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LINEAR VECTOR OPTIMIZATION. PROPERTIES OF THE EFFICIENT SETS

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ABSTRACT. It is proved that the majority (in the sense of Baire category) of the set of linear vector semi-infinite optimization problems are well-posed. The set of problems for which the set of p-efficient points coincides with the set of weakly p-efficient points is also discussed.

0. Introduction. This paper is motivated by some generic results in the scalar optimization concerning such notions as uniqueness of the solutions, well-posedness of optimization problems etc., given by Kenderov, Lucchetti and many other authors. Next we will use a similar approach so as to extend these results over the linear vector semi-infinite optimization.

First we shall set the definitions of the linear vector semi-infinite optimization:

Let T be a compact Hausdorff space and $\mathbf{R}^N$ be the usual N-dimensional Euclidean space. Let $B : T \longrightarrow \mathbf{R}^N$, $b : T \longrightarrow \mathbf{R}$ be continuous mappings and let $p_1, p_2, \dots, p_l$ be elements in $\mathbf{R}^N$.

Define $P : \mathbf{R}^N \longrightarrow \mathbf{R}^l$ as

$P(x) = (\langle p_1, x\rangle, \langle p_2, x\rangle, \dots, \langle p_l, x\rangle)$, $x \in \mathbf{R}^N$.

For each triplet

$\sigma \in \Theta = (B, b, P) \in C(T)^N \times C(T) \times \mathbf{R}^{Nl}$

we consider (as in [1,2]) the closed subset $Z(\sigma)$ of $\mathbf{R}^N$ described by side-conditions as follows:

$Z(\sigma) = \{x \in \mathbf{R}^N : \langle B(t), x\rangle \le b(t) \text{ for every } t \in T\}$.

The following linear vector optimization problems can be defined:

a) LVM(σ) – determine p-efficient points subjected to side conditions;

b) LVW(σ) – determine weakly p-efficient points subjected to side conditions.

In the finite dimensional space $\mathbf{R}^l$ we consider the partial ordering generated by the usual positive cone $\mathbf{R}^l_+$. This means that, for $z^1, z^2 \in \mathbf{R}^l$, $z^1 \le z^2$ if this inequality holds for the corresponding coordinates of z^1 and z^2.

Definition 0.1. *A point $x_0 \in Z(\sigma)$ is called p-efficient if for each $x \in Z(\sigma)$ such that $P(x) \leq P(x_0)$ holds $P(x) = P(x_0)$.*

Definition 0.2. *A point $x_0 \in Z(\sigma)$ is called weakly p-efficient if for each $x \in Z(\sigma)$ such that $P(x) \leq P(x_0)$ holds $\langle p_j, x\rangle = \langle p_j, x_0\rangle$ for some $j \in \{1, 2, \dots, l\}$.*

Obviously each p-efficient point is also weakly p-efficient but the converse statement is not true in general.

In the set Θ we consider the norm: $\|\sigma\| = \|B\|_\infty + \|b\|_\infty + \|P\|_{\mathbb{R}^{Nl}}$, where $\|\cdot\|$ is the usual sup norm. This norm turns Θ into a Banach space.

The multivalued mapping $F : \Theta \longrightarrow \mathbf{R}^N$ maps every $\sigma \in \Theta$ to the set of weakly p-efficient points,

$F(\sigma) = \{x \in Z(\sigma) : x \text{ is a weakly } p\text{-efficient point}\}$.

It has been proved that $F(\sigma)$ is a closed subset of $\mathbf{R}^N$ [3].

Now, taking into account [4], we shall give a definition of well-posedness in the linear vector semi-infinite optimization.

Definition 0.3. *The problem* LVW(σ) *is well-posed if the mapping F is continuous at the point σ, (i.e. F is upper semi-continuous and lower semi-continuous at the point σ).*

A similar definition can be stated for the problem LVM(σ).

These definitions do not contain a property similar to the uniqueness in the scalar optimization. Later we shall find conditions for the linear vector semi-infinite optimization which is similar to the uniqueness in the scalar case. Also, we shall investigate the set of well-posed problems.

1. Well-posedness of linear vector optimization problems. Let us define sets

$L_M = \{\sigma \in \Theta : \text{LVM}(\sigma) \text{ has a solution}\}$ and

$L_W = \{\sigma \in \Theta : \text{LVW}(\sigma) \text{ has a solution}\}$.

Let A be a subset of some topological space X. int$A$ denotes the set of all interior points of A.

Now let us remind the theorem given in [5] which clarifies the relation between sets L_M and L_W.

Theorem 1.1. *Let compact T contain at least N points. Then*

$$\emptyset \neq \text{int} L_M \subset L_M \subset L_W \subset \overline{\text{int} L_M}.$$

It is worth mentioning that sets L_M and L_W do not coincide.

We need some additional definitions and statements:

Definition 1.2. *We say that Slater condition is fulfilled for $\sigma \in \Theta$ if there exists $x \in \mathbf{R}^N$ such that $\langle B(t), x\rangle < b(t)$ for every $t \in T$.*

Now we define the set:
$L_z = \{\sigma \in \Theta : Z(\sigma) \neq \emptyset\}$.
It is not difficult to prove (see for instance [6]) that:

Proposition 1.3. *The Slater condition is fulfilled (for $\sigma \in \Theta$) iff $\sigma \in \text{int} L_z$.*

The set $\text{int} L_z$ is an open subset of Θ, which is dense in L_z.

Lemma 1.4. *Let (X, q) be a complete metric space and let V be an open subset of X. Then there exists a metric $r(x, y)$ for the set V according to which V is a complete metric space.*

Let us consider the following metric on the set $\text{int} L_z$:
$r(x, y) = \|x - y\| + |1/f(x) - 1/f(y)|$, where
$f(x) = \inf\{\|x - y\|, y \in \Theta \backslash \text{int} L_z\}$.

This is a metric which turns $\text{int} L_z$ into a complete metric space and generates in it the same topology as the original norm in Θ. Now we restrict our considerations over the set $\text{int} L_z$ with the metric $r(x, y)$.

The replacement of the vector optimization problem by a family of scalar optimization problems is called scalarization. Various scalarization techniques are treated in [2,7]. Next follows a theorem, very useful for our purposes. It may be used as a basis for many scalarization procedures.

Theorem 1.5. *Let $\sigma \in \Theta$. Then $x \in F(\sigma)$ iff there exist $\alpha = (\alpha_1, \ldots, \alpha_l)$, $\alpha_i \geq 0$, $i = 1, 2, \ldots, l$ and $\sum_{i=1}^{l} \alpha_i = 1$ such that x is a solution of the problem*

$$LM(\alpha\sigma) : \min\{\langle \alpha, P(x) \rangle : x \in Z(\sigma)\}.$$

Let us define
B_i – the closed ball in the $\mathbf{R}^N$ with the radius i,
$F_i : \Theta \longrightarrow B_i$, $F_i(\sigma) = F(\sigma) \cap B_i$,
$A_i = \{\sigma \in \text{int} L_z : F_i(\sigma) \neq \emptyset\}$, $i = 1, 2, \ldots$.

Proposition 1.6. *A_i is a closed subset of the set $\text{int} L_z$ for every $i = 1, 2, \ldots$.*

P r o o f. Let us fix i and consider the convergent sequence $\{\sigma_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \sigma_n = \sigma_0$ such that for every n holds $\sigma_n \in A_i$, i.e. there exists $x_n \in F_i(\sigma_n)$.

Having in mind Theorem 1.5 we obtain that for every n there exists $\alpha^n = (\alpha_1^n, \ldots, \alpha_l^n)$, $\alpha_i^n \geq 0$, $i = 1, \ldots, l$ and $\sum_{i=1}^{l} \alpha_i^n = 1$ such that x_n is a solution of $L_M(\alpha^n\sigma)$. Without loss of generality we can consider that $\lim_{n\to\infty} \alpha^n = \alpha^0$ and $\lim_{n\to\infty} x_n = x_0$. It is obvious that α^0 belongs to the unit sphere in $\mathbf{R}_+^l$ and that $\|x_0\| \leq i$. □

Now we shall prove that $x_0 \in F_i(\sigma_0)$. For this aim we need the following theorem given in [1]:

Theorem 1.7. *If the set $Z(\sigma)$ is compact then the mapping $Z : \Theta \longrightarrow \mathbf{R}^N$ is u.s.c. at the point σ. If Slater condition is fulfilled, then the mapping Z is l.s.c. at this point.*

To complete the proof of the proposition we take some $x \in Z(\sigma_0)$. $\operatorname{int} L_z$ is a complete metric space, therefore $\sigma_0 \in \operatorname{int} L_z$.

Using Theorem 1.7 especially the l.s.c. of the mapping Z we obtain the sequence $\{y_n\}_{n\geq 1}$ such that $y_n \in Z(\sigma_n)$, $n = 1, 2, \ldots$ and $\lim_{n\to\infty} y_n = x$. For every $n = 1, 2, \ldots$ we know that $\langle \alpha^n, P^n(x_n)\rangle \leq \langle \alpha^n, P^n(y_n)\rangle$ from what follows that $\langle \alpha^0, P^0(x_0)\rangle \leq \langle \alpha^0, P^0(x)\rangle$. But $x \in Z(\sigma_0)$ was chosen arbitrarily and therefore x_0 is a solution of $\mathrm{LM}(\alpha^0\sigma_0)$. Now using Theorem 1.5 we obtain that $x_0 \in F_i(\sigma_0)$. The proposition is proved. □

Proposition 1.8. *The mapping $F_i : A_i \longrightarrow B_i$ has a closed graph for every $i = 1, 2, \ldots$.*

Proof. We have to prove that if $\lim_{n\to\infty}(\sigma_n, x_n) = (\sigma_0, x_0)$, where $x_n \in F_i(\sigma_n)$, $n = 1, 2, \ldots$ and $\sigma_n \in A_i$, $n = 0, 1, 2, \ldots$, then $x_0 \in F_i(\sigma_0)$. The rest of the proof is the same as in the previous proposition. □

We need the lemma:

Lemma 1.9. *Let X and Y be topological spaces and let $T : X \longrightarrow Y$ be a multivalued mapping with a closed graph. Let there exist an open neighbourhood V of the point $x \in X$ such that $T(V)$ is relatively compact subset of Y. Then the mapping T is u.s.c. at the point x.*

Having in mind that $F(\sigma)$ is a closed subset of $\mathbf{R}^N$ and the previous proposition we come to:

Proposition 1.10. *For every $i = 1, 2, \ldots$ the multivalued mapping $F_i : A_i \longrightarrow B_i$ is upper semi-continuous at every point $\sigma \in A_i$.*

Now by means of the famous theorem of Fort [8] we are in a position to formulate the following:

Theorem 1.11. *For every $i = 1, 2, \ldots$ there exists a dense and G_δ subset M_i of A_i such that for every $\sigma \in M_i$ the mapping $F_i : A_i \longrightarrow B_i$ is u.s.c. and l.s.c. at the point σ.*

The assertion in this theorem, which plays an important role in our assumptions, does not mean exactly well-posedness of the problems $\mathrm{LVW}(\sigma)$ but something very close to it.

2. Properties of the efficient sets. Let us consider some notations which motivate the forthcoming definition. When $l = 1$, then problems $\mathrm{LVM}(\sigma)$ and $\mathrm{LVW}(\sigma)$

– and sets L_M and L_W coincide respectively. Thus we obtain the well known linear semi-infinite optimization problem:

$$\text{LM}(\sigma): \begin{cases} \text{minimize } \langle p, x\rangle \\ \text{subject to} \\ \langle B(t), x\rangle \leq b(t) \text{ for every } t \in T \end{cases}$$

Various results are proved for the problems $\text{LM}(\sigma)$ similar to that given in [9].

Theorem 2.1. *Let the compact T contain at least N points. Then the set of points $\sigma \in \Theta$ for which $\text{LM}(\sigma)$ is Hadamard well-posed contains an open and dense subset of the set $L = \{\sigma \in \Theta : \text{LM}(\sigma)$ has a solution$\}$.*

We cannot expect a similar result for the linear vector semi-infinite optimization but the above considerations suggest that:

Definition 2.2. *We shall say that the point $\sigma \in \Theta$ is "nice" if the set of p-efficient points coincides with the set of weakly p-efficient points.*

Next we show that the definition given above is essential, i.e. we shall prove that most points (in the sense of Baire category) are "nice" in the case when T is a finite set.

Some definitions follow:

$T_{\{x\}} = \{t \in T : \langle B(t), x\rangle = b(t)\}$

is the set of active restrictions at the point $x \in Z(\sigma)$.

$D_\sigma(x) = \{z \in \mathbf{R}^N :$ there exists $\delta > 0$ such that $x + \delta z \in Z(\sigma)\}$

is the cone of feasible directions.

We mention some theorems given in [9].

Theorem 2.3. *Let $\sigma \in \Theta$. Then for every $x \in Z(\sigma)$ $D_\sigma(x) \subset \{B(t) : t \in T_{\{x\}}\}^*$. If $\sigma \in \text{int} L_z$, then $\overline{D_\sigma(x)} = \{B(t) : t \in T_{\{x\}}\}^*$.*

Theorem 2.4. *Let $\sigma \in \Theta$ and 0^{N+1} be the origin of $\mathbf{R}^{N+1}$. Then $\sigma \in \text{int} L_z$ if and only if $\text{int} Z(\sigma) \neq \emptyset$ and $0^{N+1} \notin \{(B(t), b(t)) : t \in T\}$.*

We define the cone:

$K(\sigma) = \text{co}(\text{cone}\{p_1, p_2, \ldots, p_l\})$.

Using the definition of *p*-efficient (weakly *p*-efficient), points reported in [10], and some trivial reasonings we reformulate the definition 0.1 (0.2) into:

Definition 2.5. *Let $\sigma \in \Theta$ and the cone $K^*(\sigma)$ be pointed ($\text{int} K^*(\sigma) \neq \emptyset$). The point $x \in Z(\sigma)$ is p-efficient (weakly p-efficient) if $(x - K^*(\sigma)) \cap Z(\sigma) = \{x\}$ ($(x - \text{int} K^*(\sigma)) \cap Z(\sigma) = \emptyset$).*

To the end of the paragraph we shall require that $T = \{t_1, t_2, \ldots, t_q\}$.

Lemma 2.6. *Let $\sigma \in \Theta$ and $y_0 \in Z(\sigma)$. There exists $\varepsilon > 0$ such that for every $y \in O_\varepsilon(y_0)$ holds $T_{\{y\}} \subset T_{\{y_0\}}$.*

Proof. Let us assume the opposite, i.e. there exist sequences $\{\varepsilon_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ and $\{t_n\}_{n\geq 1}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, $y_n \in O_{\varepsilon_n}(y_0)$ and $t_n \in T_{\{y_n\}}\backslash T_{\{y_0\}}$, $n = 1, 2, \ldots$. Since T is a finite set there exists $t_0 \in T$ such that $t_0 \in T_{\{y_{n_m}\}}\backslash T_{\{y_0\}}$, whereby $\langle B(t_0), y_{n_m}\rangle = b(t_0)$. This means that $\langle B(t_0), y_0\rangle = b(t_0)$, i.e. $t_0 \in T_{\{y_0\}}$, which is a contradiction. This completes the proof. □

Next we prove the main result of the article.

Theorem 2.7. *Let the compact T be a finite set. Then the set of points $\sigma \in \Theta$ which are not "nice", is of the first Baire category.*

Proof. Let $\sigma \in \mathrm{int}L_z$ be such that the sets of weakly p-efficient points and p-efficient points do not coincide. This means that there exists $y_0 \in Z(\sigma)$ which is weakly p-efficient but not p-efficient, i.e. we can find $x_0 \neq y_0$ such that $x_0 \in (y_0 - K^*(\sigma)) \cap Z(\sigma)$ and $P(x_0) \neq P(y_0)$.

We consider the number ε which was determined for the point $y_0 \in Z(\sigma)$ in Lemma 2.6. Therefore for every $y \in O_\varepsilon(y_0)$ holds $T_{\{y\}} \subset T_{\{y_0\}}$ and using Theorem 2.3 we obtain $\overline{D_\sigma(y_0)} \subset \overline{D_\sigma(y)}$. By Theorem 2.4 we can construct the sequence $\{x_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} x_n = x_0$ and $x_n \in \mathrm{int}Z(\sigma)$, $n = 1, 2, \ldots$.

We consider the sequence $\{y_n\}_{n\geq 1}$, where $y_n = x_n - y_0$, $n = 1, 2, \ldots$

It is obvious that for every n, $y_n \in \mathrm{int}D_\sigma(y_0)$, i.e. for every $y \in O_\varepsilon(y_0)$ and $n = 1, 2, \ldots$, $y_n \in \mathrm{int}D_\sigma(y)$.

Let $\langle y_0 - x_0, p_i\rangle > 0$ for $i = 1, 2, \ldots, j < l$ and $\langle y_0 - x_0, p_i\rangle = 0$ for $i = j+1, \ldots, l$.

Consider the sequence $\{P_m\}_{m\geq 1}$, $P_m = (p_1, \ldots, p_j, p_{j+1} + (y_0 - x_0)/m, \ldots, p_l + (y_0 - x_0)/m)$. Since $\lim_{n\to\infty} y_n = x_0 - y_0$, then $\langle y_n, p_i\rangle > 0$, $i = 1, 2\ldots, j$ for enough large n. We find the subsequence $\{y_{n_m}\}_{m\geq 1}$ such that $\langle y_{n_m}, p_i\rangle < (\|x_0 - y_0\|^2 - \nu)/m$, for some small $\nu > 0$ and $i = j+1, \ldots, l$.

Then for every $i = j + 1, \ldots, l$,

$$\langle -y_{n_m}, p_i + (y_0 - x_0)/m\rangle =$$

$$\langle -y_{n_m}, p_i\rangle + \langle y_{n_m}, (x_0 - y_0)/m\rangle(\nu - \|y_0 - x_0\|^2 + \langle y_{n_m}, x_0 - y_0\rangle)/m > 0,$$

whenever $m \geq m_0$ for some large m_0.

This entails immediately that $\mathrm{int}K^*(\sigma_m) \neq \emptyset$ and $y_{m_n} \in -\mathrm{int}K^*(\sigma_m)$, where $\sigma_m = (B, b, P_{n_m})$, $m \geq 1$.

It is evident that for every m and for every $y_0 \in O_\varepsilon(y_0)$ holds $y_{n_m} \in \mathrm{int}D_{\sigma_m}(y_0) \subset D_{\sigma_m}(y)$ and $y_{n_m} \in D_{\sigma_m}(y) \cap -\mathrm{int}K^*(\sigma_m)$ which shows that y is neither weakly nor p-efficient point.

We put $i = \max\{\|y\|, y \in \overline{O_\varepsilon(y_0)}\}$, then $\sigma \in A_i$. We assume that $\sigma \in M_i$. According to Theorem 1.11 it follows that the mapping $F_i : A_i \longrightarrow B_i$ is continuous at the point σ. But $\lim_{m\to\infty}\sigma_m = \sigma$ which contradicts the lower semi-continuity of mapping F_i.

To complete the proof we have to apply the density results given in Theorem 1.1. The theorem is proved. □

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