p^1$ we have (see for instance [4])

$$(7)\qquad n\|f - \sigma_n(f)\|_p = O(1)\|\tilde{f}'\|_p\,,\ 1 \le p \le \infty \quad \text{while if } f \in L_\infty\,,$$

then [4]

$$(8)\qquad \|J_n(f)\|_p = O(1)\left[\frac{1}{n}\sum_{k=0}^{n}|f(t_k)|^p\right]^{1/p},\ 1 \le p \le \infty\,.$$

Let $P \in T_n$, then [5]

$$(9)\quad J_n(P_n,x)-\sigma_n(P_n,x) = \frac{1}{n+1}\left[\tilde{P}_n'(x).\cos(n+1)x - P_n'(x).\sin(n+1)x\right] .$$

From (8) and (9) by using (4) and (5), we get ($1 \le p \le \infty$)

$$(10)\qquad \|J_n(f)\|_{\frac{1}{n},p} = c\|f\|_{\frac{1}{n},p}\,,\ f \in L_\infty .$$

$$(11)\qquad \|J_n(P_n,x)-\sigma_n(P_n,x)\|_{\frac{1}{n},p} \le \frac{c}{n+1}\left[\|\tilde{P}_n'\|_p + \|P_n'\|_p\right] ,\ P_n \in T_n .$$

Now consider the Dirichlet kernel

$$D_n(u) := \frac{1}{2}+\sum_{k=1}^{n}\cos ku = \frac{\sin(n+1/2)u}{2\sin u/2} .$$

Define $V_{n,2n}(u) := \frac{1}{n+1}\{D_n(u) + D_{n+1}(u) + \cdots + D_{2n}(u)\}$. Then

$$(12)\qquad V_{n,2n}(u) = D_n(u) + \frac{2n+1}{n+1}\sum_{k=n+1}^{2n}\left(1-\frac{k}{2n+1}\right)\cos ku .$$

It is easy to get that

$$(13)\qquad V_{n,2n}(u) = \frac{2n+1}{n+1}K_{2n}(u) - \frac{n}{n+1}K_{n-1}(u) .$$

From (13), we obtain

$$(14)\qquad \|V_{n,2n}(\cdot)\|_1 \le 3 .$$

Let $V_n^d(f,x) = \frac{2}{2n+1}\sum_{k=0}^{2n} f(x_k)V_{n,2n}(x-x_k)$ be the Valee-Poussin operator of function $f \in L_\infty$. Let $V_n(f,x) = \frac{1}{\pi}\int_0^{2\pi} f(u)V_{n,2n}(u-x)\,du$ the Valee-Poussin operator of function $f \in L_1$. The locally modulus of continuity of bounded function is defined by

$$\omega(f,x,\delta) := \sup_{x',x''\in[x-\frac{\delta}{2},x+\frac{\delta}{2}]}\{|f(x')-f(x'')|\} .$$

While the average modulus of smoothness in L_p spaces is (see [3] p.7)

$$\tau(f,\delta)_p := \|\omega(f,\cdot,\delta)\|_p\,,\ 1 \le p \le \infty .$$

The classical L_p-modulus of smoothness is defined by

$$\omega(f,\delta)_p := \sup_{|t|\le\delta}\|f(x+t) - f(x)\|_p .$$

From the definition, we can see that $(1 \leq p \leq \infty)$

$$\tau(f,\delta)_p \leq 2\|f\|_{\delta,p}, \quad f \in L_\infty . \tag{15}$$

For $\lambda > 0,\ 1 \leq p \leq \infty$, we have ([3, p.9,10])

$$\tau(f,\lambda\delta)_p \leq (2(\lambda+1))^2\,\tau(f,\delta)_p\,, \ f \in L_\infty \tag{16}$$

$$\tau(g,\delta)_p \leq \delta\|g'\|_p\,, \ g \in W_p^1 . \tag{17}$$

Let $f \in L_\infty$, then [3, p.14]

$$\omega(f,\delta)_p \leq \tau(f,\delta)_p \leq \omega(f,\delta)_\infty . \tag{18}$$

We shall consider the appropriate Peetre K-functionals, as follows: $(t > 0)$

$$\begin{aligned}
K(f,t;L_p,W_p^1) &:= \inf\{\|f-g\|_p + t\|g'\|_p : g \in W_p^1\}\,,\\
K(f,t;L_{t,p},W_p^1) &:= \inf\{\|f-g\|_{t,p} + t\|g'\|_p : g \in W_p^1\}\,,\\
K(f,t;L_p,W_p^1,\tilde{W}_p^1) &:= \inf\{\|f-g\|_p + t\|g'\|_p + t\|\tilde{g}'\| : g \in W_p^1 \cap \tilde{W}_p^1\}\,,\\
K(f,t;L_{t,p},W_p^1,\tilde{W}_p^1) &:= \inf\{\|f-g\|_{t,p} + t\|g'\|_p + t\|\tilde{g}'\|_p : g \in W_p^1 \cap \tilde{W}_p^1\}\,.
\end{aligned}$$

It is easy to see that

$$K(f,t;L_p,W_p^1) \leq K(f,t;L_{t,p},W_p^1) \leq K(f,t;L_{t,p},W_p^1,\tilde{W}_p^1)\,. \tag{19}$$

2. Main results. We give direct and inverse inequalities of Jakson polynomials and Valee-Poussin discrete operators in terms of K-functionals.

Theorem 1. *If* $f \in L_\infty$, *then* $(1 \leq p \leq \infty)$

$$\|f - J_n(f)\|_{\frac{1}{n},p} \leq cK(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1)\,.$$

Theorem 2. *If* $f \in L_\infty$, *then* $(1 \leq p \leq \infty)$

$$K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1) \leq c\|f - J_n(f)\|_{\frac{1}{n},p} + \frac{c}{n}\sum_{k=1}^{n}\|f - J_k(f)\|_p\,.$$

From theorem 1 and theorem 2, we obtain

Corollary 1. *If* $f \in L_\infty,\ 0 < \alpha < 1,$ *then* $(1 \leq p \leq \infty)$

$$\|f - J_n(f)\|_{\frac{1}{n},p} = O(\frac{1}{n^\alpha}) \ \text{ iff } K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1) = O(\frac{1}{n^\alpha})\,.$$

Theorem 1′. *If* $f \in L_\infty$, *then* $(1 \le p \le \infty)$

$$\|f - V_n^d(f)\|_{\frac{1}{n},p} \le cK(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1).$$

Theorem 2′. *If* $f \in L_\infty$, *then* $(1 \le p \le \infty)$

$$K(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) \le c\|f - V_n^d(f)\|_{\frac{1}{n},p} + \frac{c}{n}\sum_{k=1}^{n} \|f - V_k^d(f)\|_p .$$

From theorem 1′ and theorem 2′, we get

Corollary 1′. *If* $f \in L_\infty$, $0 < \alpha < 1$, *then* $(1 \le p \le \infty)$

$$\|f - V_n^d(f)\|_{\frac{1}{n},p} = O(\frac{1}{n^\alpha}) \text{ iff } K(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) = O(\frac{1}{n^\alpha}).$$

Theorem 3. *If* $f \in L_\infty$, *then*

$$\text{(20)}\quad \begin{aligned} &(i)\quad K(f,\tfrac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) \le c\left[\tau(f,\tfrac{1}{n})_p + \omega(\tilde{f},\tfrac{1}{n})_p\right], \quad p = 1, \infty \\ &(ii)\quad \tau(f,\tfrac{1}{n})_p \le K(f,\tfrac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) \le c_p\tau(f,\tfrac{1}{n})_p, \quad 1 < p < \infty \end{aligned}$$

Corollary 2. *If* $f \in L_\infty$, $0 < \alpha < 1$, $1 < p < \infty$, *then*

$$\|f - J_n(f)\|_{\frac{1}{n},p} = O(\frac{1}{n^\alpha}) \text{ iff } \tau(f, \frac{1}{n})_p = O(\frac{1}{n^\alpha}).$$

From (1), theorem 1 and theorem 3, we see that our results generalize some of the results of V.A.Popov and J.Szabados [4].

3. Auxiliary results. The best approximation of functions $f \in L_p[-\pi, \pi]$ with trigonometric polynomials T_n in the metric of space L_p is given by $E_n(f)_p := \inf\{\|f - T\|_p : T \in T_n\}$, and the best approximation of function $f \in L_\infty[-\pi, \pi]$ with polynomials from T_n in the metric of space $L_{\delta,p}$ is given by $E_n(f)_{\delta,p} := \inf\{\|f - T\|_{\delta,p} : T \in T_n\}$.

The best one-sided approximation of function $f \in L_\infty$ with polynomials from T_n in the metric of space L_p is given by

$$\tilde{E}_n(f)_p := \inf\{\|P^+ - P^-\|_p : P^\pm \in T_n \text{ and } P^-(x) \le f(x) \le P^+(x),\ x \in [-\pi, \pi]\}.$$

For $f \in L_\infty$, we have [1] $(1 \le p \le \infty)$,

$$\tilde{E}_n(f)_p \le E_n(f)_{\frac{2\pi}{n},p} \le c\tilde{E}_n(f)_p, \tag{21}$$

while if $f \in W_p^1$, then ([3] ,p.166)

$$\tilde{E}_n(f)_p \le \frac{c}{n}\|f'\|_p, \quad 1 \le p \le \infty. \tag{22}$$

From (2), (3), (21) and (22), we have

$$E_n(f)_{\frac{1}{n},p} = \frac{c}{n}\|f'\|_p, \; f \in W_p^1, \quad 1 \le p \le \infty. \tag{23}$$

Next Bernstein inequalities are given since we use them further on

$$\|P'\|_p \le n\|P\|_p, \quad P \in T_n, 1 \le p \le \infty, \tag{24}$$

$$\|\tilde{P}'\|_p \le n\|P\|_p, \quad P \in T_n, 1 \le p \le \infty. \tag{25}$$

From the semi-additivity of $E_n(f)_{\frac{1}{n},p}$ and (23), we have

$$E_n(f)_{\frac{1}{n},p} \le cK(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1), \; 1 \le p \le \infty. \tag{26}$$

Lemma 1. *If $f \in L_\infty$, then there is $g \in W_p^1 \cap \tilde{W}_p^1$ such that*

$$\begin{array}{lll} (i) & \|f-g\|_{\frac{1}{n},p} \le c\tau(f, \frac{1}{n})_p & ; \\ (ii) & \|g'\|_p \le n\omega(f, \frac{1}{n})_p & ; \\ (iii) & \|\tilde{g}'\|_p \le n\omega(\tilde{f}, \frac{1}{n})_p & , 1 \le p \le \infty. \end{array} \tag{27}$$

Proof. We shall use Steklov function $g(x) := n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x+u)\,du$. Then

$$|f(x)-g(x)| \le n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} |f(x+u)-f(x)|\,du$$

Hence by using (16), we obtain

$$\begin{aligned} \|f-g\|_{\frac{1}{n},p} &\le \left(\int_0^{2\pi}\left(\sup\left\{n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} |f(t+u)-f(t)|\,du \,:\, t \in U(\frac{1}{n},x)\right\}\right)^p dx\right)^{1/p} \\ &\le \left(\int_0^{2\pi}\left(\sup\left\{n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \omega(f,t,u)\,du \,:\, t \in U(\frac{1}{n},x)\right\}\right)^p dx\right)^{1/p} \end{aligned}$$

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Lemma 5. *If* $T \in T_n$, *then*

$$V_n^d(T(t),x) - T(x) = \frac{1}{n+1}\left[\tilde{T}'(x).\cos(2n+1)x \;-\; T'(x).\sin(2n+1)x\right] . \tag{34}$$

Proof. Because of the linearity of the operator $V_n^d(f)$, it is enough to prove that $(m \le n)$

(i) $V_n^d(\cos mt, x) - \cos mx = \frac{1}{n+1}\left[\cos(2n+1)x.(\widetilde{\cos mx})' - \sin(2n+1)x.(\cos mx)'\right]$,

(ii) $V_n^d(\sin mt, x) - \sin mx = \frac{1}{n+1}\left[\cos(2n+1)x.(\widetilde{\sin mx})' - \sin(2n+1)x.(\sin mx)'\right]$.

Now using that $\sum_{k=0}^{2n} \cos ix_k = \begin{cases} 0 & \text{if } i = 1,2,\dots,2n \\ 2n+1 & \text{if } i = 0, 2n+1 \end{cases}$ and

$\sum_{k=0}^{2n} \sin ix_k = 0\,,\quad i = 0,1,\dots\,,$ from (12) we obtain

$$\begin{aligned}
V_n^d(\cos mt, x) - \cos mx &= -\cos mx + \frac{2}{2n+1}\sum_{k=0}^{2n} \cos mx_k\, D_n(x - x_k) \\
&+\frac{2}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}(1 - \frac{i}{2n+1}) \cos mx_k.\cos i(x - x_k) \\
&= \frac{2}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}(1 - \frac{i}{2n+1}) \cos mx_k.\{\cos ix.\cos ix_k + \sin ix.\sin ix_k\} \\
&= \frac{1}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}(1 - \frac{i}{2n+1}) [\cos ix.\{\cos(m+i)x_k + \cos(m-i)x_k\} \\
&\qquad + \sin ix.\{\sin(m+i)x_k + \sin(m-i)x_k\}] \\
&= \frac{2n+1}{n+1}(1 - \frac{2n+1-m}{2n+1}) \cos(2n+1-m)x \\
&= \frac{2n+1}{n+1}\,\frac{m}{2n+1}\{\cos(2n+1)x.\cos mx + \sin(2n+1)x.\sin mx\} \\
&= \frac{1}{n+1}\left[\cos(2n+1)x.(\widetilde{\cos mx})' - \sin(2n+1)x.(\cos mx)'\right]
\end{aligned}$$

Similarly we can get (ii). ∎

Lemma 5. *If* $T \in T_n$, *then*

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(ii) $V_n^d(\sin mt,x) - \sin mx = \frac{1}{n+1}\left[\cos(2n+1)x.(\widetilde{\sin mx})' - \sin(2n+1)x.(\sin mx)'\right]$.

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V_n^d(\cos mt,x) - \cos mx &= -\cos mx + \frac{2}{2n+1}\sum_{k=0}^{2n} \cos mx_k\, D_n(x-x_k) \\
&+ \frac{2}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}\left(1-\frac{i}{2n+1}\right)\cos mx_k.\cos i(x-x_k) \\
&= \frac{2}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}\left(1-\frac{i}{2n+1}\right)\cos mx_k.\{\cos ix.\cos ix_k + \sin ix.\sin ix_k\} \\
&= \frac{1}{n+1}\sum_{k=0}^{2n}\sum_{i=n+1}^{2n}\left(1-\frac{i}{2n+1}\right)[\cos ix.\{\cos(m+i)x_k + \cos(m-i)x_k\} \\
&\qquad + \sin ix.\{\sin(m+i)x_k + \sin(m-i)x_k\}] \\
&= \frac{2n+1}{n+1}\left(1-\frac{2n+1-m}{2n+1}\right)\cos(2n+1-m)x \\
&= \frac{2n+1}{n+1}\,\frac{m}{2n+1}\{\cos(2n+1)x.\cos mx + \sin(2n+1)x.\sin mx\} \\
&= \frac{1}{n+1}\left[\cos(2n+1)x.(\widetilde{\cos mx})' - \sin(2n+1)x.(\cos mx)'\right]
\end{aligned}$$

Similarly we can get (ii). ■

$$\begin{aligned}
&\leq E_n(f)_{\frac{1}{n},p} + \frac{c}{n}\|P_n'(f)\|_p + \frac{c}{n}\|\tilde{P}_n'(f)\|_p \\
&\quad + \frac{c}{n}\|P_n'(f)\|_p + \frac{c}{n}\|\tilde{P}_n'(f)\|_p + c\|f - P_n(f)\|_{\frac{1}{n},p} \\
&\leq (1+c)E_n(f)_{\frac{1}{n},p} + cK(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) \\
&\leq cK(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1)\ .
\end{aligned}$$

Similarly from (34) and (35), we can prove theorem 1′. ■

P r o o f o f t h e o r e m 2. We shall use the following inequality

$$K(f, \frac{1}{n}; L_p, W_p^1, \tilde{W}_p^1) \leq \frac{c}{n}\sum_{k=1}^{n} E_k(f)_p\,, \tag{36}$$

which we can prove by the standard technique of telescopic sums and Bernstein inequalities (24) and (25). Let $g \in T_n$ be such that $\|f - g\|_{\frac{1}{n},p} = E_n(f)_{\frac{1}{n},p}$ and $h \in T_n$ such that (see (31))

$$2\inf\left\{\|f - T\|_p + \frac{1}{n}\|T'\|_p + \|\tilde{T}'\|_p\ :\ T \in T_n\right\}$$

$$\geq \|f - h\|_p + \frac{1}{n}\|h'\|_p + \|\tilde{h}'\|_p\,.$$

Then from (24), (25), (32) and (36) ($J_n(f) \in T_n$), we obtain (we follow the proof of theorem 4.2 in [2])

$$\begin{aligned}
K(f, \frac{1}{n}; L_{\frac{1}{n},p}, W_p^1, \tilde{W}_p^1) &\leq \|f - g\|_{\frac{1}{n},p} + \frac{1}{n}\|g'\|_p + \frac{1}{n}\|\tilde{g}'\|_p \\
&\leq E_n(f)_{\frac{1}{n},p} + \frac{1}{n}\|g' - h'\|_p + \frac{1}{n}\|\tilde{g}' - \tilde{h}'\|_p + \frac{1}{n}\|h'\|_p + \frac{1}{n}\|\tilde{h}'\|_p \\
&\leq E_n(f)_{\frac{1}{n},p} + 2\|g - h\|_p + \frac{1}{n}\|h'\|_p + \frac{1}{n}\|\tilde{h}'\|_p \\
&\leq E_n(f)_{\frac{1}{n},p} + 2\|f - g\|_p + 2\|f - h\|_p + \frac{1}{n}\|h'\|_p + \frac{1}{n}\|\tilde{h}'\|_p \\
&\leq E_n(f)_{\frac{1}{n},p} + 2\|f - g\|_{\frac{1}{n},p} + 2\|f - h\|_p + \frac{1}{n}\|h'\|_p + \frac{1}{n}\|\tilde{h}'\|_p \\
&\leq 3E_n(f)_{\frac{1}{n},p} + cK(f, \frac{1}{n}; L_p, W_p^1, \tilde{W}_p^1) \\
&\leq 3\|f - J_n(f)\|_{\frac{1}{n},p} + \frac{c}{n}\sum_{k=1}^{n} E_k(f)_p \\
&\leq c\|f - J_n(f)\|_{\frac{1}{n},p} + \frac{c}{n}\sum_{k=1}^{n}\|f - J_k(f)\|_p\,.
\end{aligned}$$

Similarly by using (34) and (35), we can prove theorem 2′. ■

Proof of theorem 3. Using the function g from Lemma 1 and (18), we get

$$\begin{aligned} K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1) &\le \|f-g\|_{\frac{1}{n},p}+\frac{1}{n}\|g'\|_p+\frac{1}{n}\|\tilde{g}'\|_p \\ &\le c\tau(f,\frac{1}{n})_p+\omega(f,\frac{1}{n})_p+\omega(\tilde{f},\frac{1}{n})_p \\ &\le c\tau(f,\frac{1}{n})_p+\omega(\tilde{f},\frac{1}{n})_p \end{aligned}$$

which proves (i). From M.Riesz inequality $\|\tilde{f}\|_p \le c_p\|f\|_p$, $f\in L_p$, $1<p<\infty$ ([6], p.253). From (19), we have

$$K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1) \le K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1) \le c_pK(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1).$$

By using (27), (i), (ii), we get (the function g is Steklov function)

$$\begin{aligned} K(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1,\tilde{W}_p^1) &\le \|f-g\|_{\frac{1}{n},p}+\frac{1}{n}\|g'\|_p+\frac{1}{n}\|\tilde{g}'\|_p \\ &\le \|f-g\|_{\frac{1}{n},p}+\frac{1}{n}\|g'\|_p+\frac{c_p}{n}\|g'\|_p \\ &\le c_p\tau(f,\frac{1}{n})_p. \end{aligned}$$

From (15) and (17), $(1<p<\infty)$, we have

$$\begin{aligned} \tau(f,\frac{1}{n})_p &\le \tau(f-g,\frac{1}{n})_p+\tau(g,\frac{1}{n})_p \\ &\le 2\|f-g\|_{\frac{1}{n},p}+\frac{1}{n}\|g'\|_p \\ &\le 2\{\|f-g\|_{\frac{1}{n},p}+\frac{1}{n}\|g'\|_p\}, \end{aligned}$$

If we take the infimum of $g\in W_p^1\cap\tilde{W}_p^1$, we obtain

$$\tau(f,\frac{1}{n})_p \le cK(f,\frac{1}{n};L_{\frac{1}{n},p},W_p^1),$$

which completes the proof of (ii).

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