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FINITE HINKEL-CLIFFORD TRANSFORMATIONS OF THE FIRST KIND OF ARBITRARY ORDER

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ABSTRACT. In this paper, two series expansions of Fourier-Bessel type involving the functions $C_\nu(x) = x^{-\nu/2}J_\nu(2\sqrt{x})$ and $C^*_\nu(x) = (-1)^\nu x^\nu C_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν, are exhaustively investigated. This suggests the introduction of the finite Hankel-Clifford transformation of the first kind and arbitrary real order ν throughout

$$H_{1,\nu}\{f(x)\} = \bar{f}_{1,\nu}(n) = \int_0^a x^\nu \mathfrak{C}_\nu(\lambda_n x) f(x)dx, \quad a > 0$$

where

$$\mathfrak{C}_\nu(z) = \begin{cases} C_\nu(z), & \text{if} \quad \nu \geq 0 \\ C^*_\nu(z), & \text{if} \quad \nu \leq 0 \end{cases}$$

and λ_n denote the n-tn positive root of the equation $\mathfrak{C}_\nu(\lambda a) = 0$. The operational calculs generated is used in solving some partial differential equations with contain the Kerpinski operator $\Lambda_\nu = xD^2 + (1+\nu)D$, for any real value of the paramer-er ν, removing the classical restriction $\nu \geq -1/2$ imposed to the finite Hankel transforms.

1. Introduction. The Bessel-Clifford function $C_\nu(x)$ of the first kind of order ν [5,7,8,8,18] verifies the differential equation

$$xy'' + (1+\nu)y' + y = 0 \tag{1}$$

and is closely related to the Bessel one by

$$C_\nu(x) = x^{-\nu/2}J_\nu(2\sqrt{x}) \tag{2}$$

This function possesses the following series expansion

$$C_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\Gamma(\nu+r+1)} \tag{3}$$

It is to be observed that when $\nu = n$ is an integer, we have

$$C_{-n}(x) = (-1)^n x^n C_n(x) \tag{4}$$

A second solution of (1) is given by

$$D_\nu(x) = \frac{C_\nu(x)\cos\nu\pi - x^{-\nu}C_{-\nu}(x)}{\sin\nu\pi}, \tag{5}$$

for any ν not an integer. Because of (4), if ν is zero or an integer we define

$$D_n(x) = \lim_{\nu\to n} D_\nu(x).$$

Thus, in any case, $C_\nu(x)$ and $D_\nu(x)$ consistute a fundamental system of (1). On the other hand, the functions

$$C_\nu^*(x) = (-1)^\nu x^\nu C_\nu(x), \qquad D_\nu^*(x) = (-1)^\nu x^\nu D_\nu(x) \tag{6}$$

are linearly independent solutions of this other equation

$$xy'' + (1-\nu)y' + y = 0. \tag{7}$$

In view of (1), (6) and (7), note that the mutiplication by x^ν implies only the change of sign in the parameter ν.

We shall frequently need the formulas [9, p.69]

$$D^r C_\nu(x) = (-1)^r C_{\nu+r}(x), \quad r = 0,1,2,\dots \tag{8}$$

$$D^r\left[x^{\nu+r}C_{\nu+r}(x)\right] = x^\nu C_\nu(x), \quad r = 0,1,2,\dots \tag{9}$$

and the asymptotic expansions [9, p.144]

$$C_\nu(z) = \mathbf{O}(1), \quad \text{as } z \to 0 \tag{10}$$

and

$$C_\nu(z) = \frac{z^{-\nu/2-1/4}}{2\sqrt{\pi}}\left\{e^{1(2\sqrt{z}-\nu\pi/2-\pi/4)} + e^{-i(2\sqrt{z}-\nu\pi/2-\pi/4)}\right\} + \mathbf{O}(z^{-\nu/2-3/4}), \tag{11}$$

as $|z| \to \infty$, $|\arg z| < \pi$.

Consider now the Sturm-Liouville problem [17,p.446]

$$\begin{cases} (\Lambda_{\nu,x} + \lambda)\varphi(x) = 0 \\ \varphi(1) = 0 \end{cases} \tag{12}$$

where

(13) $$\Lambda_{\nu,x} = \Lambda_\nu = xD^2 + (1+\nu)D = x^{-\nu}Dx^{\nu+1}D, \quad D = \frac{d}{dx}$$

For $\lambda > 0$ it follows from (3) that $C_\nu(-\lambda) > 0$ and $C_\nu(0) = \Gamma^{-1}(\nu+1)$. Hence, the equation

(14) $$C_\nu(\lambda) = 0, \quad \nu \geq 0,$$

has not any nonpositive real zero. Let us arrange the positive roots of (14) in ascending order of magnitude: $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ ([9, p.152]). Thus, $\{C_\nu(\lambda_n x)\}_{n\in\mathbb{N}}$ is the fammily of eigenfunctions of problem (12), λ_n standing for the n-th positive zero of (14).

The orthogonality condition [9, p.161]

(15) $$\int_0^1 x^\nu C_\nu(\lambda_m x)C_\nu(\lambda_n x)dx = \begin{cases} 0, & \text{if } m \neq n \\ \lambda_n C_{\nu+1}^2(\lambda_n); & \text{if } m = n \end{cases}$$

suggests to express an arbitrary function $f(x)$ defined in $0 < x < 1$ as a series expansion

(16) $$f(x) = \sum_{m=1}^{\infty} a_m C_\nu(\lambda_m x)$$

whose coefficients are given by

(17) $$a_m = \frac{1}{\lambda_m C_{\nu+1}^2(\lambda_m)} \int_0^1 t^\nu C_\nu(\lambda_m t) f(t) dt.$$

The series (16) will be called the Fourier-Bessel-Clifford series of function $f(x)$, due to its analogy with Fourier-Bessel expansion concerning the eigenfunction system $\{J_\nu(j_m x)\}_{m\in\mathbf{N}}$ ([19, p.576]).

The main objective of this paper is to establish rigorously the validity of the convergence of two series of type (16), as well as to study the finite integral transformations associated with these expansions, their properties and applications.

2. The Fourier-Bessel-Clifford series expansions. Although the convergence proof of the series (16) runs parallely to that of the Fourier-Bessel expansion [19, Ch.XVIII], some original aspects are worthwhile to be analised.

Considering now the partial sum

(18) $$S_n(x) = \sum_{m-1}^{n} a_m C_\nu(\lambda_m x).$$

If $T_n(t,x)$ denotes the finite sum

(19) $$T_n(t,x) = T_n(x,t) = \sum_{m=1}^{n} \frac{C_\nu(\lambda_m t) C_\nu(\lambda_m x)}{\lambda_m C_{\nu+1}^2(\lambda_m)},$$

by substituting in (18) the coefficients a_m by (17), we get

$$S_n(x) = \int_0^1 t^\nu T_n(x,t) f(t) dt. \tag{20}$$

In order to investigate the convergence of (16), we need some results in advance.

Lemma 1.

$$\lim_{n\to\infty} \int_0^1 t^\nu T_n(t,x) dt = 1, \quad 0 < x < 1, \tag{21}$$

$$\lim_{n\to\infty} \int_0^x t^\nu T_n(t,x) dt = 1/2, \quad 0 < x < 1, \tag{22}$$

$$\lim_{n\to\infty} \int_x^1 t^\nu T_n(t,x) dt = 1/2, \quad 0 < x < 1, \tag{23}$$

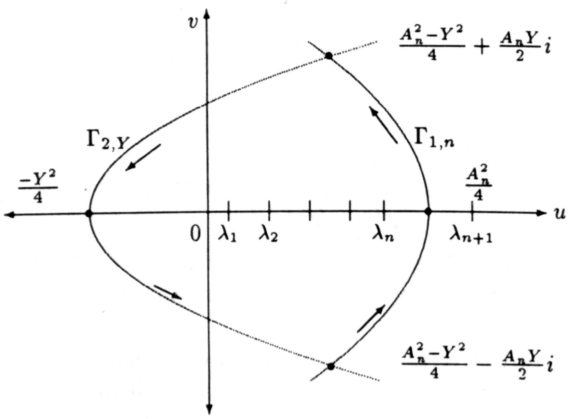

Fig. 1

P r o o f. Let Y be any fixed positive real number and choose A_n such that $A_n > 0$ and $\lambda_n < \frac{A_n^2}{4} < \lambda_{\nu+1}$, λ_n denoting the n-th positive zero of equation (14). The integral countour $\Gamma = \Gamma_{1,n} \cup \Gamma_{2,Y}$ of the figure 1 is suggested by J. Betancort in [1], where $\Gamma_{1,n}$ and $\Gamma_{2,Y}$ are, respectively, the finite parabolic arcs:

$$\Gamma_{1,n} = \{W = u + iv \in \mathbb{C} : \mathrm{Re} 2\sqrt{w} = A_n\} =$$

$$\{w = u + iv \in \mathbb{C} : u = \frac{A_n^2 - \lambda^2}{4}, v = \frac{A_n \lambda}{2}, \lambda \in [-Y, Y]\},$$

$$\Gamma_{2,Y} = \{w = u + iv \in \mathbb{C} : \mathrm{Im} 2\sqrt{w} = Y\} =$$

$$\{w = u + iv \in \mathbb{C} : u = \frac{\lambda^2 - Y^2}{4}, v = \frac{Y\lambda}{2}, \lambda \in [-A_n, A_n]\}.$$

Next we study the function

$$-\frac{C_\nu(wx)}{wC_\nu(w)}. \tag{24}$$

The residue at the pole $w = \lambda_m$, which is the m-th root of the equation $C_\nu(w) = 0$, is

$$\frac{C_\nu(\lambda_m x)}{\lambda_m C_{\nu+1}(\lambda_m)}.$$

Moreover, (24) has another pole at the origin, whose residue is -1.

We make use of Cauchy theorem of residues to obtain

$$\frac{1}{2\pi i}\oint_\Gamma \frac{-C_\nu(wx)}{wC_\nu(w)}dw = -1 + \sum_{m=1}^{n} \frac{C_\nu(\lambda_m x)}{\lambda_m C_{\nu+1}(\lambda_m)},$$

that is to say

$$\int_0^1 t^\nu T_n(t,x)dt = 1 - \frac{1}{2\pi i}\oint_\Gamma \frac{C_\nu(wx)}{wC_\nu(w)}dw, \tag{25}$$

by taking into account (19) and (9).

On the other hand, we infer immediately from (11)

$$\lim_{|w|\to\infty}\left|\frac{C_\nu(w)}{w^{-\nu/2-1/4}\{e^{i(2\sqrt{w}-\nu\pi/2-\pi/4)} + e^{-i(2\sqrt{w}-\nu\pi/2-\pi/4)}\}}\right| = \frac{1}{2\sqrt{\pi}}.$$

Consequently, there exist positive constants c_1 and c_2 so that

$$\begin{aligned} |C_\nu(w)| &\le c_1|w|^{-\nu/2-1/4}\cdot \exp|\mathrm{Im}(2\sqrt{w})| \\ |C_\nu(w)| &\ge c_2|w|^{-\nu/2-1/4}\cdot \exp|\mathrm{Im}(2\sqrt{w})| \end{aligned}, \tag{26}$$

for $|w|$ adequately large.

First of all, note that the use of (26) justifies the following result on $\Gamma_{2,Y}$:

$$\left|\int_{\Gamma_{2,Y}} \frac{C_\nu(wx)}{wC_\nu(w)}dw\right| \le \int_{-A_n}^{A_n} \frac{c_1 e^{-(1-\sqrt{x})|Y|}}{c_2 x^{\nu/2-1/4}|w|}|\lambda/2 + iY/2|d\lambda \le$$

$$\le \frac{4c_1 e^{-(1-\sqrt{x})Y}}{c_2 x^{\nu/2-1/4}}A_n \longrightarrow 0,$$

as $Y \to \infty$, independently of the value of n. Therefore, taking limits as $Y \to \infty$ in (25), we obtain

$$\int_0^1 t^\nu T_n(t,x)dt = 1 - \frac{1}{2\pi i}\int_{\Gamma_n} \frac{C_\nu(wx)}{wC_\nu(w)}dw, \tag{27}$$

where now Γ_n denotes the infinite parabola

$$\Gamma_n = \{w \in \mathbb{C} : \mathrm{Re}2\sqrt{w} = A_n\} =$$
$$\{w = u + iv \in \mathbb{C} : u = \tfrac{A_n^2-\lambda^2}{4}, v = \tfrac{A_n\lambda}{2}, -\infty < \lambda < \infty\} \tag{28}$$

Finaly, with the same sort of arguments we come to

$$\left|\int_{\Gamma_n} \frac{C_\nu(wx)}{wC_\nu(w)}dw\right| \le \frac{4c_1}{c_2 A_n(1-\sqrt{x})x^{\nu/2+1/4}} \to 0,$$

as $n \to \infty$, since $A_n \simeq (n + \nu/2 + 1/4)\pi$ ([9, p.156] and [19, p.584]). Letting $n \to \infty$ in (27) we deduce the desired result.

To establish (22), by invoking (9) and (19) we can write

$$\int_0^x t^\nu T_n(t,x)dt = \sum_{m=1}^n \frac{x^{\nu+1}C_\nu(\lambda_m x)C_{\nu+1}(\lambda_m x)}{\lambda_m C_{\nu+1}^2(\lambda_m)}. \tag{29}$$

Note that the general term of the sum on the right-hand side of (29) turns out to be the residue of the function

$$\frac{\pi w^\nu x^{\nu+1}\{C_\nu(w)D_\nu(xw) - C_\nu(xw)D_\nu(w)\}C_{\nu+1}(xw)}{C_\nu(w)}$$

at the simple pole $w = \lambda_m$, which is the m-th zero of equation (14). By applying again the Cauchy theorem it is easy to see that

$$\int_0^x t^\nu T_n(t,x)dt = \frac{x^{\nu+1}}{2i}\oint_\Gamma W^\nu\{C_\nu(w)D_\nu(xw) - C_\nu(xw)D_\nu(w)\}\frac{C_{\nu+1}(xw)}{C_\nu(w)}dw. \tag{30}$$

In order to evaluate last integral as $Y \to \infty$ and $n \to \infty$ independently, let$x$ be arbitrary fixed in $0 < x < 1$. Then, $|w|$ and $x|w|$ take larger values as $|w| \to \infty$. Therefore, the asymptotic expression (11) can be used to replace the right-hand side in (30) by

$$\frac{1}{4\pi i}\left\{\oint_\Gamma \frac{dw}{w} - \oint \frac{\cos(4\sqrt{wx} - 2\sqrt{w} - \nu\pi/2 - \pi/4}{w\cos(2\sqrt{w} - \nu\pi/2 - \pi/4)}dw\right\}. \tag{31}$$

It is evident that

$$\oint \frac{dw}{w} = 2\pi i,$$

while the second integral in (31) contributes $\mathbf{O}(1/n)$ over the arc $\Gamma_{1,n}$ and $\mathbf{O}(1/Y)$ over $\Gamma_{2,Y}$. Then, (22) is quikly inferred by taking in mind the above considerations, provided that $n \to \infty$ and $Y \to \infty$.

Making use of the aforementioned integral contour $\Gamma = \Gamma_{1,n} \cup \Gamma_{2,Y}$, the same procedure allows us to find the following integral representations of Hankel-Schläfli type [19, p.582] for $T_n(x,t)$, along the infinite parabola Γ_n given by (28):

$$T_n(x,t) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{tC_\nu(xw)C_{\nu+1}(tw) - xC_\nu(tw)C_{\nu+1}(xw)}{(t-x)C_\nu^2(w)}dw, \tag{32}$$

$0 < x < 1, 0 < t < 1, x \neq t$;

$$T_n(x,t) = \frac{1}{2i}\int_{\Gamma_n} \frac{w^\nu\{C_\nu(w)D_\nu(xw) - C_\nu(xw)D_\nu(w)\}C_\nu(tw)}{C_\nu(w)}dw, \tag{33}$$

$0 \le t < x < 1$; and

$$T_n(x,t) = \frac{1}{2i}\int_{\Gamma_n} \frac{w^\nu\{C_\nu(w)D_\nu(tw) - C_\nu(tw)D_\nu(w)\}C_\nu(xw)}{C_\nu(w)}dw, \tag{34}$$

$0 \le x < t < 1$.

We can exploit these formulas to get some interesting boundness for $T_n(x,t)$:

Lemma 2. *For certain positive constants K_1, K_2 and K_3 we have*

$$|T_n(x,t)| \le \frac{K_1(xt)^{-\nu/2-1/4}}{|t-x|(2-\sqrt{x}-\sqrt{t})}, \quad 0 < x,t < 1, \quad x \neq t$$

$$|T_n(x,t)| \le \frac{K_2(xt)^{-\nu/2-1/4}}{|\sqrt{x}-\sqrt{t}|},$$

$$\left|\int_0^t t^\nu T_n(x,t)(t-x)dt\right| \le \frac{K_3}{A_n(2-\sqrt{x}-\sqrt{t})}\left(\frac{t}{x}\right)^{\nu/2+1/4}.$$

Through a reasoning similar to the one given in [19, p.589] with respect to the Fourier-Bessel series and with the help of Lemma 2, we can assert.

Lemmma 3. *Let $\nu \ge 0$ and assume that $0 < a < b < 1$. If $f(t)$ is a function defined in the interval $(0,1)$ such that*

$$\int_0^1 t^{\nu/2-1/4}|f(t)|dt < \infty,$$

then one has

$$\lim_{n\to\infty}\int_a^b t^\nu T_n(x,t)f(t)dt = 0$$

for all $x \notin [a, b]$.

Next, we enunciate the most meaningful result of this section.

Theorem 1. *Let* $\nu \geq 0$. *Assume that* $f(t)$ *is a function defined and absolutely integrable on* $(0, 1)$ *such that*

$$\int_0^1 t^{\nu/2-1/4}|f(t)|dt < \infty.$$

Then, the Fourier-Bessel-Clifford series expansion

$$\sum_{m-1}^{\infty} a_m C_\nu(\lambda_m x)$$

whose coefficients are given by

$$a_m = \frac{1}{\lambda_m C_{\nu+1}^2(\lambda_m)} \int_0^1 t^\nu C_\nu(\lambda_m t) f(t) dt, \quad m = 1, 2, 3, \dots ,$$

converges and its sum is $\frac{1}{2}[f(x+0) + f(x-0)]$ *in a neiborhood of every point* $t = x$ *where* $f(t)$ *is of bounded variation.*

Proof. In view of (18) and (20) we can set

$$(35) \quad \begin{aligned} &S_n(x) - f(x-0) \int_0^x t^\nu T_n(x,t)dt - f(x+0) \int_x^1 t^\nu T_n(x,t)dt = \\ &\int_0^x t^\nu T_n(x,t)[f(t) - f(x-0)]dt + \int_x^1 t^\nu T_n(x,t)[f(t) - f(x+0)]dt \end{aligned}$$

Now taking limits as $n \to \infty$, we conclude that the left-hand side of (35) tends to

$$\lim_{n\to\infty} S_n(x) = \frac{1}{2}[f(x+0) + f(x-0)],$$

by virtue of (22) and (23), whereas the right-hand side vanishes by invoking Lemma 3 and another analytical considerations.

Remark 1. Expression (25) justifies rigorously the following Fourier-Bessel-Clifford expansion

$$1 = \sum_{n=1}^{\infty} \frac{C_\nu(\lambda_n x)}{\lambda_n C_{\nu+1}(\lambda_n)},$$

obtained formally by N.Hayek [9, p.167].

Remark 2. In the sequel $L_\nu^2(0,1)$ denote the linear space of all functions $f(t)$ square integrable on $(0,1)$ in relation with the weight function t^ν, which is such that

$$\int_0^1 t^\nu |f(t)|^2 dt < \infty.$$

Proceeding as in [20,10,2] we can establish

Theorem 2. *Let $\nu \geq 0$. If $f(t) \in L_\nu^2(0,1)$, then*

$$\lim_{n\to\infty} \int_0^1 t^\nu |f(t) - S_n(t)|^2 dt = 0.$$

In other words, the collection of eigenfunctions $\{C_\nu(\lambda_n x)\}_{n\in\mathbf{N}}$ is a complete orthogonal system in $L_\nu^2(0,1)$.

Remark 3. Now we pose the Sturm-Liouville problem

$$\left\{ \begin{array}{l} (\Lambda_\nu^* + \lambda)\varphi(x) = 0, \quad \nu \geq 0, \\ \varphi(1) = 0 \end{array} \right. \tag{36}$$

where

$$\Lambda_\nu^* = xD^2 + (1-\nu)D = Dx^{\nu+1}Dx^{-\nu}. \tag{37}$$

Because of (6) the general solution of (36) is expressed by

$$\varphi(x) = A(\lambda)C_\nu^*(\lambda x) + B(\lambda)D_\nu^*(\lambda x),$$

for arbitrary constants $A(\lambda)$ and $B(\lambda)$. If we choose the roots of $C_\nu^*(\lambda)$ and impose the boundary condition $\varphi(1) = 0$, it is inferred that $B(\lambda) = 0$. Observe, in this point, that $C_\nu^*(\lambda)$ possesses the same positive roots λ_m as the function $C_\nu(\lambda)$, since the additional zero $\lambda = 0$ occuring when $\nu > 0$ supplies no eigenfunction.

The family $\{C_\nu^*(\lambda_m x)\}_{n\in\mathbf{N}}$ also defines an orthogonal system of eigenfunction on the interval $(0,1)$ with respect to the weight function $x^{-\nu}$, verifying

$$\int_0^1 x^{-\nu} C_\nu^*(\lambda_m x) C_\nu^*(\lambda_n x) dx = \left\{ \begin{array}{ll} 0, & \text{if} \quad m \neq n, \\ \lambda_n^{2\nu+1} C_{\nu+1}^2(\lambda_n), & \text{if} \quad m = n \end{array} \right. .$$

Last orthogonality condition permits to express, at least formally, an arbitrary function $g(x)$ by means of the expansion

$$g(x) = \sum_{m=1}^{\infty} a_m^* C_\nu^*(\lambda_m x), \tag{38}$$

the coefficients being evaluated through

$$a_m^* = \frac{1}{\lambda_m^{2\nu+1} C_{\nu+1}^2(\lambda_m)} \int_0^1 t^{-\nu} C_\nu^*(\lambda_m t) g(t) dt. \tag{39}$$

We refer to (38)-(39) as the complementary Fourier-Bessel-Clifford expansion by its analogy with series (16)-(17), as we shall see later on.

Using argument we can prove a result analogous to Theorem 1.

Theorem 3. *Let $\nu \geq 0$. If $g(t)$ is a function absolutely integrable in the interval $(0,1)$ such that*

$$\int_0^1 t^{-\nu/2-1/4}|g(t)|dt < \infty,$$

then the Fourier-Bessel-Clifford series (38), the coefficients A_m^* being computed through (39), converges and its sum is $\frac{1}{2}[g(x+0)+g(x-0)]$ in a neighborhood of every point $t = x \in (0,1)$ where $g(t)$ is of bounded variation.

Remark 4. It is worth emphasizing here that the expansions (16)-(17) and (38)-(39) coincide only when ν is an integer. It is obvious for $\nu = 0$. Assume that $\nu = r$ denote a positive integer, the same reasoning being appropriate for any negative integer. In view of (4) and the first formula in (6), the expansion (16) of negative integer order $\nu = -r$ becomes

$$f(x) = \sum_{m=1}^{\infty} a_m C_{-r}(\lambda_m x) = \sum_{m=1}^{\infty} a_m C_r^*(\lambda_m x), \tag{40}$$

since $C_{-r}(x) = C_r^*(x)$, $r = 0, 1, 2, \ldots$, where

$$a_m = \frac{1}{\lambda_m C_{-r+1}^2(\lambda_m)} \int_0^1 t^{-r} C_{-r}(\lambda_m t) f(t) dt. \tag{41}$$

On the other hand, (1) and (8) provide the recurrence relation

$$xC_{\nu+2}(x) - (\nu+1)C_{\nu+1}(x) + C_\nu(x) = 0. \tag{42}$$

If we put $\nu = r - 1$ and invoke (4) and (14), we led to

$$C_{-r+1}^2(\lambda_m) = [(-1)^{r-1}\lambda_m^{r-1}C_{r-1}(\lambda_m)]^2 = \lambda_m^{2r}C_{r+1}^2(\lambda_m).$$

By this reason, (41) adopts definitively the form

$$a_m = \frac{1}{\lambda_m^{2\nu+1}C_{r+1}^2(\lambda_m)} \int_0^1 t^{-r} C_r^*(\lambda_m t) f(t) dt \tag{43}$$

which implies that $a_m = a_m^*$.

By comparing now (40)-(43) with (38)-(39) we can conclude that the Fourier-Bessel-Clifford series (16)-(17) of negative integer order agrees with its complementary expansion of positive integer order. In other words, the series expansion (38)-(39)

of positive integer order can be employed to replace the coresponding Fourier-Bessel-Clifford expansion of negative integer index.

3. The finite Hankel-Clifford integral transformation of first kind. The theory developed in the preceding section enable us to introduce immediately the finite Hankel-Clifford integral transformation of the first kind order $\nu \geq 0$ of a function defined in $0 \leq x \leq a$ through

$$b_{1,\nu}\{f(x)\} = \bar{f}_{1,\nu}(n) = \int_0^a x^\nu C_\nu(\lambda_n x) f(x) dx. \tag{44}$$

The corresponding inversion formula

$$b_{1,\nu}^{-}1\{\bar{f}_{1,\nu}(n)\} = f(x) = \frac{1}{a^{\nu+2}} \sum_{n=1}^{\infty} \frac{\bar{r}_{1,\nu}(n)}{\lambda_n C_{\nu+1}^2(\lambda_n a)} C_\nu(\lambda_n x) \tag{45}$$

is suggested by the associated series espansion of Fourier-Bessel-Clifford type. Here λ_n denote the n=th root of the equation $C_\nu(\lambda a) = 0$.

Theorem 4. Inversion theorem. *Let* $\nu \geq 0$. *If* $f(t)$ *is a function absolutely integrable in* $(0, a)$ *such that*

$$\int_0^a t^{\nu/2-1/4} |f(t)| dt < \infty,$$

then the series expansion (45) *converges and its sum is*

$$b_{1,\nu}^{-1}\{\bar{f}_{1,\nu}(n)\} = \frac{1}{2}[f(x+0) + f(x-0)]$$

in a neighborhood of every point $t = x \in (0, a)$ *where* $f(t)$ *is of bounded variation.*

Proof. Note that the integral which defines the finite transformation (44) exists always. Indeed, by the virtue of (10) and (11) we have

$$\left| \int_0^a x^\nu C_\nu(\lambda_n x) f(x) dx \right| \leq K \int_0^a x^{\nu/2-1/4} |f(x)| dx,$$

for certain constant $K > 0$. The remaining of the assertion follows immediately from Theorem 1 concerning the ordinary convergence of the Fourier-Bessel-Clifford series.

Taking into account (8), (9) and (14) and integrating by parts twice, we get

$$b_{1,\nu}\{\Lambda_\nu f(x)\} = \int_0^a C_\nu(\lambda_n x) D x^{\nu+1} D f(x) dx =$$

$$= \left[x^{\nu+1} C_\nu(\lambda_n x) f'(x)\right]_0^a + \lambda_n \int_0^a x^{\nu+1} C_{\nu+1}(\lambda_n x) f'(x) dx =$$

$$\lambda_n\left[x^{\nu+1}C_{\nu+1}(\lambda_n x)f(x)\right]_0^a - \lambda_n\int_0^a x^\nu C_\nu(\lambda_n x)f(x)dx.$$

Under the assumptions $\nu \geq 0$ and $f(x) \in C^2([0,a])$, we find

$$b_{1,\nu}\{\Lambda_\nu f(x)\} = -\lambda_n b_{1,\nu}\{f(x)\} + \lambda_n a^{\nu+1}C_{\nu+1}(\lambda_n a)f(a).$$

Moreover, when $f(a) = 0$ it follows

$$b_{1,\nu}\{\Lambda_\nu f(x)\} = -\lambda_n b_{1,\nu}\{f(x)\}. \tag{46}$$

The complementary finite Hankel-Clifford integral transformation of first kind and order $\nu \geq 0$ of a function $g(x)$ defined in $0 \leq x \leq a$ is given by means of

$$b^*_{1,\nu}\{g(x)\} = \bar{g}^*_{1,\nu}(n) = \int_0^a x^{-\nu}C^*_\nu(\lambda_n x)g(x)\,dx, \tag{47}$$

whose inversion formula is expressed through

$$b^{*\,-1}_{1,\nu}\{\bar{g}^*_{1,\nu}(n)\} = g(x) = \frac{1}{a^{\nu+2}}\sum_{n=1}^\infty \frac{\bar{g}^*_{1,\nu}(n)}{\lambda_n^{2\nu+1}C^2_{\nu+1}(\lambda_n a)}C^*_\nu(\lambda_n x). \tag{48}$$

Starting from Theorem 3, an assertion similar to Theorem 4 is proved in relation with the finite transformation (47).

Theorem 5 (Inversion theorem). *Let $\nu \geq 0$. If $g(t)$ is a function absolutely integrable in the interval $(0,a)$ such that*

$$\int_0^a t^{-\nu/2-1/4}|g(t)|\,dt < \infty,$$

then the series expansion (48) *converges towards the sum*

$$b^{*\,-1}_{1,\nu}\{\bar{g}^*_{1,\nu}(n)\} = \frac{1}{2}[g(x+0) + g(x-0)]$$

in a neighborhood of every point $t = x \in (0,a)$ where $g(t)$ is of bounded variation.

If we assume that $\nu \geq 0$, $g(t) \in C^2([0,a])$ and $g(0) = g(a) = 0$, we get the most interesting operational rule of the transformation $b^*_{1,\nu}$ namely

$$b^*_{1,\nu}\{\Lambda^*_\nu g(x)\} = -\lambda_n b^*_{1,\nu}\{g(x)\}. \tag{49}$$

When the finite transformations $b_{1,\nu}$ and $b^*_{1,\nu}$ and their respective inversion formulas are studied in detail, it is easily seen that most of the porperties of both transforms differ only with respect to the sign of parameter ν. In fact:

(i) Note that the differential operators Λ_ν and Λ_ν^* of order $\nu \geq 0$ fulfil the relation $\Lambda_\nu^* = \Lambda_{-\nu}$, by virtue of (13) and (37).

(ii) Recall that $C_\nu(x)$ is a solution of (1), whereas $C_\nu^*(x)$ verifies (7). That is, the multiplication of $C_\nu(x)$ by x^ν has only repercussions on the sign of parameter ν.

(iii) On the one hand, $\{C_\nu(\lambda_m x)\}_{m\in\mathbb{N}}$ forms an orthogonal system with respect to the weight function x^ν; on the other hand, $\{C_\nu^*(\lambda_m x)\}_{m\in\mathbb{N}}$ constitues another orthogonal system with regard to $x^{-\nu}$.

(iv) We notice that the hypotheses in the paramount results (Theorems 1,3,4 and 5) distinguish exclusively in the sign of parameter ν.

(v) As an immediate consequence of Remark 4, it follows that the finite transformations (44), (47) and their inverses coincide for every integer value of νq namely,

$$b_{1,-r} = b_{1,r}^* \quad \text{and} \quad b_{1,-r}^{-1} = b_{1,r}^{*\ -1}, \quad r = 0, 1, 2, \ldots$$

All the above considerations lead us to introduce the notation

$$\mathfrak{C}_{\nu+s}(z) = \begin{cases} C_{\nu+s}(z), & \text{if} \quad \nu \geq 0 \\ C_{-\nu-s}^*(z), & \text{if} \quad \nu \leq 0 \end{cases} \qquad (s = 0, 1)$$

and

$$h_{1,\nu} = \begin{cases} b_{1,\nu}, & \text{if} \quad \nu \geq 0 \\ b_{1,-\nu}^*, & \text{if} \quad \nu \leq 0 \end{cases} .$$

In this fasion, the finite Hankel-Clifford integral transformation of the first kind and arbitrary real order is defined according to

$$h_{1,\nu}\{f(x)\} = \bar{f}_{1,\nu}(n) = \int_0^a x^\nu \mathfrak{C}_\nu(\lambda_n x) f(x)\, dx. \tag{50}$$

Its appropriate inversion formula will be given by

$$h_{1,\nu}^{-1}\{\bar{f}_{1,\nu}(n)\} = f(x) = \frac{1}{a^{\nu+2}} \sum_{n=1}^{\infty} \frac{\bar{f}_{1,\nu}(n)}{\lambda_n \mathfrak{C}_{\nu+1}^2(\lambda_n a)} \mathfrak{C}_\nu(\lambda_n x). \tag{51}$$

Thus, Theorems 4 and 5 are summarized in the unique assertion

Theorem 6. *Let ν be an arbitrary real number. If $f(t)$ is a function absolutely integrable in $(0, a)$ and satisfying the condition*

$$\int_0^a t^{\nu/2-1/4} |f(t)|\, dt < \infty,$$

then the series (51) *converges, its sun being*

$$h_{1,\nu}^{-1}\{\bar{f}_{1,\nu}(n)\} = \frac{1}{2}[f(x+0) + f(x-0)],$$

in a neighborhood of every point $t = x \in (0, a)$ where $f(t)$ is of bounded variation.

The operational formulas (46) and (49) can be regrouped in the expression

$$h_{1,\nu}\{\Lambda_\nu f(x)\} = -\lambda_n h_{1,\nu}\{f(x)\}, \tag{52}$$

which holds for any real ν, in accordance with the above convention.

To illustrate the applications of the finite Hankel-Clifford transformations, we wish to solve the following problem involving the Kepinski-Myller-Lebedev partial differential equation ([11], [19, p.99]), of course, now in a finite interval:

$$\begin{aligned} & x\frac{\partial^2 u}{\partial x^2} + (1+\nu)\frac{\partial u}{\partial x} - \mu\frac{\partial u}{\partial t} = 0, \quad 0 < x < a, \quad t > 0 \\ & u(a,t) = 0, \quad t \geq 0 \\ & u(x,0) = f(x), \quad 0 \leq x \leq a, \end{aligned} \tag{53}$$

where ν stands for an arbitrary parameter and $\mu > 0$.

Set $\bar{u}(n,t) = h_{1,\nu}\{u(x,t)\}$. By using the operational ryle (52), equation (53) converts into

$$-\lambda_n \bar{u}(n,t) - \mu\frac{\partial}{\partial t}\bar{u}(n,t) = 0,$$

whose solution is

$$\bar{u}(n,t) = \bar{f}_{1,\nu}(n) \cdot \exp(-\lambda_n t/\mu),$$

where $\bar{f}_{1,\nu}(n) = h_{1,\nu}\{f(x)\}$ and λ_n represents the n-th positive zero of the equation $C_\nu(\lambda a) = 0$. The inversion formula (51) provides us with the required solution

$$u(x,t) = \frac{1}{a^{\nu+2}} \sum_{n=1}^{\infty} \frac{\bar{f}_{1,\nu}(n) \cdot \exp(-\lambda_n t/\mu)}{\lambda_n \mathfrak{C}_{\nu+1}^2(\lambda_n a)} \mathfrak{C}_\nu(\lambda_n x). \tag{54}$$

That (54) is exactly the solution we need, can be shown through a procedure analogous to the described in [4, p.191].

Remark 5. The Fourier-Bessel series expansion ([4, p.181], [7, p.91], [19, p.576], [20], [21]) and the finite Hankel transformations ([16], [17, p.446]) and their applications have to be restructed to the case $\nu \geq -1/2$. In this work, the simultaneous research of the Fourier-Bessel-Clifford series expansions (6) and (38) and finite Hankel-Clifford transformations (44) and (47) allows us to carry out a unified approach. This culminates in the introduction of the finite transformation (50)-(51) of arbitrary order, which can be used in solving a class of Kepinski partial differential equaiton no matter what the real value of ν may be.

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