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ON PROCESSES ASSOCIATED WITH A SUPER – CRITICAL MARKOV BRANCHING PROCESS

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ABSTRACT. We study a super-critical Markov branching process (MBP) conditioned on the event of non-extinction. Such a process is a non-branching Markov process. We discuss limiting behaviour of such a process. Further, we construct an associated bivariate process which is based on the finite and infinite lines of descent of particles of the MBP. We show that this process is a bivariate super-critical MBP. The bivariate process retains the branching property when conditioned on events of extinction and non-extinction. Asymptotic results for the bivariate process are established. This approach also gives easier proofs of some of the known results.

Introduction. We consider an one-dimensional, super-critical Markov branching process (MBP) $\{X(t); t \geq 0\}$ defined over a Harris-tree probability space $(\omega, \mathcal{F}, \mathcal{P})$. Let $A = \{X(t) \neq 0$, for all $t \geq 0\}$ and $B = A^c$ be the events of non-extinction and extinction, respectively. Let $X(t, A)$ and $X(t, B)$ be the restrictions of $X(t)$ to the sets A and B, respectively. Let $(A, \mathcal{F}_A, \mathcal{P}_A)$ and $(B, \mathcal{F}_B, \mathcal{P}_B)$ be the conditional probability spaces over which $\{X(t, A); t \geq 0\}$ and $\{X(t, B); t \geq 0\}$ are defined respectively.

Following the arguments in Athreya and Ney ([3], p.52), it can be shown that the process $\{X(t, B); t \geq 0\}$ is a sub-critical MBP. The process $\{X(t, A); t \geq 0\}$ is, however, a non-branching Markov process; see Section 2. Further in Section 2, we study the process $\{X(t, A); t \geq 0\}$ in some detail and establish some asymptotic properties. However, it is more easy to establish central limit theorems for $\{X(t, A); t \geq 0\}$ via the process $\{Z(t); t \geq 0\}$ which is introduced below.

In Section 3, we study the bivariate process $\{Z(t); t \geq 0\}$ where $Z(t) = (Z^{(1)}(t), Z^{(2)}(t))$ for all $t \geq 0$ and $Z^{(1)}(t)$ and $Z^{(2)}(t)$ denote the number of particles among $X(t)$ with infinite and finite lines of descent, respectively. We show that $\{Z(t); t \geq 0\}$ is a bivariate, super-critical MBP. An earlier result in this direction is the branching property of $\{Z^{(1)}(t); t \geq 0\}$; see Athreya and Karlin ([2]) and Yakymiv ([9]). Let $Z(t, A)$ and $Z(t, B)$ be the restrictions of $Z(t)$ to the sets A and B, respectively. It turns out that each of the processes $\{Z(t,A); t \geq 0\}$ and $\{Z(t,B); t \geq 0\}$ is a MBP. The study of these associated processes allows us to obtain a number of results for $\{X(t,A); t \geq 0\}$ which are analogs of the results for a one-dimensional Galton-Watson branching process ($GWBP$). We refer to Gadag and Rajarshi ([5]) for an earlier work in this direction. Our results improve the understanding of the branching process. We also refer to Yakymiv ([9]) for the study of the associated processes obtained using a retrospective study of the pedigree from a reference time.

In Section 4, we establish a central limit theorem for the process $\{X(t,A); t \geq 0\}$ via corresponding result on the process $\{Z(t,A); t \geq 0\}$.

2. Properties of the process $\{X(t,A); t \geq O\}$. Let $\{X(t); t \geq 0\}$ be a MBP as described in Section 1. Let the expected life length of a particle in the process be b^{-1}, $0 < b < \infty$. By $f(s) = \sum_{k=0}^{\infty} p_k^{s^k}$, we denote the offspring probability generating function ($p.g.f.$). We assume that $P_0 > 0$ and $p_0 + p_1 < 1$. Let $u(s) = b[f(s) - s]$, $\lambda = u'(1)$ and $m(t) = E[X(t)|X(0) = 1]$. The transition probability function of the above Markov process would be denoted by $p_{ij}(t)$, whereas the $p.g.f.$ of $X(t)$ given that $X(0) = 1$ is denoted by $F(s,t)$. Let q be the smallest non-negative solution of the equation $u(s) = 0$.

Following Muthsam ([7]), we observe that the process $\{X(t,A); t \geq 0\}$ is a Markov process whose transition probability function is given by $p_{ij}^A(s, s+t) = [(1-q^j)/(1-q^i)]p_{ij}(t)$. If $P_A[X(0,A) = k] = 1$ and if $u'(1) < \infty$, we have

$$E_A[X(t,A)] = m(t,A) = \frac{k[m(t) - q^k m(t,B)]}{1-q^k}, \tag{1}$$

where $m(t,B) = e^{-\beta t}$ with $\beta = -u'(q)$. It now follows that the process $\{X(t,A); t \geq 0\}$ is non–branching Markov process.

3. The bivariate branching process $\{Z(t); t \geq O\}$. Let the particles on the infinite lines of descent be known as type one particles and those on finite lines of descent be known as type two particles. Since $\{X(t); t \geq 0\}$ is a Markov process (MP), we have the following representation for the process $\{Z(t); t \geq 0\}$:

$$Z(s+t) = \sum_{k=1}^{Z^{(1)}(t)} Z_k(s,1) + \sum_{k=1}^{Z^{(2)}(t)} Z_k(s,2) \tag{2}$$

with $Z_k(s,i) = (Z_k^{(1)}(s,i), Z_k^{(2)}(s,i)$, where $Z_k^{(j)}(s,i)$ represents the number of type j descendants at the epoch $s+t$ of the k–th member of the $Z(i)(t)$ particles living at t. In the expression (2), it is understood that the sum of the type $\sum_1^0$ equals zero.

The following result which gives the branching property of the process $\{Z(t); t \geq 0\}$ is the main result of this paper.

Theorem 3.1. *The process $\{Z(t); t \geq 0\}$ is a two-type, super-critical MBP which is nonsingular, non-positive regular and reducible.*

Proof. The fact that the process $Z(t); t \geq 0\}$ is a two–type MBP follows from the representation in (2). Now, arguing as in Theorem 2.1 of Gadag and Rajarshi ([5]), we find that the $p.g.f.$ of the process $Z(t); t \geq 0\}$ is given by

$$(3)\ F_a(\underset{\sim}{s}, t) = (F_a^{(1)}(\underset{\sim}{s}, t), F_a^{(2)}(\underset{\sim}{s}, t)) = \left\{ \frac{F(s_1, (1-q) + s_2 q, t) - F(s_2 q, t)}{1-q}, \frac{F(s_2 q, t)}{q} \right\},$$

with $\underset{\sim}{s} = (s_1, s_2)$ and $\max\{|s_1|, |s_2|\} \leq 1$. Here and throughout, the suffix $'a'$ indicates an expression for the associated process. We observe that the classification of the particles into two types is independent of the distribution of the life length and depends only on the offspring probability distribution. Hence, the infinitesimal generating function of the process $\{Z(t);\ t \geq 0\}$ is given by

$$(4) \qquad u_a(\underset{\sim}{s}) = (u_a^{(1)}(\underset{\sim}{s}), u_a^{(2)}(\underset{\sim}{s})) = \left(\frac{u(s_1(1-q) + s_2 q) - u(s_2 q)}{1-q}, \frac{u(s_2 q)}{q} \right).$$

The nonsingularity, non-positive regularity and the reducibility follow from (3). Using (3), after making some computations, we find that the largest eigen-value of A_a, the generator of the mean matrix of $\{Z(t); t \geq 0\}$ is λ. □

Remark 3.1. If $\underset{\sim}{u}_a$ and $\underset{\sim}{v}_a$ are the right and left eigen-vectors of A_a, corresponding to the eigen-value λ such that the inner products $<\underset{\sim}{u}_a, \underset{\sim}{v}_a> = <\underset{\sim}{u}_a, \underset{\sim}{1}> = 1$ (where $\underset{\sim}{1} = (1,1)$), then we have $\underset{\sim}{u}_{a'} = (1,0)$ and $\underset{\sim}{v}_a = (1, q/(1-q))$.

Corollary 3.1. *The process $\{Z^{(1)}(t, A); t \geq 0\}$, defined on $(A, \mathcal{F}_A, \mathcal{P}_A)$, is a super-critical Markov branching process with λ as the criticality parameter.*

Corollary 3.2. *The process $\{Z(t, A); t \geq 0\}$, defined on $(A, \mathcal{F}_A, \mathcal{P}_A)$, is a super–critical, two–type MBP. Further, this process is nonsingular, non–positive regular and reducible and its p.g.f. is given by (3).*

Remark 3.2. Corollary 3.1. is the result on page 278 of Athreya and Karlin [2]. It now follows from Pakes ([8]) that there exists a norming function $C(t)$ such that as $t \to \infty$,

$$(5) \qquad [Z^{(1)}(t, A)/C(t)] \to W_a \quad \text{a.s.} \ \ (P_A),$$

where W_a is scalar r.v. absolutely continuous on $(0, \infty)$ and $P_A(W_a > 0) = 1$. Further, $E_A(W_a) < \infty$, if and only if

$$(6) \qquad \sum j \log j p_{*j} < \infty,$$

where $\{p{*}j; j \geq 0\}$ is the offspring probability distribution of the process $\{Z^{(1)}(t, A); t \geq 0\}$. Furthermore, the norming function $C(t)$ is an increasing function of t, such that

(i) for $\tau > 0$, $[C(t+\tau)/C(t)] \to e^{\lambda\tau}$ as $t \to \infty$ and

(ii) $KC(t)e^{-\lambda t} \to 1$, as $t \to \infty$ for some constant K, if and only if (6) holds. Without loss of generality we can set $K = 1$.

Lemma 3.1. *As* $t \to \infty$, $\{Z^{(1)}(t,A)/X(t,A)\} \to 1-q$ *a.s.* (P_A).

Proof. Let $\{X(n\delta, A); n \geq 0\}$ and $\{Z^{(1)}(n\delta, A); n \geq 0\}$ be the discrete skeletons of $\{X(t,A); t \geq 0\}$ and $\{Z^{(1)}(t,A); t \geq 0\}$, respectively, where $\delta > 0$. From Theorem I.12.2 of Athreya and Ney [3] we have as $n \to \infty$,

$$[Z^{(1)}(n\delta,A)/X(n\delta,A)] \to 1-q \quad \text{a.s. } (P_A). \tag{7}$$

Since (7) holds for all $\delta > 0$, the Lemma follows. □

Remark 3.3. From Lemma 3.1 and (5), it follows that as $t \to \infty$,

$$[X(t,A)/C(t)] \to [W_a/(1-q) \quad \text{a.s. } (P_A). \tag{8}$$

Hence,

$$[Z^{(2)}(t,A)/C(t)] \to [W_a q/(1-q)] \quad \text{a.s. } (P_A). \tag{9}$$

Theorem 3.2. *Let* $Z(0,A) = (1,0)$. *Let* $C(t)$ *and* W_a *be as defined in Remark 3.2. Then,*

(i) $[Z(t,A)/C(t)] \to \underset{\sim}{v}_a W_a$ *a.s.* (P_A) *as* $t \to \infty$.

(ii) *Further, if* $u''(1) < \infty$, *we have*

$$Var_A[\underset{\sim}{v}_a W_a] = \begin{bmatrix} 1 & \frac{q}{1-q} \\ \frac{q}{1-q} & \frac{q^2}{(1-q)^2} \end{bmatrix} [(1-q)u''(1)\lambda^{-1} - 1]. \tag{10}$$

Proof. Part (i) follows from the expression in (5) and Remarks 3.1 and 3.3. Part (ii) follows from Remark 3.2, because under the condition of part (ii), $C(t)$ can be taken to be $e^{\lambda t}$. □

4. Central limit theorems. In this section, we establish a central limit theorem for the process $\{X(t,A); t \geq 0\}$ via the central limit theorem for the process $\{Z(t,A); t \geq 0\}$. Let $W_a(t,A) = e^{-\lambda t} Z(t,A)$ and $\underset{\sim}{e}_1 = (1,0)$.

Theorem 4.1. *Let* $Z(0,A) = \underset{\sim}{e}_1$. *Then,*

$$(i) \quad [X(t,A)]^{-\frac{1}{2}}[Z(t,A) - (1-q,q)X(t,A)] \xrightarrow{d} Z^*, \tag{11}$$

$$(ii) \quad [X(t,A)]^{-\frac{1}{2}}[Z(t,A) - (I - \underset{\sim}{u}_a \underset{\sim}{v}_a)] \xrightarrow{d} Z_1^*, \tag{12}$$

where $\xrightarrow{d}$ *denotes convergence in distribution,* $Z^* \sim N_2(\underset{\sim}{Q}, Q)$*,* $Z_1^* \sim N_2(\underset{\sim}{Q}, V)$*, with*

$$Q = q(1-q) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad V = \frac{q}{1-q} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

and each of the random variables (r.v.s.) Z^* *and* Z_1^* *are independent of the r.v.* W_a.

Proof. We first note that the conditional distribution of $Z(t, A)$ given $X(t, A)$, can be viewed as a truncated multinomial random vector with index $X(t, A)$ and probability vector $(1-q, q)$, since $X(t, A) > 0$ for every $t \geq 0$.

Let ξ_i take the values $\underset{\sim}{e}_1$ or $\underset{\sim}{e}_2$ as the i-th of the $X(t, A)$ particles has an infinite line of descent or a finite line of descent respectively. Let $\eta_i = \xi_i - (1-q, q)$ for $i = 1, 2, \ldots, X(t, A)$.

By the classical central limit theorem, as $n \to \infty$, $n^{-\frac{1}{2}}[\sum_{i=1}^{n} \eta_i] \xrightarrow{d} Z^*$, where $Z^* \sim N_2(\underset{\sim}{Q}, Q)$ with Q as given in (13). This limit law is mixing in the sense of Renyi (cf. Aldous and Eagleson [1] and references cited therein). If $\{\alpha_n; n \geq 0\}$ is a sequence of positive integer valued *r.v.s.* such that $\{\alpha_n / f_{(n)}\} \xrightarrow{P} \alpha$, where α is *a.s.* positive, then, when $f_{(n)} \uparrow \infty$, $\{\alpha_n^{-\frac{1}{2}} \sum^{\alpha_n} \eta_i\}$ is also mixing (Csörgo and Fishler [4]). Using the continuous time version of this result, it follows that

$$[X(t, A)]^{-\frac{1}{2}}[\sum_{i=1}^{X(t,A)} \eta_i] \xrightarrow{d} Z^* \text{ (mixing)} , \tag{14}$$

since $[X(t, A)e^{-\lambda t}] \xrightarrow{d} [W_a/(1-q)] > 0$ *a.s.* (P_A) and $e^{\lambda t} \uparrow \infty$. It also follows from Aldous and Eagleson ([1]) that Z^* and W_a are independent *r.v.s.*

Part (i) of the theorem now follows on noting that the difference between $[X(t, A)]^{-\frac{1}{2}}[Z(t, A) - (1-q, q)X(t, A)]$ and $[X(t, A)]^{-\frac{1}{2}}[\sum_{i=1}^{X(t,A)} \eta_i]$ converges to zero in probability as $t \to \infty$. Part (ii) of the theorem follows on choosing an appropriate linear transformation in (11).

Theorem 4.2. *Let* $Z(0, A) = \underset{\sim}{e}_1$. *Let* $u''(1) < \infty$. *Then,*

$$[X(t, A)]^{-\frac{1}{2}} e^{\lambda t}[W_a(t, A) - \underset{\sim}{v}_a W_a] \xrightarrow{d} X^*, \tag{15}$$

where X^* *and* W_a *are independent and* $X^* \sim N_2(\underset{\sim}{Q}, Q^*)$ *with*

$$Q^* = \begin{bmatrix} (1-q)^2 & q(1-q) \\ q(1-q) & q^2 \end{bmatrix} u''(1)\lambda^{-1} - \begin{bmatrix} 1-q & q \\ q & -q \end{bmatrix}.$$

Proof. We note that under the condition of the theorem, $C(t)$ can be replaced by $e^{\lambda t}$. Using the representation of type 2 for the $\{Z(t,A); t \geq 0\}$ process, we observe that,

$$[Z(t,A) - e^{-\lambda t} \underset{\sim}{v}_a W_a] = \sum_{i=1}^{Z^{(1)}(t,A)} \eta_i^* + Z(t,A)(I - \underset{\sim}{u}_a \underset{\sim}{v}_a), \tag{16}$$

where η_i^*'s are independent and identically distributed random vectors, independent of $Z^{(1)}(t,A)$, having mean 0 and covariance matrix given by (3.9). Since $[e^{-\lambda t} Z^{(1)}(t,A)] \to W_a > 0$ a.s (P_A) and $e^{\lambda t} \uparrow \infty$, applying the central limit theorem as in the earlier theorem, we have

$$[Z^{(1)}(t,A)]^{-\frac{1}{2}} [\sum_{i=1}^{Z^{(1)}(t,A)} \eta_i^*] \to Z_2^* \text{ (mixing)} . \tag{17}$$

The theorem now follows from expression (12), (16), (17) and Lemma 3.1, on noting that given $Z(t,A)$, the two terms on the right hand side of (4.6) are independent.

Remark 4.1. Using a linear transformation in (15), one can obtain a result which is the analog of the theorem 2.9.2 of Jagers [6] for $GWBP$.

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