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## OPTION PRICING FORMULAE FOR SPECULATIVE PRICES MODELLED BY SUBORDINATED STOCHASTIC PROCESSES

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*Dedicated to Academician Ljubomir Iliev  
on the Occasion of his Eightieth Birthday*

**ABSTRACT.** Mandelbrot and Taylor [12,13] and Clark [2] developed two competing models to explain the non-normality of the price changes. The common feature of the models is that they are based on subordinated processes, in the Mandelbrot-Taylor model the resulting process is Lévy motion while in the Clark's model, the process admits finite variance. We exhibit option price formulae for the both models. The formulae are based on the limits of randomized versions of the Cox-Ross-Rubinstein binomial option pricing formula.

**1. Introduction.** There is no longer disagreement among specialists that the distribution of speculative price changes is longer-tailed than the normal. We refer to the seminal work of Mandelbrot [12,13,14], Fama [8], Mandelbrot and Taylor [15], Ziemba [22] for the stable Paretian model of the empirical distributions of returns on common stocks, see also Akgiray and Booth [1], Mittnik and Rachev [18] and the references there of. For the distributional aspects of returns on treasury bills, we refer to Du Mouchel [7], for commodity futures see Clark [2], Dusak [6], for exchange rates see McFarland et al [17], So [21].

Despite the non-normality of the speculative price changes in the practical use of the modern theory of contingent claim valuation, it is primarily referred to the Black-Scholes option pricing formula and its variations as the limit of the binomial pricing formula with nonrandom "up's" and "down's", see Cox and Rubinstein [4].

The aim of this paper is to derive continuous option price formulae under the assumption that the changes of the stock returns are described by a subordinated

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process as suggested in Mandelbrot and Taylor [15] and Clark [2]. In Section 2 we start with the Mandelbrot-Taylor subordinated Lévy motion as a process describing the stock movements. The corresponding new Black-Scholes type formula for option pricing (cf. Harrison and Pliska [9], Cox and Rubinstein [3], Karatzas [10]) is given in Theorem 1. Theorem 2 shows that our formula may remain unchanged even if we allow certain dependence between the price changes.

The Mandelbrot-Taylor approach is based on the fact that the involvement of the Lévy motion exhibit leptokurtosis which is typically observed in asset return data (i.e. they have fatter tails and are more peaked than the normal law) thus making them very good candidates for the distribution of price differences. Clark [2] approached the problem of non-normality of price changes distribution modelling the price movements with a subordinated to Brownian motion process with a directing process having finite variance. While in both models a subordinated process  $\xi(t) = X(\tau(t))$  is used, Mandelbrot and Taylor [15] assume that the "transaction time"  $\tau(t)$  is distributed as a positive stable process, in Clark [2],  $\tau(t)$  represents the trading volume at time  $t$  and has log-normal distribution. In section 3 we develop a formula for continuous option pricing based on the Clark's model, see Theorem 3. Our method is based on the Cox, Ross and Rubinstein [3] approximation method treating the Black-Scholes formula as the limiting case of binomial pricing formulae. Rachev and Ruschendorf [19] studied all possible weak limits of the binomial pricing formulae under the Cox-Ross-Rubinstein approach. While these limits describe pricing formulae for large class of stochastic processes modelling asset returns the Mandelbrot-Taylor and Clark subordinated processes are not among them. In contrast, in this paper our approach is based on a binomial option pricing formula with *random* up's and down's mimicking the increments of the subordinated process chosen to model the price movements. The result is formulae for continuous pricing subject to subordinated processes of price changes. It should be noted that, unlike the case of the classical Black-Scholes formula, it appears impossible to use the hedging argument throughout the whole computation. This is likely to be due to the incompleteness of the market in our case. Therefore, we do average at a certain point of our argument. However, we are taking average only with respect to the *magnitudes* of the jumps of the underlying Lévy motion, and using a hedging argument with respect to the *directions* of the jumps. It is our hope that this partial averaging will reduce the uncertainty and risk associated with any pricing by taking average.

## 2. Option pricing formula for asset returns following Lévy motion.

Mandelbrot and Taylor [15] model of non-normal price changes is based on the assumption that the price changes over a fixed number of transaction is normal, but the number of transaction in any time period is random with infinite variance. More precisely, let  $\{X(t), t \geq 0\}$  be a Brownian motion with zero drift and variance  $v^2$ , which is viewed as the process of stock log prices on the time scale measured in volume of transactions. Let  $\{\tau(t), t \geq 0\}$  be a positive  $\frac{\alpha}{2}$  - stable stochastic process with characteristic

function

$$(2.1) \quad E e^{i\theta\tau(t)} = \exp\{-\nu t|\theta|^{\alpha/2}(1 - i(\theta/|\theta|)\tan(\pi\alpha/4))\} \quad (0 < \alpha < 2, \nu > 0)$$

interpreted as the cumulative volume or number of transactions up to physical time  $t$ . Then  $\xi(t) = X(\tau(t))$  is a subordinated process to  $X(t)$  with a directing process  $\tau(t)$ , and  $\xi(t)$  represent the (log) price of the stock at time  $t$ . (Recall that by stock price changes in the financial literature one means the difference of the logarithms of the prices, in other words  $\xi(t) = \log S(t)$  where  $S(t)$  is the actual price of the stock at time  $t$ , see for example Mandelbrot [12] Harrison and Pliska [9]. The resulting process  $\xi(t)$  is now  $\alpha$ -stable Lévy motion with ch.f.

$$(2.2) \quad E e^{i\theta\xi(t)} = e^{-t|\sigma\theta|^\alpha},$$

where  $\sigma^\alpha = \nu(v^2/2)^{\alpha/2}[1 - \tan(\pi\alpha/4)]$ , see Mandelbrot and Taylor [15]. To model a stock price process whose logarithm is  $\xi(t)$  we assume that if the current price of a stock is  $S = S_0$ , the stock price  $S_1$  at the end of the first period is described by

$$(2.3) \quad S_0 = \begin{cases} S_1 = u_1 S_0 & \text{with probability } \frac{1}{2} \\ S_1 = d_1 S_0 & \text{with probability } \frac{1}{2} \end{cases}, (u_1 \geq 1 \geq d_1)$$

In contrast with the standard binomial option pricing model (cf. Cox and Rubinstein [4]) we assume that  $u_1$  and  $d_1$  are random and moreover

$$(2.4) \quad U_1 := \log u_1, \quad D_1 := \log d_1$$

have heavy tailed distributions. Continuing as in (2.3), the consecutive movements of the stock are given by

$$(2.5) \quad S_k \stackrel{d}{=} S \prod_{i=1}^k u_i^{\delta_i} d_i^{(1-\delta_i)}, \quad \text{or}$$

$$(2.6) \quad \log(S_k/S) \stackrel{d}{=} \sum_{i=1}^k (U_i \delta_i + D_i(1 - \delta_i)),$$

where  $U_i := \log u_i$ ,  $D_i := \log d_i$ , and  $\delta_i$ 's are i.i.d. Bernoulli ( $\frac{1}{2}$ ) independent of  $u_i$ 's and  $d_i$ 's. We assume that the log-increments of our stock price process are symmetrically distributed,

$$(2.7) \quad U_i = \sigma|X_i^{(n)}|, D_i = -U_i,$$

where  $n$  represents the number of movements until the terminal time  $T$  of a call and  $\{X_i^{(n)}, i = 1, \dots, n\}$  are *i.i.d.* symmetric Pareto *r.v.*'s with

$$(2.8) \quad P(|X_i^{(n)}| > x) = n^{-1} x^{-\alpha}, \quad x \geq n^{-1/\alpha}, \quad 1 < \alpha < 2.$$

(For most of commodities and for some stocks it is not a serious restriction to assume that the log-price changes are symmetrically distributed. In other words we can write,

$$(2.9) \quad \log(S_k/S) \stackrel{d}{=} \sigma \sum_{i=1}^k X_i^{(n)},$$

and thus the process

$$(2.10) \quad \xi_n(t) = \log(S_k/S), \quad T \frac{k-1}{n} < t \leq T \frac{k}{n}, \quad k = 1, \dots, n, \quad (\xi_n(0) = 0),$$

converges weakly to a symmetric  $\alpha$ -stable Lévy motion  $\xi(t)$  in  $D[0, T]$  with *ch.f.* given in (2.2), as desired in the Mandelbrot-Taylor model. Let  $r_i$  denote the "riskless interest rate" at the  $i$ th period,

$$(2.11) \quad r_i = \frac{1}{2}(u_i + d_i).$$

In contrast with the classical Cox-Ross-Rubinstein model  $r_i = r_i(\omega)$  is now random. With this in mind we continue to follow the usual arguments leading to the binomial option pricing formulae. For fixed  $\omega$ , in order to have an "equivalent portfolio" and "no riskless arbitrary opportunities", the value  $c = c^{(n)}$  of the call - with expiration date  $T$  which is just  $n$  periods away and striking price  $K$  - equals

$$(2.12) \quad c^{(n)} = \frac{2^{-n}}{r_1 \dots r_n} \left\{ (u_1 \dots u_n S - K)_+ + [(u_1 \dots u_{n-1} d_n S - K)_+ + \dots + (d_1 u_2 \dots u_n S - K)_+] + \dots + (d_1 \dots d_n S - K)_+ \right\}.$$

Similarly to the case for non-random  $u_i = u$  and  $d_i = d$  (c.f. Cox, Ross and Rubinstein (1979)) let us argue for  $c^{(n)}$  in the simplest case  $n = 1$ , *i.e.* when the expiration date is just one period away. By the end of the period the value of the call  $c_1$  equals  $(S_1 - K)_+ = b_1 c_+ + (1 - b_1) c_-$ ; where  $c_+ = (u_1 S - K)_+$ ,  $c_- = (d_1 S - K)_+$  and  $b_1$  is a Bernoulli random variable with success probability  $\frac{1}{2}$  and independent of  $u_1$ . In order to form an equivalent portfolio containing  $\Delta_1$  shares of stock and  $B_1$  dollars in riskless bonds, we must select  $\Delta_1$  and  $B_1$  to equate the end-of-period value of the portfolio,

$$b_1(u_1 S \Delta_1 + r_1 B_1) + (1 - b_1)(d_1 S \Delta_1 + r_1 B_1).$$

In other words,

$$\Delta_1 = \frac{c_+ - c_-}{(u_1 - d_1)S}, \quad B_1 = \frac{u_1 c_- - d_1 c_+}{(u_1 - d_1)S}.$$

Then the assumptions “no riskless arbitrary opportunities” and “riskless rate” (2.11) implies  $c = S \Delta_1 + B_1 = \frac{1}{2r_1}(c_+ + c_-) = \frac{1}{2r_1}((u_1 S - K)_+ + (d_1 S - K)_+)$ , which coincides with  $c^{(1)}$  in (2.12) as desired.

Formula (2.12) represent the random value  $c^{(n)}$  of the call. In contrast with the classical binomial pricing formula,  $c^{(n)}$  gives us a unique rational value for fixed  $\omega$ . Now, how much would we be willing to pay for the call at time 0? It is quite reasonable, in the absence of a unique deterministic solution, to look at the mean value  $C^{(n)} = E c^{(n)}$ . The next theorem provides an expression for the limit  $C = \lim_{n \rightarrow \infty} C^{(n)}$ .

Suppose  $Z_i$ 's are i.i.d. uniforms on (0,1) and  $\epsilon_i$ 's are independent of  $Z_i$ 's Rademacher random signs. Then  $X_i^{(n)} \stackrel{d}{=} \epsilon_i n^{-1/\alpha} Z_i^{-1/\alpha}$  and rearranging in  $(X_1^{(n)}, \dots, X_n^{(n)})$  in an increasing absolute order, say  $(X_{1,n}^{(n)}, \dots, X_{n,n}^{(n)})$ , we observe that the latter order statistics have the same joint distribution as

$$\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha} (\epsilon_1 \Gamma_1^{-1/\alpha}, \dots, \epsilon_n \Gamma_n^{-1/\alpha})$$

where  $\Gamma_1, \Gamma_2, \dots$  are Poisson arrivals with intensity 1, independent of  $\epsilon_i$ 's .

**Lemma 1.** (Binomial option pricing formula for heavy tailed distributed stock returns). *If the stock movements are described by the “discretized” Mandelbrot-Taylor model (2.8) – (2.10) then for any  $n \geq 1$*

$$(2.13) \quad C^{(n)} = E \frac{[S \exp(\sigma(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \sum_{i=1}^n \epsilon_i \Gamma_i^{-1/\alpha}) - K]_+}{2^{-n} \prod_{i=1}^n \exp(\sigma(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \Gamma_i^{-1/\alpha}) + \exp(-\sigma(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \Gamma_i^{-1/\alpha})}$$

or, equivalently,

$$(2.14) \quad C^{(n)} = E \frac{\left(S \exp(\sigma(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \sum_{i=1}^n \epsilon_i \Gamma_i^{-1/\alpha}) - K\right)_+}{E_{(\epsilon_1, \dots, \epsilon_n)} \exp(\sigma(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \sum_{i=1}^n \epsilon_i \Gamma_i^{-1/\alpha})},$$

where we use the standard notation  $E_{(\epsilon_1, \dots, \epsilon_n)}$  to denote the expectation taken with respect to  $\epsilon_1, \dots, \epsilon_n$ .

**Proof.** Recall that  $r_i = \frac{1}{2}(u_i + d_i)$ , then by the formula for  $c^{(n)}$ (cf. (2.12), (2.7), (2.11))

$$C^{(n)} = E \frac{\sum_{\{\delta_i = \pm 1, i=1, \dots, n\}} [S \exp(\sigma(\delta_1 |X_1^{(n)}| + \dots + \delta_n |X_n^{(n)}|)) - K]_+}{\prod_{i=1}^n (e^{\sigma |X_i^{(n)}|} + e^{-\sigma |X_i^{(n)}|})}.$$

We can rewrite  $C^{(n)}$  as

$$(2.15) \quad C^{(n)} = E \frac{(S e^{\sigma(X_1^{(n)} + \dots + X_n^{(n)})} - K)_+}{2^{-n} \prod_{i=1}^n (e^{\sigma|X_i^{(n)}|} + e^{-\sigma|X_i^{(n)}|})},$$

which implies (2.13) and (2.14).  $\square$

**Theorem 1.** (Pricing formula for stock returns governed by Lévy motion).  
Letting  $n \rightarrow \infty$  in the "discretized" Mandelbrot-Taylor model (2.8) – (2.11) implies  $C^{(n)} \rightarrow C$ , where

$$(2.16) \quad C = E \frac{(S \exp(\sum_{i=1}^{\infty} \sigma \epsilon_i \Gamma_i^{-1/\alpha}) - K)_+}{E_{(\epsilon_1, \epsilon_2, \dots)} \exp\{\sum_{i=1}^{\infty} \sigma \epsilon_i \Gamma_i^{-1/\alpha}\}}.$$

*Proof.* Without loss of generality set  $\sigma = 1$ . Using the representation (2.14) for  $C^{(n)}$  we let  $n \rightarrow \infty$ , then clearly  $(\frac{\Gamma_{n+1}}{n})^{1/\alpha} \sum_{i=1}^n \epsilon_i \Gamma_i^{-1/\alpha} \rightarrow \sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha}$  a.s., and thus the numerator of (2.14) converges to the numerator in (2.16). Our next step is to show that the denominator in (2.14) converges to the denominator in (2.16). Using the above limit relationship it is enough to show the following claim.

**Claim.** Suppose that  $a_1, a_2, \dots$  is a sequence of real numbers such that, as  $n \rightarrow \infty$ ,

$$(2.17) \quad \sum_{i=1}^n a_i \epsilon_i \rightarrow \sum_{i=1}^{\infty} a_i \epsilon_i \quad \text{a.s.},$$

which is equivalent to  $\sum_{i=1}^{\infty} a_i^2 < \infty$ . Suppose  $r_n \rightarrow 1$  as  $n \rightarrow \infty$ , then

$$(2.18) \quad E \exp(r_n \sum_{i=1}^n a_i \epsilon_i) \rightarrow E \exp(\sum_{i=1}^{\infty} a_i \epsilon_i).$$

**Proof of the claim.** By Hölder's inequality,

$$\begin{aligned} & |E \exp(r_n \sum_{i=1}^n a_i \epsilon_i) - E \exp(\sum_{i=1}^n a_i \epsilon_i)|^2 \\ & \leq [E \exp(2 \sum_{i=1}^{\infty} a_i \epsilon_i)] [E(-1 + \exp(r_n - 1) \sum_{i=1}^n a_i \epsilon_i)^2] \\ & =: T_1 + T_2. \end{aligned}$$

The term  $T_1$  in the above product is finite by hypothesis. As  $r_n \rightarrow 1$ , the second term  $T_2$  is bounded by

$$\begin{aligned}
 (2.18) \quad T_2 &\leq \int_0^\infty P(|r_n - 1| \left| \sum_{i=1}^n a_i \epsilon_i \right| > \log(1 + \sqrt{t})) dt \\
 &+ \int_0^1 P(|r_n - 1| \left| \sum_{i=1}^n a_i \epsilon_i \right| < \log(1 - \sqrt{t})) dt \\
 &\leq 2 \int_0^\infty \exp\left(-\frac{(\log(1 + \sqrt{t}))^2}{2(r_n - 1)^2 \sum_{i=1}^n a_i^2}\right) dt \\
 &+ 2 \int_0^1 \exp\left(-\frac{(\log \frac{1}{1 - \sqrt{t}})^2}{2(r_n - 1)^2 \sum_{i=1}^n a_i^2}\right) dt.
 \end{aligned}$$

The last inequality follows from the exponential bound for  $\sum_{i=1}^\infty b_i^2 < \infty$ ,

$$(2.19) \quad P\left(\left|\sum_{i=1}^\infty b_i \epsilon_i\right| > t\right) \leq 2 \exp\left(-\frac{t^2}{2 \sum_{i=1}^\infty b_i^2}\right),$$

see, e.g. Ledoux and Talagrand [11]. The two integrals in the RHS of (2.18) vanish as  $n \rightarrow \infty$ , since  $r_n \rightarrow 1$  and  $\sum_{i=1}^\infty a_i^2 < \infty$ , and thus

$$(2.20) \quad E \exp\left(r_n \sum_{i=1}^n a_i \epsilon_i\right) - E \exp\left(\sum_{i=1}^n a_i \epsilon_i\right) \rightarrow 0.$$

To show that

$$(2.21) \quad E \exp\left(\sum_{i=1}^n a_i \epsilon_i\right) \rightarrow E \exp\left(\sum_{i=1}^\infty a_i \epsilon_i\right)$$

we use the same arguments as before. Using the exponential bound (2.19) and  $\sum_{i=1}^\infty a_i^2 < \infty$ ,

$$\begin{aligned}
 &\left|E \exp\left(\sum_{i=1}^n a_i \epsilon_i\right) - E \exp\left(\sum_{i=1}^\infty a_i \epsilon_i\right)\right|^2 \\
 &\leq \text{const.} E\left(e^{-\sum_{i=n+1}^\infty a_i \epsilon_i} - 1\right)^2 \\
 &\leq \text{const.} \int_0^\infty \exp\left(-\frac{(\log(1 + \sqrt{t}))^2}{2 \sum_{i=n+1}^\infty a_i^2}\right) dt \\
 &+ \text{const.} \int_0^1 \exp\left(-\frac{(\log(1 + \sqrt{t}))^2}{2 \sum_{i=n+1}^\infty a_i^2}\right) dt
 \end{aligned}$$

and the latter bound vanishes as  $n \rightarrow \infty$ . Combining (2.20) and (2.21) completes the proof of the claim and, by the bounded convergence theorem, of the theorem as well.  $\square$



The construction of our “discrete” version of the Mandelbrot-Taylor model suggests that the option pricing formula (2.16) may remain unchanged even if the price changes are dependent. To see that, suppose the rate of return over the  $i$ -th period of time can have two possible values — each of them with probability  $1/2$ :

$$(2.22) \quad u_i^* - 1 = \exp(\Gamma_i^{-1/\alpha} W_i) - 1$$

and

$$(2.23) \quad d_i^* - 1 = \exp(-\Gamma_i^{-1/\alpha} W_i) - 1.$$

In (2.22) and (2.23),  $\alpha \in (0, 2)$ ,  $W_i$ 's are *i.i.d.* nonnegative *r.v.*'s, with  $E W_1^\alpha < \infty$ ,  $\Gamma_i$ 's are independent of  $W_i$ 's and represent a sequence of arrival times of a Poisson process with unit rate. The stock price after  $n$  periods of time equals

$$(2.24) \quad S_n^* = S \exp \sum_{i=1}^n (\epsilon_i U_i^* + (1 - \epsilon_i) D_i^*)$$

with  $\epsilon_i$ 's being Bernoulli ( $\frac{1}{2}$ ) *r.v.*'s independent of  $(\Gamma_i, W_i)$  and, as before,  $U_i^* = \log u_i^*$ ,  $D_i^* = \log d_i^*$ . In other words, (2.24) reads

$$(2.25) \quad \log(S_n^*/S) = \sum_{i=1}^n \epsilon_i \Gamma_i^{-1/\alpha} W_i.$$

Then, as  $n \rightarrow \infty$ , we have

$$(2.26) \quad \log(S_n^*/S) \rightarrow \sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i \stackrel{d}{=} S_\alpha(\sigma_\alpha, 0, 0).$$

Here  $S_\alpha(\sigma_\alpha, 0, 0)$  stands for a symmetric  $\alpha$ -stable law with scaling parameter  $\sigma_\alpha \geq 0$ , given by

$$(2.27) \quad \sigma_\alpha^\alpha = \frac{1}{c_\alpha} E W_1^\alpha, c_\alpha := \left( \int_0^\infty x^{-\alpha} \sin x dx \right)^{-1} = \begin{cases} \frac{\Gamma(2-\alpha) \cos(\pi\alpha/2)}{2/\pi} & \text{if } \alpha \neq 1, \\ 2/\pi & \text{if } \alpha = 1, \end{cases}$$

see, *e.g.* Samorodnitsky and Taqqu [20]. The limit relation (2.26) shows that at the terminal time the distribution of the stock price is the same as in the Mandelbrot-Taylor model. We now use the assumptions of “riskless interest rate”,  $r_i = \frac{1}{2}(u_i + d_i)$ , and “no riskless arbitrary opportunities”, to conclude that the mean-value  $C^{*(n)}$  of the

call  $c^{*(n)}$   $n$ -periods before the expiration rate equals

$$\begin{aligned}
 C^{*(n)} &= E c^{*(n)} \\
 &= E \frac{2^{-n}}{r_1 r_2 \dots r_n} \left( (e^{U_1 + \dots + U_n} S - K)_+ \right. \\
 (2.28) \quad &+ [(e^{U_1 + \dots + U_{n-1} + D_n} S - K)_+ + \dots \\
 &+ (e^{D_1 + U_2 + \dots + U_n} S - K)_+] + \dots \\
 &\left. + (e^{D_1 + \dots + D_n} S - K)_+ \right)
 \end{aligned}$$

**Theorem 2.** If the stock price after  $n$  moves is determined by (2.22) – (2.24) then

$$(2.29) \quad C^{*(n)} \rightarrow C^*$$

where

$$(2.30) \quad C^* = E \frac{(S e^{\sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i} - K)_+}{E_{(\epsilon_1, \epsilon_2, \dots)} e^{\sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i}}$$

The proof is similar to that of Theorem 1 and thus omitted.

**Remark 1.** From (2.26), (2.27) the limiting distribution for  $S_n^*$  depends on the distribution of  $W_i$ 's through the mean  $EW_1^\alpha$ . So, one should expect that  $C^*$  depend on the distribution of  $W_i$ 's only through  $EW_1^\alpha$ . To see that, we rewrite  $C^*$  in (2.30) as follows

$$C^* = E \frac{(e^{\sum_{i=1}^{\infty} \epsilon_i \Gamma_i^{-1/\alpha} W_i} S - K)_+}{\prod_{i=1}^{\infty} \frac{1}{2} [e^{\Gamma_i^{-1/\alpha} W_i} + e^{-\Gamma_i^{-1/\alpha} W_i}]}$$

Thus, it is enough to prove that the distribution of the point process  $\mathcal{N} = \{\Gamma_i^{-1/\alpha} W_i, i = 1, 2, \dots\}$  depends only on  $EW_1^\alpha$ . First it is easy to see that  $\mathcal{N}$  is a Poisson process. The next step is to show that its intensity measure  $\mu$  depends on the distribution of  $W_i$ 's only through  $EW_1^\alpha$ . In fact, for any  $\lambda > 0$

$$\begin{aligned}
 \mu((\lambda, \infty)) &= \sum_{i=1}^{\infty} P(\Gamma_i^{-1/\alpha} W_i > \lambda) \\
 &= \sum_{i=1}^{\infty} \int_0^{\infty} \frac{x^{i-1}}{(i-1)!} e^{-x} P(W_i^\alpha > x \lambda^\alpha) dx \\
 &= \lambda^{-\alpha} \int_0^{\infty} P(W_1^\alpha > x) dx = \lambda^{-\alpha} E W_1^\alpha.
 \end{aligned}$$

**3. Option pricing formula for price changes in the domain of attraction of the normal law.** While Mandelbrot [12,13] set out to explain the non-normality in price changes by assuming that they are  $\alpha$ -stable with  $\alpha < 2$ , Clark [2] presented the opposite hypothesis assuming that the price change is subordinate to Brownian motion with directing process having finite variance. Clark [2] modelled the process of stock changes by

$$(3.1) \quad \xi(t) = X(\tau(t)),$$

that is  $\xi$  is subordinated to  $X(t)$  with directing process  $\tau(t) \geq 0$ ,  $E \tau(t)^2 < \infty$ . If  $X$  and  $\tau$  have stationary independent increments,  $E X(t) = 0$ , and  $\text{Var}X(t) = v^2t$ , and  $E \tau(t) = \beta t$  then  $\xi(t)$  has stationary independent increments,  $E \xi(t) = 0$  and  $\text{Var}\xi(t) = \beta v^2t$ . The special case considered in Clark's paper is  $X$  being a Wiener process with zero mean and  $\text{Var}X(t) = \sigma_2^2t$  and  $\tau(t)$  a log-normal, that is the density of  $\tau(1)$  is

$$(3.2) \quad f(x, \mu, \sigma_1^2) = \frac{1}{2\pi\sigma_1^2x} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma_1^2}\right), x > 0.$$

The random process  $\xi$  has unit increments with density

$$(3.3) \quad f_{\xi(1)}(y) = \frac{1}{2\pi\sigma_1^2\sigma_2^2} \int_0^\infty v^{-3/2} \exp\left(-\frac{(\log v - \mu)^2}{2\sigma_1^2}\right) \exp\left(\frac{-y^2}{2v\sigma_2^2}\right) dv.$$

Since the choice of a log normal directed process is not completely justified in the Clark's paper [2] we shall only assume that  $\tau(t)$  has a finite first moment.

To model a stock price process whose logarithm is  $\xi(t)$  in (3.1) we assume the same "discretized" model of stock price as in Section 2, (2.5) and (2.6), but this time the log-increments of our stock price process are taken to be in the domain of attraction of the normal distribution. Specifically, we define the random up's ( $\tilde{u}_i$ 's) and down's ( $\tilde{d}_i$ 's) by

$$(3.4) \quad \log \tilde{u}_i := \tilde{U}_i := \sigma T^{1/2} |n^{-1/2} X_i|, \quad \log \tilde{d}_i := \tilde{D}_i := -U_i,$$

where  $X_1, \dots, X_n$  are *i.i.d.* symmetric *r.v.*'s with unit variance, and thus the "discretized" Clark's model of stock price is given by

$$(3.5) \quad \{\tilde{S}_k\}_{k=1, \dots, n} \stackrel{d}{=} \left\{ S \exp \sum_{i=1}^k (\epsilon_i \tilde{U}_i + (1 - \epsilon_i) \tilde{D}_i) \right\}_{k=1, \dots, n},$$

when  $\epsilon_i$ 's are Bernoulli ( $\frac{1}{2}$ ) independent of  $\tilde{U}_i$ 's.

The same arguments as in Section 2 yield a binomial option pricing formula – the random value ( $\bar{c}_n$ ) of a call  $n$  periods prior to the expiration date.

$$(3.6) \quad \bar{c}_n = \frac{2^{-n}}{\bar{r}_1 \dots \bar{r}_n} \left\{ (\tilde{u}_1 \dots \tilde{u}_n S - K)_+ + [(\tilde{u}_1 \dots \tilde{u}_{n-1} \tilde{d}_{n-1} S - K)_+ + \dots + (\tilde{d}_1 \tilde{u}_2 \dots \tilde{u}_n S - K)_+] + \dots + (\tilde{d}_1 \dots \tilde{d}_n S - K)_+ \right\},$$

where  $\bar{r}_i$  is the “riskless interest rate”

$$(3.7) \quad \bar{r}_i = \frac{1}{2}(\tilde{u}_i + \tilde{d}_i).$$

In contrast to the Mandelbrot-Taylor model, in the “discretized” Clark’s model, the product of the interest rate does converge to a constant.

**Lemma 2.** *If  $T$  is the expiration date corresponding to  $n$  movements in the discretized Clark’s model, then*

$$(3.8) \quad \bar{r}_1 \dots \bar{r}_n \xrightarrow{D} e^{\frac{1}{2}\sigma^2 T}.$$

*Proof.* The above limit relation follows immediately from the following claim.

**Claim.** Let  $X_1, X_2 \dots$  be a sequence of *i.i.d.* zero mean random variables with a finite variance  $\sigma^2 = E X_1^2$ . Then

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n \left( \frac{1}{2} e^{X_i, n^{-1/2}} + \frac{1}{2} e^{-X_i, n^{-1/2}} \right) = e^{\sigma^2/2} \text{ a.s.}$$

*Proof of the claim.* By the SLLN,

$$(3.9) \quad \frac{X_1^2 + \dots + X_n^2}{n} \xrightarrow[n \rightarrow \infty]{} \sigma^2 \text{ a.s.}$$

and thus, as  $i \rightarrow \infty$ ,

$$(3.10) \quad i^{-1/2} X_i \rightarrow 0 \text{ a.s.}$$

In particular, it follows from (3.10) that

$$(3.11) \quad \lim_{n \rightarrow \infty} n^{-1/2} \max_{i \leq n} |X_i| = 0 \text{ a.s.}$$

Fix any  $\omega$  for which both (3.9) and (3.11) hold. Clearly,

$$(3.12) \quad \lim_{a \rightarrow 0} \frac{\log \frac{(e^a + e^{-a})}{2}}{a^2/2} = 1.$$

It follows then from (3.11) and (3.12) that for any  $0 < \epsilon < 1$  there is an  $N = N(\omega, \epsilon)$  such that for every  $n > N$ , and every  $i \leq n$ ,

$$(3.13) \quad \log\left(\frac{e^{X_i n^{-1/2}} + e^{-X_i n^{-1/2}}}{2}\right) \in \left((1 - \epsilon)\frac{X_i^2}{2n}, (1 + \epsilon)\frac{X_i^2}{2n}\right).$$

It follows now that for every  $n > N$

$$(3.14) \quad \sum_{i=1}^n \left(\log\frac{e^{X_i n^{-1/2}} + e^{-X_i n^{-1/2}}}{2}\right) \in \left((1 - \epsilon)\frac{X_1^2 + \dots + X_n^2}{2n}, (1 + \epsilon)\frac{X_1^2 + \dots + X_n^2}{2n}\right).$$

Thus by (3.9),

$$\begin{aligned} (1 - \epsilon)\frac{\sigma^2}{2} &\leq \liminf_{n \rightarrow \infty} \sum_{i=1}^n \left(\log\frac{e^{X_i n^{-1/2}} + e^{-X_i n^{-1/2}}}{2}\right) \\ &\leq \limsup_{n \rightarrow \infty} \sum_{i=1}^n \left(\log\frac{e^{X_i n^{-1/2}} + e^{-X_i n^{-1/2}}}{2}\right) \leq (1 + \epsilon)\frac{\sigma^2}{2}. \end{aligned}$$

Since this is true for any  $1 > \epsilon > 0$ , we conclude that

$$(3.15) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\log\frac{e^{X_i n^{-1/2}} + e^{-X_i n^{-1/2}}}{2}\right) = \frac{\sigma^2}{2}.$$

The claim now follows from (3.15).  $\square$

**Remark 2.** Relation (3.6) strongly resembles the limit relation for the interest rate in the Cox-Ross-Rubinstein model. Recall that in their model  $\hat{r}_i = \hat{r}$  is a constant and  $\hat{r}^n = r^T$  where  $r - 1$  is the interest rate over a fixed unit of calendar year, and so over the elapsed time  $T$ ,  $r^T$  is the total return.

The next theorem gives us the option pricing formula for the mean value

$$\tilde{C} = \lim_{n \rightarrow \infty} E\tilde{c}^n$$

of the call under Clark's model.

**Theorem 3.** (Pricing formula for stock returns governed by the subordinated process  $X(\tau(t))$ ). *Letting  $n \rightarrow \infty$  in the "discretized" Clark model (3.6) - (3.7) implies*

$$(3.16) \quad \tilde{C} = \lim_{n \rightarrow \infty} E\tilde{c}^n \stackrel{d}{=} e^{-1/2\sigma^2 T} E(Se^{\sigma\sqrt{T}N} - K)_+,$$

where  $N$  has standard normal distribution.

Proof. From (3.6) and (3.4) a simple conditioning argument implies that

$$(3.17) \quad \tilde{C}^{(n)} = E\tilde{c}^{(n)} = E \frac{(S e^{\sigma T^{1/2} n^{-1/2} (X_1 + \dots + X_n)} - K)_+}{\tilde{r}_1 \cdots \tilde{r}_n}.$$

It is sufficient to show (3.16) for  $T = 1$ . In other words, we need to prove that if  $X_1, X_2, \dots$  are *i.i.d* symmetric random variables with a finite variance  $\sigma^2 = EX_1^2$ , then

$$(3.18) \quad E \frac{(S e^{n^{-1/2} (X_1 + \dots + X_n)} - K)_+}{\prod_{i=1}^n (\frac{1}{2} e^{X_i n^{-1/2}} + \frac{1}{2} e^{-X_i n^{-1/2}})} \rightarrow e^{-(1/2)\sigma^2} E(S e^{\sigma N} - K)_+.$$

Let  $\epsilon_1, \epsilon_2, \dots$  be a sequence of *i.i.d.* Rademacher random variables, independent of  $X_1, X_2, \dots$ . Then

$$(3.19) \quad \begin{aligned} & E \frac{(S e^{n^{-1/2} (X_1 + \dots + X_n)} - K)_+}{\prod_{i=1}^n (\frac{1}{2} e^{X_i n^{-1/2}} + \frac{1}{2} e^{-X_i n^{-1/2}})} \\ &= E \frac{(S e^{n^{-1/2} (\epsilon_1 X_1 + \dots + \epsilon_n X_n)} - K)_+}{E_{(\epsilon_1, \dots, \epsilon_n)} e^{n^{-1/2} (\epsilon_1 X_1 + \dots + \epsilon_n X_n)}} \\ &= E \frac{E_{(\epsilon_1, \dots, \epsilon_n)} (S e^{n^{-1/2} (\epsilon_1 X_1 + \dots + \epsilon_n X_n)} - K)_+}{E_{(\epsilon_1, \dots, \epsilon_n)} e^{n^{-1/2} (\epsilon_1 X_1 + \dots + \epsilon_n X_n)}} := E Z_n. \end{aligned}$$

Observe that  $Z_n \leq S$  a.s.  $\forall n$ . Therefore, bounded convergence theorem would apply (3.18) once we prove that

$$(3.20) \quad Z_n \rightarrow e^{-\frac{1}{2}\sigma^2} E(S e^{\sigma N} - K)_+ \quad \text{a.s.}$$

Since we have already proved that the denominator in  $Z_n$  converges a.s. to  $e^{\frac{1}{2}\sigma^2}$  it remains to show that

$$(3.21) \quad E_{(\epsilon_1, \dots, \epsilon_n)} (S e^{n^{-1/2} (\epsilon_1 X_1 + \dots + \epsilon_n X_n)} - K)_+ \rightarrow E(S e^{\sigma N} - K)_+ \quad \text{a.s.}$$

As before, it is enough to prove convergence in (3.21) for  $\omega$ 's for which both (3.9) and (3.11) hold. For the simplicity of notation, we will assume that  $\epsilon_1, \epsilon_2, \dots$  live on some other probability space  $(\Omega_1, \mathcal{F}_1, P_1)$ . Denote

$$(3.22) \quad a_1^{(n)} := n^{-1/2} X_i(\omega), i = 1, \dots, n, n = 1, 2, \dots$$

The first step is to show that the sequence  $\sum_{i=1}^n \epsilon_i a_i^{(n)}$ ,  $n = 1, 2, \dots$  converges in

distribution to  $\sigma N$ , as  $n \rightarrow \infty$ . For the corresponding characteristic functions we get

$$\begin{aligned}
 (3.23) \quad E_1 \exp(i\theta \sum_{j=1}^n \epsilon_j a_j^{(n)}) &= \prod_{j=1}^n E_1 e^{i\theta \epsilon_j a_j^{(n)}} \\
 &= \prod_{j=1}^n \left( \frac{1}{2} e^{i\theta a_j^{(n)}} + \frac{1}{2} e^{-i\theta a_j^{(n)}} \right) \\
 &= \prod_{j=1}^n \left( \frac{1}{2} e^{i\theta X_j n^{-1/2}} + \frac{1}{2} e^{-i\theta X_j n^{-1/2}} \right) \\
 &= \prod_{j=1}^n \cos(\theta X_j n^{-1/2}),
 \end{aligned}$$

and the same argument as in the proof of the lemma above shows that

$$(3.24) \quad E_1 e^{i\theta \sum_{j=1}^n \epsilon_j a_j^{(n)}} \rightarrow e^{-\frac{\sigma^2 \theta^2}{2}}.$$

Now, (3.24) will imply (3.21) if we show that the sequence

$$(3.25) \quad (S e^{\sum_{j=1}^n \epsilon_j a_j^{(n)}} - K)_+, n = 1, 2, \dots,$$

is uniformly integrable. To this end we show that

$$(3.26) \quad \sup_{n \geq 1} E_1 (S e^{\sum_{j=1}^n \epsilon_j a_j^{(n)}} - K)_+^2 < \infty.$$

In fact,

$$\begin{aligned}
 (3.27) \quad E_1 (S e^{\sum_{j=1}^n \epsilon_j a_j^{(n)}} - K)_+^2 &\leq S^2 E_1 e^{2 \sum_{j=1}^n \epsilon_j a_j^{(n)}} \\
 &= S^2 E_1 e^{2n^{-1/2} \sum_{j=1}^n \epsilon_j X_j} = S^2 \prod_{i=1}^n \left( \frac{1}{2} e^{2n^{-1/2} X_j} + \frac{1}{2} e^{-2n^{-1/2} X_j} \right)
 \end{aligned}$$

Clearly, this is an expression of the same form as in the claim of Lemma 2 and so by the same argument, it converges to a finite limit as  $n \rightarrow \infty$ . This proves (3.26).

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