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INSERTION OF $E_p(\lambda)$ TO L_∞ FOR THE BEST APPROXIMATION IN HAAR'S SYSTEM OF FUNCTIONS IF $0< p <1$

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Introduction. Let $\{X_n(t)\}_{n=1}^{\infty}$ be a Haar's orthonormal system of functions defined in $[0,1]$ as follows:

$$X_i(t) = 1, \text{ if } t \in [0,1] \text{ and for } n = 2^m + k,$$

where $k = 1, 2, \ldots, 2^m$ and $m = 0, 1, \ldots$

$$X_n(t) = \begin{cases} \sqrt{2^m}, & \text{for } t \in (\frac{2k-2}{2^{m+1}}, \frac{2k-1}{2^{m+1}}), \\ -\sqrt{2^m}, & \text{for } t \in (\frac{2k-1}{2^{m+1}}, \frac{2k}{2^{m+1}}), \\ 0, & \text{for } t \notin [\frac{k-1}{2^m}, \frac{k}{2^m}]. \end{cases}$$

At the points of discontinuity the Haar's functions are equal to the arithmetic mean of left- and right-hand limits, and furthermore $X_n(0) = \lim_{t\to 0+} X_n(t)$, $X_n(1) = \lim_{t\to 1-} X_n(t)$ ([2]).

For $0 < p < \infty$ we shall denote by $L_p[0,1]$ the space of all measurable functions f defined in $[0,1]$, such that

$$\|f\|_p = \left\{\int_0^\infty |f(x)|^p dx\right\}^{\frac{1}{p}} < \infty.$$

It is known that L_p is a Banach space with the norm $\|f\|_p$ if $1 \le p \le \infty$, and for $0 < p < 1$, L_p is a Fréchet space with the metric

$$d_p(f,g) = \|f - g\|_p^p.$$

For $f \in L_p, (0 < p \le \infty)$ we shull denote by

$$E_n^{(p)}(f) = \inf_{\{a_k\}} \|f - \sum_{k=1}^n a_k . X_k\|_p, \quad (n = 1, 2, \ldots),$$

where $\|f\|_\infty = \underset{0\le x\le 1}{\text{ess sup}}|f(x)|$.

It is obvious that $E_n^{(p)}(f)$ expresses the best approximation of the function $f \in L_p$ by Haar's polynomials of not more than n-th degree.

Let $0 < p < \infty$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be the sequence of real positive numbers such that $\lambda_n \searrow 0$. The symbol $E_p(\lambda)$ will stand for the class of all functions $f \in L_p$ with the property $E_n^{(p)}(f) = O(\lambda_n)$.

The following theorem is valid (see [2]):

Theorem A. *If $1 \le p < \infty$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequencs of positive real numbers, such that $\lambda_n \searrow 0$, then*

$$E_p(\lambda) \subset L_\infty \quad \text{iff} \quad \sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n < \infty \quad \text{holds.}$$

The goal of this paper is to prove the same theorem also for the case $0 < p < 1$.

Lemma 1. *Suppose that $0 < p < 1, q \ge 1$ and $\{\nu_k\}_{k=1}^{\infty}$ is an increasing sequence of integers. Then*

$$\sum_{i=\nu_k}^{\nu_{k+1}-1} i^{\frac{q}{p}-1} < \int_{\nu_k}^{\nu_{k+1}} x^{\frac{q}{p}-1}dx \quad \text{holds for} \quad k = 1, 2.\ldots.$$

Proof. From the conditions of Lemma 1 it follows that $\frac{q}{p} - 1 > 0$, so the function $f(x) = x^{\frac{q}{p}-1}$ is increasing in the interval $[\nu_k, \nu_{k+1}]$. With the help of Lagrange's meanvalue theorem we can easily verify that

$$i^{\frac{q}{p}-1} < \int_i^{i+1} x^{\frac{q}{p}-1}dx \quad (i = \nu_k, \nu_k + 1, \ldots, \nu_{k+1} - 1).$$

After the addition of these inequalities we obtain

$$\sum_{i=\nu_k}^{\nu_{k+1}-1} i^{\frac{q}{p}-1} < \sum_{i=\nu_k}^{\nu_{k+1}-1} \int_i^{i+1} x^{\frac{q}{p}-1}dx = \int_{\nu_k}^{\nu_{k+1}} x^{\frac{q}{p}-1}dx. \ \square$$

The definition of Haar's functions implies the lemma:

Lemma 2. *Let $0 < p < \infty$ and $2^m < n < 2^{m+1}$. Then*

$$\|X_n\|_p = 2^{m(\frac{1}{2}-\frac{1}{p})} \quad \text{holds.}$$

Lemma 3. *Suppose that $0 < p < 1$ and $\{a_i\}_{i=1}^{\infty}$ is a sequence of real numbers. Then*

$$|\sum_{i=1}^{\infty} a_i|^p \leq \sum_{i=1}^{\infty} |a_i|^p \quad \textit{holds.}$$

P r o o f. It is enough to show that $(|a_1| + |a_2|)^p \leq |a_1|^p + |a_2|^p$. We can propose that $a_2 \neq 0$ and $|a_1| \geq |a_2|$.

Since $0 < p < 1$ then the function $f(x) = x^{p-1}$ is decreasing in $[1, \infty]$, so from the meanvalue theorem for a suitable $c \in (x, x+1)$ we obtan

$$(1-x)^p - x^p = pc^{p-1} \leq x^{p-1} \leq 1, \quad \text{that is}$$

$$(1-x)^p - x^p \leq 1, \quad \text{for all} \quad x \in [1, \infty).$$

The substitution $x = \frac{|a_1|}{|a_2|} \geq 1$ implies the desired inequality. □

Lemma 4. *Let $0 < p < q < \infty$ and $\{\lambda_n\}_{n=1}^{\infty}$, respectively $\{\mu_n\}_{n=1}^{\infty}$ be sequences of real positive terms such that $\lambda_n \downarrow 0$ and $\mu_n \downarrow 0$. The sufficient condition for the insertion $E_p(\lambda) \subset E_q(\mu)$ is*

$$n^{\frac{1}{p}-\frac{1}{q}} \lambda_n + \Big[\sum_{k=n+1}^{\infty} k^{\frac{q}{p}-2} \lambda_k^q \Big]^{\frac{1}{q}} = O(\mu_n).$$

P r o o f. It follows from [1] (see Theorem 2.4). □

Main Result.

Theorem. *Let $0 < p < 1$ and $\{\lambda_n\}_{n=1}^{\infty}$, be a sequence of positive real numbers such that $\lambda_n \downarrow 0$ and $\lambda_n n^{\frac{1}{p}-2} \downarrow 0$. Then $E_p(\lambda) \subset L_\infty$ iff*

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-1} \lambda_n < \infty. \tag{1}$$

P r o o f. Suppose that $0 < p < 1$ and (1) is valid. Then all the more

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-2} \lambda_n < \infty. \tag{2}$$

holds.

Let us define a sequence $\{\mu_n\}_{n=1}^{\infty}$ as follows:

$$\mu_n = n^{\frac{1}{p}-1} \lambda_n + \sum_{k=n+1}^{\infty} k^{\frac{1}{p}-2} \lambda_k, \quad (n = 1, 2, \ldots).$$

Connections (1) and (2) imply that $\mu_n \downarrow 0$, so from Lemma 4 for the sequence $\{\mu_n\}_{n=1}^{\infty}$ and $q=1$ we obtain $E_p(\lambda) \subset E_1(\mu)$. The question is whether $\sum_{n=1}^{\infty} \mu_n < \infty$ holds because afterwards from Theorem A we shall obtain $E_1(\mu) \subset L_\infty$, thus $E_p(\lambda) \subset L_\infty$. However from (1) we can derive

$$\sum_{n=1}^{\infty} \mu_n = \sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n + \sum_{n=1}^{\infty}\sum_{k=n+1}^{\infty} k^{\frac{1}{p}-2}\lambda_k <$$

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} k^{\frac{1}{p}-2}\lambda_k = 2\sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n < \infty.$$

Conversely. Let $\sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n = \infty$ and $0<p<1$. We shall define a sequence $\{\nu_j\}_{j=1}^{\infty}$ as follows:

$$\nu_1 = 1 \quad \text{and} \quad \nu_{j+1} = \min\{n : \lambda_n \le \frac{1}{2}\lambda_{\nu_j}\}, \quad (j = 1, 2, \ldots).$$

Then obviously

$$\lambda_{\nu_{j+1}} \le \frac{1}{2}\lambda_{\nu_j} \quad \text{and} \quad \lambda_{\nu_{j+1}-1} \ge \frac{1}{2}\lambda_{\nu_j}. \tag{3}$$

Furthermore Lemma 1 and the fact that $\{\lambda_n\}_{n=1}^{\infty}$ is a non-decreasing sequence imply

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-1}\lambda_n = \sum_{k=1}^{\infty}\sum_{i=\nu_k}^{\nu_{k+1}-1} i^{\frac{1}{p}-1}\lambda_i < \sum_{k=1}^{\infty}\lambda_{\nu_k}\int_{\nu_k}^{\nu_{k+1}} x^{\frac{1}{p}-1}dx < \sum_{k=1}^{\infty} p\lambda_\nu \nu_{k+1}^{\frac{1}{p}}.$$

Thus

$$\sum_{k=1}^{\infty} p\lambda_{\nu_k}\nu_{k+1}^{\frac{1}{p}} = \infty. \tag{4}$$

Let us denote by

$$m_k = \max\{n : 2^n < \nu_{k+1}\}, \quad (k = 1, 2, \ldots). \tag{5}$$

One can easily realize that the proof will be fulfilled if there exists a function $f \in E_p(\lambda)$ such that $f \notin L_\infty$. Let us show tat the function to be found can be defined by

$$f(x) = \sum_{k=1}^{\infty} 2^{m_k(\frac{1}{p}-\frac{1}{2})}\lambda_{\nu_k} X_{2^{m_k}+1}(x).$$

Taking into account Lemma 3, Lemma 2 and (3) we have

$$\|f(x)\|_p^p = \int_0^1 \sum_{k=1}^{\infty} 2^{m_k(\frac{1}{p}-\frac{1}{2})} \lambda_{\nu_k} X_{2^{m_k}+1}(x)^p dx \leq$$

$$\sum_{k=1}^{\infty} 2^{m_k(1-\frac{p}{2})} \lambda_{\nu_k}^p \|X_{2^{m_k}+1}(x)^p\|_p^p =$$

$$\sum_{k=1}^{\infty} \lambda_{\nu_k}^p \quad \leq \quad \sum_{k=0}^{\infty} \lambda_{\nu_1}^p (2^{-p})^k \quad < \quad \infty, \quad \text{i.e. } f \in L_p.$$

Moreover it is necessary to prove that $E_p(\lambda) = O(\lambda_n)$.
Let n be a constant integer and

$$2^{m_{k-1}} + 1 \leq n < 2^{m_k} \quad (m_k \text{ is defined in } (5)).$$

Choose constants a_i $(i = 1, 2, \ldots, n)$ as follows:

$$a_j = \begin{cases} 2^{m_j(\frac{1}{p}-\frac{1}{2})} \lambda_{\nu_j}, & \text{for } i = 2^{m_j} + 1 \ (j = 1, 2 \ldots, k-1) \\ 0 & \text{for other } i. \end{cases}$$

Then from Lemma 3, Lemma 2 and (3) we obtain

$$(E_n^{(p)}(f))^p \leq \| \sum_{j=1}^{\infty} 2^{m_j(\frac{1}{p}-\frac{1}{2})} \lambda_{\nu_j} X_{2^{m_k}+1} - \sum_{i=1}^{\infty} a_i X_i \|_p^p =$$

$$\| \sum_{j=k}^{\infty} 2^{m_j(\frac{1}{p}-\frac{1}{2})} \lambda_{\nu_j} X_{2^{m_j}+1} \|_p^p \leq \sum_{j=k}^{\infty} 2^{m_j(1-\frac{p}{2})} \lambda_{\nu_j}^p \|X_{2^{m_j}+1}\|_p^p =$$

$$\sum_{j=k}^{\infty} \lambda_{\nu_j}^p \leq \lambda_{\nu_k}^p + \sum_{j=0}^{\infty} \lambda_{\nu_{k+1}}^p (2^{-p})^j = \lambda_{\nu_k}^p + \lambda_{\nu_{k+1}}^p \frac{1}{1-2^{-p}}$$

There $2^{m_{k-1}} + 1 < n < 2^{m_k} < \nu_{k+1}$ holds as we can see from (5) and since the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is non-decreasing, $\lambda_{\nu_{k+1}} \leq \lambda_n$ holds as well. The second part of (3) and the inequality $\nu_{k+1} - 1 \geq n$ imply that $\lambda_{\nu_k} < 2\lambda_n$.

Thus $(E_n^{(p)}(f))^p \leq \lambda_n^p(2^p + \frac{1}{1-2^{-p}})$, what is equivalent to the equality $E_n^{(p)}(f) = O(\lambda_n)$.

We shall prove that $f \notin L_\infty$. Assume that $f \in L_\infty$ and let us define functions $H_n(x) = \sum_{k=1}^{n} 2^{\frac{k}{2}} X_{2^k+1}(x)$ for $n = 1, 2, \ldots$.

Evidently we have:

$$(6) \qquad H_n(x) = \begin{cases} 2^{n+1} - 2, & \text{for} \quad x \in (0, \frac{1}{2^{n+1}}), \\ -2, & \text{for} \quad x \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}), \ \text{for every} \ k = 1, 2 \ldots, n, \\ 0 & \text{for} \ x \in (\frac{1}{2}, 1) \end{cases}$$

For every integer n there exists an integer $i(n)$ such that $m_{i(n)} \leq n < m_{i(n)+1}$, i.e. by the orthonormality of Haar's system we have

$$(7) \qquad \int_0^1 f(x) H_n(x) dx = \sum_{k=1}^{i(n)} 2^{m_k \frac{1}{p}} \lambda_{\nu_k}.$$

Furtheremore by Hölder's inequality

$$(8) \qquad \int_0^1 f(x) H_n(x) dx \leq \|f\|_\infty \|H_n\|_1$$

holds.

From (6) it is evident that

$$(9) \qquad \|H_n\|_1 = \int_0^1 |H_n(x0) dx < 2,$$

so $\|H_n\|_1$ is bounded for $n = 1, 2 \ldots$.

The connection (8) with (7) and (9) implies that

$$\sum_{k=1}^{i(n)} 2^{m_k \frac{1}{p}} \lambda_{\nu_k} \leq 2\|f\|_\infty \ \text{holds for arbitrary} \ n.$$

Since $i(n) \to \infty$ as $n \to \infty$ and $2^{m_k} < \nu_{k+1} < 2^{m_k+1}$, therefore

$$\|f\|_\infty \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{m_k \frac{1}{p}} \lambda_{\nu_k} \geq \frac{1}{2} \sum_{k=1}^{\infty} 2^{-p} \nu_{k+1}^{\frac{1}{p}} \lambda_{\nu_k}.$$

From (4) we can easily realize that $\|f\|_\infty = \infty$ what is a contradiction with the assumption $f \in L_\infty$. □

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