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ON THE RATE OF APPROXIMATION OF RANDOM FUNCTIONS

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ABSTRACT. An essential problem of the theory of approximation of random functions is the problem of determining the classes of approximation functions. An early result due to Ky Fan [6] asserts that a random function can be approximated by random splines of order one. Later D.Dugue [4] improves his result generalizing the Weierstrass' theorem. Thus the class of random polynomials turns to be a class of special interest. O.Onicescu and V.Istratescu [9] consider random Bernstein polynomials as a main tool of approximation of real-valued random functions continuous in probability on $[0,1]$. The paper deals with the estimation of the rate of approximation by Bernstein polynomials for random functions continuous in probability or in mean-square on the compact interval $[0,1]$. Various types of moduli of continuity determined with respect to various types of limits based on probability are introduced and explored for quantitative measuring the errors. The estimates of the errors are obtained in terms of the corresponding moduli. The results are generalizations of Popoviciu's [8] and Kamolov's [5] theorems.

1. Introduction. The theory of approximation of random functions (r.f.'s) is a natural extension of the classical approximation theory. Further on we shall call them "random theory" and "deterministic theory" respectively. It is interesting to trace the development of the "random theory". The scope of results there obtained is extremely wide. Some recent results due to Andrus & Brown [1] and Brown & Schreiber [3] go so far that generalize some results in the "deterministic theory" from an abstract point of view, with the help of generalized random functions. Another approach helps us to handle real problems as one of determining the classes of approximation functions. An early result in this direction is the theorem of Ky Fan [6] (which is in fact a generalization of E.Borel's approach within the "deterministic theory"). This theorem asserts (see also [4]) that a random function (r.f.) can be approximated by random splines of

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order one. Later D.Dugue [4] improves the result which actually is a generalizion of Weierstrass' theorem. Thus the class of random polynomials becomes a class of special interest. The text of Bharucha-Reid & Sambandham [2] is concerned mainly from the algebraic point of view but the approximation problem is also mentioned (p.16, 17). A paper of essential importance by O.Onicescu & V. Istratescu [9] gives references to a number of problems related to the topic. They consider random Bernstein polynomials as a main tool of approximation of real-valued r.f.'s which are continuous in probability. The use of Bernstein polynomials is natural on [0, 1]. What is left to be determined is the rate of approximation. Two types of continuous r.f.'s are treated in the paper, i.e. continuous in probability (in pr.) and continuous in mean-square (in msqr.). One is supposed to generalize the resuts from the "deterministic theory" in this respect once some appropriate tools are introduced and developed. Such a tool appears to be the various types of moduli of continuity determined with respect to (w.r.t.) various types of limits based on probability. This tool enables us to obtain quantitative results for the measuring errors. Part 2 of the present paper deals with the introduction of some concepts, definitions and properties. We introduce a random modulus of continuity for r.f.'s continuous in pr. A mean-square modulus of continuity is used, the latter being introduced by Kamolov [5]. Some properties of the two types of moduli are summarized as well. For r.f.'s continuous in pr. or in msqr. on [0, 1] the rate of approximation by random Bernstein polynomials is estimated in terms of the corresponding modulus of continuity.

The theorems obtained in Part 3 of the paper deal with the problems thus mentioned. Theorem 2 and Theorem 3 are generalizations of the theorem of Popoviciu [8] from the "deterministic theory" and Kamolov's theorem [5] from the "random theory".

2. Definitions, concepts and properties. Let $(\Omega, \mathcal{U}, P)$ be a probability space and T be a topological space.

Definition 1. *By a random function X_T in T we mean any function defined on T with values in the space $\mathcal{R}$ of all random variables defined on $(\Omega, \mathcal{U}, P)$ with phase space $(\mathfrak{X}, \mathfrak{Y})$, that is a measurable space of the Borel pr. space on the line.*

In other words the r.f. is a family $X_T = (X_t, t \in T)$ of random variables defined on the given probability space. We shall use the following notations: $X(t)$ or X_t for a random variable at t; $X(t,\omega)$ the value of $X(t)$ at ω, $\omega \in \Omega$ or the so called sample function, trajectory or path. $X(t,\omega) \in \mathfrak{X}_t$ where $\mathfrak{X}_t$ is a replica of the range space $\mathfrak{X}$.(Thus the r.f. maps Ω to $\mathfrak{X}_T = \prod_{t\in T} \mathfrak{X}_t$). Further T is a compact interval $T = [a, b]$, $a < b$ in the Euclidean line (whenever convenient it is replaced by $[0, 1]$ with no restriction of generality).

Definition 2.

A. *A r.f. X_T is called continuous in pr. at t_0, $t_0 \in T$ if for any $\varepsilon > 0, \eta > 0$, there exists $\delta(\varepsilon, \eta) > 0$ such that $P\{\omega\colon |X(t,\omega) - X(t_0,\omega)| \geq \varepsilon\} \leq \eta$ for $|t - t_0| \leq \delta(\varepsilon, \eta)$.*

B. *The r.f. X_T is continuous in pr. on T, if it is continuous in pr. at any t, $t \in T$.*

C. *The r.f. X_T is uniformly continuous in pr. on T if, for any $\varepsilon > 0$, $\eta > 0$, there exists $\delta(\varepsilon, \eta) > 0$ such that $P\{\omega: |X(t_1, \omega) - X(t_2, \omega)| \geq \varepsilon\} \leq \eta$ for any t_1, t_2 arbitrary chosen from T with the property $|t_1 - t_2| \leq \delta(\varepsilon, \eta)$.*

(Remark. Without loss of generality one may set $\eta = \varepsilon$).

A well known result in the theory of stochastic processes guarantees the existence of a r.f. $\tilde{X}_T$ defined on the same pr.space, which is eqiuvalent to X_T, i.e. $X(t, \omega) = \tilde{X}(t, \omega)$ (almost surely, for all $t \in T$) and which is separable and measurable (see [7]). Below we shall consider this separable measurable version and we shall denote it by X_T.

A theorem due to Slutsky [4] fits in Part B with Part C of Defition 2, i.e.

A r.f. X_T continuous in pr. on a compact interval T is uniformly continuous on T.

Our interest in r.f.'s continuous in pr. is due to the fact that we want to disregard fixed points of discontinuity. The appearance of moving points of discontinuity typical of any trajectory is prevented provided $X(t, \omega)$ is almost surely (a.s.) sample continuous. Sample continuity of X_T means that the process has neither fixed nor (outside a null event) moving discontinuity points. Obviously the a.s. sample continuity of X_T implies a.s. continuity of X_T and therefore continuity in pr., but the converse is not necessarily true. In mathematical analysis, and in particular, in the approximation theory, moduli of functions are used to characterize some properties of the functions.

Next we are going to introduce analogues of the classical modulus of continuity for a random function continuous in a certain sense. Various types of limits based on probability lead to various types of continuity and therefore various types of moduli of continuity.

Definition 3. *The random modulus of continuity of a.s. finite r.f. $X(t, \omega)$, $t \in T = [a, b]$ is a random function on $J = [0, b - a]$, $\delta \in J$, given by*

$$W_X(\delta, \omega) = \sup\{|X(t_1, \omega) - X(t_2, \omega)|: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\}. \tag{1}$$

Theorem 1. *The random function X_T a.s. finite is uniformly continuous in pr. on $T = [a, b]$ iff its random modulus of continuity is continuous in pr. at 0, i.e. it satisfies $W_X(\delta, \omega) \xrightarrow{P} 0$ as $\delta \to 0$. (Here $\xrightarrow{P}$ stands for limit in probability.)*

P r o o f. Suppose $X(t, \omega)$ is uniformly continuous in pr. on $T = [a, b]$, then for any $\varepsilon > 0$, $\eta > 0$, there exists $\delta = \delta(\varepsilon, \eta) > 0$ and for any $t_i \in T$, $i = 1, 2$ such that $|t_1 - t_2| \leq \delta$ $P\{\omega: |X(t_1, \omega) - X(t_2, \omega)| \geq \varepsilon\} \leq \eta$ holds. Hence for δ thus chosen we obtain that

$$P\{\omega: W_X(\delta, \omega) \geq \varepsilon\} = P\{\omega: \sup |X(t_1, \omega) - X(t_2, \omega)| \geq \varepsilon: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\}$$

$$\leq \sup P\{\omega: |X(t_1,\omega) - X(t_2,\omega)| \geq \varepsilon: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\} \leq \eta$$

or $p_m = P\{\omega: W_X(\delta,\omega) \geq \varepsilon\} \leq \eta$, where $m = [1/\delta]$ ($[\cdot]$ denotes an integer part). Then as $\delta \to 0$, $p_m \to 0$ and there exists a subsequence $m' = [1/\delta']$ such that $\sum p'_m \to 0$. It follows by Borel-Cantelli lemma that there exists a finite integer-valued r.v. ν_ε such that $W_X(\delta',\cdot) \leq \varepsilon$ holds outside a fixed null event for all $m' \geq \nu_\varepsilon$ or $\delta' \leq (b-a)/\nu_\varepsilon$.

Therefore as $\delta \to 0$, $W_X(\delta,\omega) \xrightarrow{P} 0$.

Suppose that the random modulus of continuity of the r.f. X_T is uniformly continuous in pr. at 0, i.e. as $\delta \to 0$, $W_X(\delta,\omega) \xrightarrow{P} 0$ uniformly. Then for any $\varepsilon > 0$, $\eta > 0$ $P\{\omega: W_X(\delta,\varepsilon) \geq \varepsilon\} \leq \eta$ for δ close to 0. This means that

$$P\{\omega: \sup |X(t_1,\omega) - X(t_2,\omega)| \geq \varepsilon: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\} \leq \eta.$$

If we choose δ from the above , i.e. δ corresponds to ε, η or $\delta = \delta(\varepsilon,\eta)$, then

$$P\{\omega: |X(t_1,\omega) - X(t_2,\omega)| \geq \varepsilon: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\}$$

$$\leq P\{\omega: \sup |X(t_1,\omega) - X(t_2,\omega)| \geq \varepsilon: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\} \leq \eta.$$

Hence $X(t,\omega)$ is uniformly continuous in pr. on T.

In this paper we do not treat the second order properties of a r.f. but with the help of the mean-square type of limit one can define mean-square type of continuity.

Further $\mathbf{E}$ stands for the expectation operator.

Definition 4.

A. *A r.f. X_T is called continuous in mean-square (msqr.) at t_0, $t_0 \in T$, if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\mathbf{E}\{[X(t,\omega) - X(t_0,\omega]^2\} \leq \varepsilon$, for $|t - t_0| \leq \delta(\varepsilon)$.*

B. *The r.f. X_T is continuous in msqr. on T, if it is continuous in msqr. at any t, $t \in T$.*

Definition 5. *A mean-square modulus of continuity of a r.f. real-valued and continuous in msqr. on $T = [0,1]$ is defined as follows*

$$W_X(\delta) = \sup\{\mathbf{E}^{1/2}\{[X(t_1,\omega) - X(t_2,\omega)]^2\}: |t_1 - t_2| \leq \delta,\ t_1, t_2 \in T\} \tag{2}$$

for $\delta \in [0,1]$.

Next we are going to consider the problem of approximation of r.f.'s X_T on a compact set T. Henceforth without loss of generality, we set $T = [0,1]$.

A theorem of Dugue [4] states that there exists a sequence of random polynomials $\{P_n\}_{n\geq 1}$ which converges uniformly in probability to X_T. This result is an analogue or generalization of Weierstrass' theorem and plays the same key role for the approximation theory of r.f.'s. The "deterministic theory" gives preference to the class of polynomials called Bernstein polynomials.

Definition 6. *The random Bernstein polynomial of order n of a r.f.* $X(t,\omega)$ *is a r.f. defined as follows*

$$B_n^X(t,\omega) = B_n(t,\omega) = \sum_{k=0}^{n} X(\frac{k}{n},\omega)b(n,t;k),$$

where $b(n,t;k) = \binom{n}{k}t^k(1-t)^{n-k}$ *is the binomial mass function for* $0 \le k \le n$.

O. Onicescu and V. Istratescu [9] proved the following result:

If $X(t,\omega)$ is a r.f. continuous in pr. on $[0,1]$ and almost surely bounded, then the sequence of the corresponding Bernstein polynomials converges uniformly in pr. to $X(t,\omega)$.

A theorem due to Kamolov [5] asserts that

If $X(t,\omega)$ is a r.f. continuous in msqr. on $[0,1]$, then

$$\mathbf{E}^{1/2}\{[X(t,\omega) - B_n^X(t,n)]^2\} < (1+\frac{2}{e})W_X(\sqrt{\frac{t(1-t)}{n}})$$

for $\delta \in (0,1)$ provided $W_X(\delta) \not\equiv 0$ and the inequality is the best possible one. This means that the approximation of X_T by Bernstein polynomials is of a rate neither larger nor smaller than the above.

3. Examination of the rate of approximation of a random function on [0,1] by Bernstein polynomials. Two theorems are obtained for r.f.'s continuous in pr. or continuous in msqr. respectively. These results should be treated as complementary ones.

Theorem 2. *Suppose* $X(t,\omega)$ *is a r.f. a.s. finite continuous in pr. on* $T = [0,1]$ *and* $\omega \in \Omega$, $\{B_n(t,\omega)\}_{n\ge1}$ *are the corresponding Bernstein polynomials and* $W_X(\delta,\omega)$ *is the random modulus of continuity determined by (1).*

A. *For a fixed* $t, t \in T$

$$|X(t,\omega) - B_n(t,\omega)| \le [1 + t(1-t)]W_X(n^{-1/2},\omega)$$

takes place;

B. *There exists an absolute constant* n_0 *such that for all* $n \ge n_0$ *and any* $t, t \in T$ *the inequality holds*

$$|X(t,\omega) - B_n(t,\omega)| \le 2W_X(\sqrt{\frac{t(1-t)}{n}},\omega);$$

C. *There exists an upper bound not depending on the choice of t such that*

$$|X(t,\omega) - B_n(t,\omega)| \le \frac{5}{4}W_X(n^{-1/2},\omega).$$

(Remark. Part C turns to be an analogue of the classical result due to Popoviciu (see [8]) in the "deterministic theory".)

P r o o f. Suppose t_1 and t_2 are two arbitrarily chosen points in T and δ, $\delta \in (0,1)$. We set n to $n = n(t_1, t_2; \delta) = \big[|t_1 - t_2|/\delta\big]$. Then we have

$$|X(t_1,\omega) - X(t_2,\omega)|$$

$$\le \sum_{k=0}^{n-1} |X(t_1+k\delta,\omega) - X(t_1+(k+1)\delta,\omega)| + |X(t_1+n\delta,\omega) - X(t_2,\omega)| \le (n+1)W_X(\delta,\omega)$$

and further we obtain that

$$|X(t,\omega) - B_n(t,\omega)| \le \sum_{k=0}^{n} |X(t,\omega) - X(\frac{k}{n},\omega)| b(n,t;k)$$

$$\le W_X(\delta,\omega)\sum_{k=0}^{n}[1+n(t,\frac{k}{n};\delta)]b(n,t;k) \le W_X(\delta,\omega)[\sum_{k=0}^{n} b(n,t;k) + \sum_{k=1}^{n} n(t,\frac{k}{n};\delta)b(n,t;k)]$$

$$\le W_X(\delta,\omega)\big[1+\delta^{-1}\sum_{k=1}^{n}|t-\frac{k}{n}|b(n,t;k)\big] \le W_X(\delta,\omega)\big[1+\delta^{-2}\sum_{k=1}^{n}(t-\frac{k}{n})^2 b(n,t;k)\big]$$

$$\le W_X(\delta,\omega)\big[1+\delta^{-2}t(1-t)/n\big].$$

One can easily realize that

$$\sum_{k=1}^{n}(t-\frac{k}{n})^2 b(n,t;k) = Var\{\frac{x}{n}\}$$

($Var\{\cdot\}$ stands for a random variable variance), where X is the binomial random variable with parameters n and t, i.e. $X \in B_i(n,t)$. Therefore $\mathbf{E}\{X/n\} = t$

$$Var\{\frac{x}{n}\} = \frac{1}{n^2}Var\{X\} = \frac{nt(1-t)}{n^2} = \frac{t(1-t)}{n}.$$

(The expression $t(1-t)$ is maximized for $t = 1/2$ and finally $Var\{X/n\} \le 1/4n$.)

The three parts of the theorem are obtained in accordance with the choice of δ, i.e. $\delta = n^{-1/2}$, $\delta = \sqrt{t(1-t)/n}$ and $\delta = n^{-1/2}$ taking into account the final estimate for $t = 1/2$.

Theorem 3. *Suppose $X(t,\omega)$ is a r.f. a.s. finite continuous in msqr. on $T = [0,1]$ for $\omega \in \Omega$, $\{B_n(t,\omega)\}_{n\ge1}$ are the corresponding Bernstein polynomials and $W_X(\delta)$ is the mean-square modulus of continuity determined by (2).*

A. *For a fixed t, $t \in T$ the inequality*

$$\mathbf{E}^{1/2}\{[X(t,\omega) - B_n(t,\omega)]^2\} \le [1 + 2t(1-t)/e]W_X(n^{-1/2})$$

holds;

B. *There exists an absolute constant n_0 such that for all $n \geq n_0$ and any t, $t \in T$ the inequality holds*

$$\mathbf{E}^{1/2}\{[X(t,\omega) - B_n(t,\omega)]^2\} \leq \left(1 + \frac{2}{e}\right) W_X\left(\sqrt{\frac{t(1-t)}{n}}\right);$$

C. *There exists an upper bound not depending on the choice of t such that*

$$\mathbf{E}^{1/2}\{[X(t,\omega) - B_n(t,\omega)]^2\} \leq \left(1 + \frac{1}{2e}\right) W_X\left(n^{-1/2}\right).$$

*(Remark. Part **B** is the theorem of Kamolov [5]. We include it here in order to accomplish the statement.)*

Proof. Carring out the very same calculations from Theorem 2 and making use of the proof of Part **B** by Kamolov [5], we obtain that

$$\mathbf{E}^{1/2}\{[X(t,\omega) - B_n(t,\omega)]^2\} \leq W_X(\delta)\left[1 + \frac{2}{e}\delta^{-2}\frac{t(1-t)}{n}\right].$$

Then for $\delta = n^{-1/2}$ we obtain Part **A** and Part **C**, the last being valid for any t from the unit interval where the inequality $t(1-t) \leq 1/4$ holds. Certainly if $\delta = \sqrt{t(1-t)/n}$, then the result of Kamolov is obtained.

The continuity in msqr. implies continuity in pr. and both the results are concurrent ones. Parts **C** of both theorems draw a comparison between the upper bound constants $5/4 = 1.25$ and $1 + 1/2e = 1.184$. Their values are similar but in fact they scale different moduli of continuity which is of great importance.

4. Final remarks. So far we have not taken into account the fact that the version X_T we work with is a representative of a class of equivalent r.f.'s from $\mathcal{R}$ w.r.t. a certain type of limit based on probability (which determines the type of continuity). It is well known fact that the space of such classes of equivalence (identified as $\mathcal{R}$) is a complete metric space with a corresponding distance. Concerning limits in probability the semidistance $d(X,Y) = \mathbf{E}\{|X-Y|/(1+|X-Y|)\}$ is used, while regarding limits in r-th mean the distance is defined as

$$d(X,Y) = \begin{cases} \mathbf{E}|X-Y|^r, & \text{if } r < 1, \\ \mathbf{E}^{1/r}|X-Y|^r, & \text{if } r \geq 1, \end{cases}$$

where X, Y stand for random variables for the two cases.

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